1.3 Initial Conditions; Initial-Value Problems

As we noted in the preceding section, we can obtain a particular solution of an nth order differential equation simply by assigning specific values to the n constants in the general solution. However, in typical applications of differential equations you will be asked to find a solution of a given equation that satisfies certain *preassigned* conditions.

Example 1. Find a solution of

$$y' = 3x^2 - 2x$$

that passes through the point $(1, 3)$.

SOLUTION In this case, we can find the general solution by integrating:

$$y = \int \left(3x^2 - 2x\right)\, dx = x^3 - x^2 + C.$$

The general solution is $y = x^3 - x^2 + C$.

To find a solution that passes through the point $(2, 6)$, we set $x = 2$ and $y = 6$ in the general solution and solve for C:

$$6 = 2^3 - 2^2 + C = 8 - 4 + C \qquad \text{which implies} \qquad C = 2.$$

Thus, $y = x^3 - x^2 + 2$ is a solution of the differential equation that satisfies the given condition. In fact, it is the only solution that satisfies the condition since the general solution represented all solutions of the equation and the constant C was uniquely determined. ■

Example 2. Find a solution of

$$x^2y'' - 2xy' + 2y = 4x^3$$

which passes through the point $(1, 4)$ with slope 2.

SOLUTION As shown in Example 4 in the preceding section, the general solution of the differential equation is

$$y = C_1x^2 + C_2x + 2x^3.$$

Setting $x = 1$ and $y = 4$ in the general solution yields the equation

$$C_1 + C_2 + 2 = 4 \quad \text{which implies} \quad C_1 + C_2 = 2.$$

The second condition, slope 2 at $x = 1$, is a condition on y'; we want $y'(1) = 2$. We calculate y':

$$y' = 2C_1x + C_2 + 6x^2,$$

and then set $x = 1$ and $y' = 2$. This yields the equation

$$2C_1 + C_2 + 6 = 2 \quad \text{which implies} \quad 2C_1 + C_2 = -4.$$

Now we solve the two equations simultaneously:

$$\begin{aligned} C_1 + C_2 &= 2 \\ 2C_1 + C_2 &= -4 \end{aligned}$$

We get: $C_1 = -6$, $C_2 = 8$. A solution of the differential equation satisfying the two conditions is

$$y(x) = -6x^2 + 8x + 2x^3.$$

It will follow from our work in Chapter 3 that this is the only solution of the differential equation that satisfies the given conditions. ■

INITIAL CONDITIONS Conditions such as those imposed on the solutions in Examples 1 and 2 are called *initial conditions.* This term originated with applications where processes are usually observed over time, starting with some initial state at time $t = 0$.

Example 3. The position $y(t)$ of a weight suspended on a spring and oscillating up and down is governed by the differential equation

$$y'' + 9y = 0.$$

(a) Show that the general solution of the differential equation is:

$$y(t) = C_1 \sin 3t + C_2 \cos 3t.$$

(b) Find a solution that satisfies the initial conditions $y(0) = 1$, $y'(0) = -2$.

SOLUTION

(a)

$$\begin{aligned} y &= C_1 \sin 3t + C_2 \cos 3t \\ y' &= 3C_1 \cos 3t - 3C_2 \sin 3t \\ y'' &= -9C_1 \sin 3t - 9C_2 \cos 3t \end{aligned}$$

Substituting into the differential equation, we get

$$y'' + 9y = (-9C_1 \sin 3t - 9C_2 \cos 3t) + 9\,(C_1 \sin 3t + C_2 \cos 3t) = 0.$$

Thus $y(t) = C_1 \sin 3t + C_2 \cos 3t$ is the general solution.

(b) Applying the initial conditions, we obtain the pair of equations

$$y(0) = 1 = C_1 \sin 0 + C_2 \cos 0 = C_2 \quad \text{which implies} \quad C_2 = 1,$$

$$y'(0) = -2 = 3C_1 \cos 0 - 3C_2 \sin 0 \quad \text{which implies} \quad C_1 = -\tfrac{2}{3}.$$

A solution which satisfies the initial conditions is: $y(t) = -\frac{2}{3}\sin 3t + \cos 3t$. ■

