Exercises, Section 2.4

Exercises 2.4.1

Find the orthogonal trajectories for the family of curves.

1. \( y = Cx^3 \).
2. \( x = Cy^4 \).
3. \( y = Cx^2 + 2 \).
4. \( y^2 = 2(C - x) \).
5. \( y = C \cos x \)
6. \( y = Ce^x \)
7. \( y = \ln(Cx) \)
8. \( (x + y)^2 = Cx^2 \)

Find the orthogonal trajectories for the family of curves.

9. The family of parabolas symmetric with respect to the \( x \)-axis and vertex at the origin.
10. The family of parabolas with vertical axis and vertex at the point \( (1, 2) \).
11. The family of circles that pass through the origin and have their center on the \( x \)-axis.
12. The family of circles tangent to the \( x \)-axis at \( (3, 0) \).

Show that the given family is self-orthogonal.

13. \( y^2 = 4C(x + C) \).
14. \( \frac{x^2}{C^2} + \frac{y^2}{C^2 - 4} = 1 \).

Exercises 2.4.2

1. A certain radioactive material is decaying at a rate proportional to the amount present. If a sample of 50 grams of the material was present initially and after 2 hours the sample lost 10\% of its mass, find:

   (a) An expression for the mass of the material remaining at any time \( t \).
   
   (b) The mass of the material after 4 hours.
(c) The half-life of the material.

2. What is the half-life of a radioactive substance if it takes 5 years for one-third of the material to decay?

3. The half-life of a certain radioactive material is 2 hours. How long will it take for a given amount of the material to decay to $1/10$ of its original mass?

4. The half-life of radium-226 is 1620 years.
   
   (a) If an original sample of 100 grams of radium-226 was present initially, how much will remain after 500 years?
   
   (b) How long will it take for the sample to be reduced to 25 grams?

5. The size of a certain bacterial colony increases at a rate proportional to the size of the colony. Suppose the colony occupied an area of 0.25 square centimeters initially, and after 8 hours it occupied an area of 0.35 square centimeters.
   
   (a) Estimate the size of the colony $t$ hours after the initial measurement.
   
   (b) What is the expected size of the colony after 12 hours?
   
   (c) Find the doubling time of the colony.

6. A biologist observes that a certain bacterial colony triples every 4 hours and after 12 hours occupies 1 square centimeter.
   
   (a) How much area did the colony occupy when first observed?
   
   (b) What is the doubling time for the colony?

7. In 1980 the world population was approximately 4.5 billion and in the year 2000 it was approximately 6 billion. Assume that the world population at each time $t$ increases at a rate proportional to the population at time $t$. Measure $t$ in years after 1980.
   
   (a) Find the growth constant and give the world population at any time $t$.
   
   (b) How long will it take for the world population to reach 9 billion (double the 1980 population)?
   
   (c) The world population for 2002 was reported to be about 6.2 billion. What population does the formula in (a) predict for the year 2002?

8. It is estimated that the arable land on earth can support a maximum of 30 billion people. Extrapolate from the data given in Exercise 7 to estimate the year when the food supply becomes insufficient to support the world population.
Exercises 2.4.3

1. A thermometer is taken from a room where the temperature is 72°F to the outside where the temperature is 32°F. After 1/2 minute, the thermometer reads 50°F.
   (a) What will the thermometer read after it has been outside for 1 minute?
   (b) How many minutes does the thermometer have to be outside for it to read 35°F?

2. A metal ball at room temperature 20°C is dropped into a container of boiling water (100°C). Given that the temperature of the ball increases 2° in 2 seconds, find:
   (a) The temperature of the ball after 6 seconds in the boiling water.
   (b) How long it will take for the temperature of the ball to reach 90°C.

3. An object at a temperature of 50°F is placed in an oven whose temperature is kept at 150°F. After 10 minutes, the temperature of the object is 75°F. Find:
   (a) An expression for the temperature of the object at any time t.
   (b) The time required for the object to reach 100°F.
   (c) The time required for the object to reach 200°F.

4. Suppose that a corpse is discovered at 10 p.m. and its temperature is determined to be 85°F. Two hours later, its temperature is 74°F. If the ambient temperature is 68°F, estimate the time of death.

5. An object with an initial temperature of 150°C is placed in a room which is kept at a constant temperature of 35°C. The object’s temperatures at 12:15 and 12:20 are 120°C and 90°C, respectively.
   (a) At what time was the object placed in the room?
   (b) At what time will the object’s temperature be 40°C?