Any n-th order differential equation with independent variable x and unknown function y can be written in the form

$$F\left(x, y, y', y'', \ldots, y^{(n-1)}, y^{(n)}\right) = 0. \tag{1}$$

by moving all the non-zero terms to the left-hand side. Since we are talking about an n-th order equation, $y^{(n)}$ must appear explicitly in the expression F. Each of the other arguments may or may not appear explicitly. For example, the third-order differential equation

$$x^2 y''' - 2xy'' = y^2 e^{xy}$$

written in the form of equation (1) is

$$x^2 y''' - 2xy'' - y^2 e^{xy} = 0$$

and $F(x, y, y', y'', y''') = x^2 y''' - 2xy'' - y^2 e^{xy}$. Note that y' does not appear explicitly in the equation. However, it is there "implicitly." For example, $y'' = (y')'$, $y''' = (y')''$.

n-th ORDER INITIAL-VALUE PROBLEM An *n-th order initial-value problem* consists of an n-th order differential equation

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

together with n (initial) conditions of the form

$$y(c) = k_0,\ y'(c) = k_1,\ y''(c) = k_2,\ \ldots,\ y^{(n-1)}(c) = k_{n-1}.$$

where c and $k_0,\ k_1,\ \ldots,\ k_{n-1}$ are given numbers.

It is important to understand that to be an n-th order initial-value problem there must be n conditions (same n) of exactly the form indicated in the definition. For example, the problem:

- Find a solution of the differential equation

  $$y'' + 9y = 0$$

  satisfying the conditions $y(0) = 0$, $y(\pi) = 0$ is not an initial-value problem; the two conditions are not of the form in the definition, namely $y(c) = \alpha$, $y'(c) = \beta$.

Similarly, the problem:

- Find a solution of the differential equation

$$y''' - 3y'' + 3y' - y = 0$$

satisfying the conditions $y(0) = 1,\ y'(0) = 2$ is not an initial-value problem; a third order equation requires three conditions: $y(c) = k_0,\ y'(c) = k_1,\ y''(c) = k_2$.

EXISTENCE AND UNIQUENESS The fundamental questions in any course on differential equations are:

(1) Does a given initial-value problem *have* a solution? That is, do solutions to the problem *exist*?

(2) If a solution does exist, is it *unique*? That is, is there exactly one solution to the problem or is there more than one solution.

The initial-value problems in Examples 1, 2, and 3 each had a unique solution; values for the arbitrary constants in the general solution were uniquely determined.

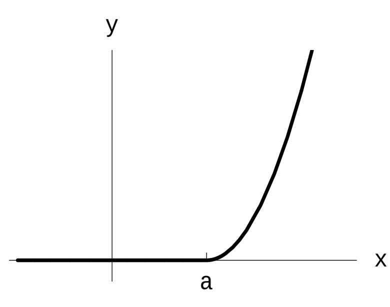
Example 4. The function $y = x^2$ is a solution of the differential equation $y' = 2\sqrt{y}$ and $y(0) = 0$. Thus the initial-value problem

$$y' = 2\sqrt{y}; \qquad y(0) = 0.$$

has a solution. However, $y \equiv 0$ also satisfies the differential equation and $y(0) = 0$. Thus, the initial-value problem does not have a *unique* solution. In fact, for any positive number a, the function

$$y_a(x) = \begin{cases} 0, & x \le a \\ (x-a)^2, & x > a \end{cases}$$

is a solution of the initial-value problem. ■



Example 5. The one-parameter family of functions $y = Cx$ is the general solution of

$$y' = \frac{y}{x}.$$

There is no solution that satisfies $y(0) = 1$; the initial-value problem

$$y' = \frac{y}{x}, \quad y(0) = 1$$

does not have a solution. ■

The questions of *existence and uniqueness* of solutions will be addressed in the specific cases of interest to us. A general treatment of existence and uniqueness of solutions of initial-value problems is beyond the scope of this course.

Exercises 1.3

1. (a) Show that each member of the one-parameter family of functions

$$y = Ce^{5x}$$

is a solution of the differential equation $y' - 5y = 0$.

(b) Find a solution of the initial-value problem $y' - 5y = 0,\ y(0) = 2$.

2. (a) Show that each member of the two-parameter family of functions

$$y = C_1e^{2x} + C_2e^{-x}$$

is a solution of the differential equation $y'' - y' - 2y = 0$.

(b) Find a solution of the initial-value problem

$$y'' - y' - 2y = 0; \quad y(0) = 2,\ y'(0) = 1.$$

3. (a) Show that each member of the one-parameter family of functions

$$y = \frac{1}{Ce^x + 1}$$

is a solution of the differential equation $y' + y = y^2$.