Exercises 2.4.4

1. (a) Solve the initial value problem
   \[ \frac{dv}{dt} + rv = -g, \quad v(0) = v_0 \]
   in terms of r, g, and v_0.
   (b) Show that \( v(t) \to -mg/k \) as \( t \to \infty \). This is called the terminal velocity of the object.
   (c) Integrate \( v \) to obtain the height \( y \), assuming an initial height \( y(0) = y_0 \).
2. An object with mass 10 kg is dropped from a height of 200 m. Given that its drag coefficient is $k = 2.5 \text{ N/(m/s)}$, after how many seconds does the object hit the ground?

3. An object with mass 50 kg is dropped from a height of 200 m. It hits the ground 10 seconds later. Find the object’s drag coefficient $k$.

4. An object with mass 10 kg is projected upward (from ground level) with initial velocity 60 m/s. It hits the ground 8.4 seconds later.
   (a) Find the object’s drag coefficient $k$.
   (b) Find the maximum height.
   (c) Find the velocity with which the object hits the ground.

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**Exercises 2.4.5**

1. A tank with a capacity of $2 \text{ m}^3$ (2000 liters) is initially full of pure water. At time $t = 0$, salt water with salt concentration 5 grams/liter begins to flow into the tank at a rate of 10 liters/minute. The well-mixed solution in the tank is pumped out at the same rate.
   (a) Set up, and then solve, the initial-value problem for the amount of salt in the tank at time $t$ minutes.
   (b) Find the time when the salt concentration in the tank becomes 4 grams/liter.

2. A 100 gallon tank is initially full of water. At time $t = 0$, a 20% hydrochloric acid solution begins to flow into the tank at a rate of 2 gallons/minute. The well-mixed solution in the tank is pumped out at the same rate.
   (a) Set up, and then solve, the initial-value problem for the amount of hydrochloric acid in the tank at time $t$ minutes.
   (b) Find the time when the hydrochloric acid concentration becomes 10%.

3. A room measuring $10 \text{ m} \times 5 \text{ m} \times 3 \text{ m}$ initially contains air that is free of carbon monoxide. At time $t = 0$, air containing 3% carbon monoxide enters the room at a rate of $1 \text{ m}^3/\text{minute}$, and the well-circulated air in the room leaves at the same rate.
   (a) Set up, and then solve, the initial-value problem for the amount of carbon monoxide in the room at time $t$ minutes.
   (b) Find the time when the carbon monoxide concentration in the room reaches 2%.

4. A tank with a capacity of $1 \text{ m}^3$ (1000 liters) is initially half full of pure water. At time $t = 0$, 4% salt solution begins to flow into the tank at a rate of 30 liters/minute. The well-mixed solution in the tank is pumped out at a rate of 20 liters/minute.
(a) Set up, and then solve, the initial-value problem for the amount of salt in the tank between time $t = 0$ and the time when the tank becomes full.
(b) Find the salt concentration of the solution in the tank during this process.

5. A 100 gallon tank is initially full of pure water. At time $t = 0$, water containing salt at concentration 15 grams/gallon begins to flow into the tank at a rate of 1 gallon/minute, while the well-mixed solution in the tank is pumped out at a rate of 2 gallons/minute.

(a) Set up, and then solve, the initial-value problem for the amount of salt in the tank between time $t = 0$ and the time when the tank becomes empty.
(b) Find the maximum amount of salt in the tank during this process.

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**Exercises 2.4.6**

1. A rumor spreads through a small town with a population of 5,000 at a rate proportional to the product of the number of people who have heard the rumor and the number who have not heard it. Suppose that 100 people initiated the rumor and that 500 people heard it after 3 days.

   (a) How many people will have heard the rumor after 8 days?
   (b) How long will it take for half the population to hear the rumor.

2. A flu virus is spreading through a city with a population of 25,000. The disease is spreading at a rate proportional to the product of the number of people who have it and the number who don’t. Suppose that 100 people had the flu initially and that 400 people had it after 10 days.

   (a) How many people will have the flu after 20 days?
   (b) How long will it take for half the population to have the flu?

3. Let $y$ be the logistic function (4). Show that $dy/dt$ increases for $y < M/2$ and decreases for $y > M/2$. What can you conclude about $dy/dt$ when $y = M/2$?

4. Solve the logistic equation by means of the change of variables

   $$\begin{align*}
y(t) &= v(t)^{-1}, \\
y'(t) &= -v(t)^{-2}v'(t).
\end{align*}$$

Express the constant of integration in terms of the initial value $y(0) = y_0$.

5. Suppose that a population governed by a logistic model exists in an environment with carrying capacity of 800. If an initial population of 100 grows to 300 in 3 years, find the intrinsic growth rate $k$. 