(b) Find a solution of the initial-value problem $y' + y = y^2;\ y(1) = -1$.

4. (a) Show that each member of the three-parameter family of functions

$$y = C_2x^2 + C_1x + C_0$$

is a solution of the differential equation $y''' = 0$.

(b) Find a solution of the initial-value problem

$$y''' = 0; \quad y(1) = 1,\ y'(1) = 4,\ y''(1) = 2.$$

(c) Find a solution of the initial-value problem

$$y''' = 0; \quad y(2) = y'(2) = y''(2) = 0.$$

5. (a) Show that each member of the two-parameter family of functions

$$y = C_1 \sin 3x + C_2 \cos 3x$$

is a solution of the differential equation $y'' + 9y = 0$.

(b) Find a solution of the initial-value problem

$$y'' + 9y = 0; \quad y(\pi/2) = y'(\pi/2) = 1.$$

6. (a) Show that each member of the two-parameter family of functions

$$y = C_1 x^2 + C_2 x^2 \ln x$$

is a solution of the differential equation $x^2 y'' - 3xy' + 4y = 0$.

(b) Find a solution of the initial-value problem

$$x^2 y'' - 3xy' + 4y = 0; \; y(1) = 0, \; y'(1) = 1.$$

(c) Is there a member of the two-parameter family which satisfies the initial condition $y(0) = y'(0) = 0$?

(d) Is there a member of the two-parameter family which satisfies the initial condition $y(0) = 0, y'(0) = 1$? If not, why not?

7. (a) Show that each member of the two-parameter family of functions

$$y = C_1 x + C_2 x^{1/2}$$

is a solution of the differential equation $2x^2 y'' - xy' + y = 0$.

(b) Find a solution of the initial-value problem

$$2x^2 y'' - xy' + y = 0; \; y(4) = 1, \; y'(4) = -2.$$

(c) Is there a member of the two-parameter family which satisfies the initial condition $y(0) = 1, \; y'(0) = 2$? If not, why not?

8. Each member of the two-parameter family of functions

$$y = C_1 \sin x + C_2 \cos x$$

is a solution of the differential equation $y'' + y = 0$.

(a) Determine whether there are one or more members of this family that satisfy the conditions

$$y(0) = 0, \quad y(\pi) = 0.$$

(b) Show that the zero function, $y \equiv 0$, is the only member of the family that satisfies the conditions

$$y(0) = 0, \quad y(\pi/2) = 0.$$

9. Given the differential equation $(y')^2 - xy' + y = 0$.

   (a) Show that the family of straight lines $y = Cx - C^2$ is the general solution of the equation

   (b) Show that $y = \frac{1}{4}x^2$ is a solution of the equation. Note that this function is not included in the general solution of the equation; it is a singular solution of the equation.

10. Given the differential equation $x(y')^2 - 2yy' + 4x = 0$.

    (a) Show that the one-parameter family $y = \dfrac{x^2 + C^2}{C}$ is the general solution of the equation

    (b) Show that each of $y = 2x$ and $y = -2x$ is a solution of the equation. Note that these functions are not included in the general solution of the equation; they are singular solutions of the equation.

Find the differential equation of the given family.

11. $y = Cx^3 + 1.$

12. $y = Cx^2 + 3$

13. $y^3 = Cx^2 + 3.$

14. $y^2 = Cx^4 - 2$

15. $y = Ce^{2x} + e^{-2x}$

16. $y = Ce^x + \sin x.$

17. $y = C_1x + C_2.$

18. $y = C_1e^x + C_2e^{-2x}.$

19. $y = (C_1 + C_2x)e^{2x}$

20. $y = C_1x + C_2x^2.$

21. $y = C_1x + C_2x^{-1}.$

22. $y = C_1 \cos 3x + C_2 \sin 3x$.

23. $y = C_1 \sin(3x + C_2)$.

24. $y = C_1 e^{2x} \cos 3x + C_2 e^{2x} \sin 3x$.

25. $y = C_1 + C_2 x + C_3 x^2$.

26. $y = C_1 x + C_2 x^2 + C_3 x^3$.

27. Refer to Example 4. Verify that the function $y_a(x)$ is a solution of the initial-value problem $y' = 2\sqrt{y},\ y(0) = 0$.
