1.2 n-Parameter Family of Solutions and General Solution; Particular Solutions

Introduction. You know from your experience in previous mathematics courses that the calculus of functions of several variables (limits, graphing, differentiation, integration and applications) is more complicated than the calculus of functions of a single variable. By extension, therefore, you would expect that the study of partial differential equations would be more complicated than the study of ordinary differential equations. This is indeed the case! Since the intent of this material is to introduce some of the basic theory and methods for differential equations, we shall confine ourselves to ordinary differential equations from this point forward. Hereafter the term *differential equation* shall be interpreted to mean *ordinary differential equation.* Partial differential equations are studied in subsequent courses. ■

We begin by considering the simple first-order differential equation

$$y' = f(x)$$

where f is some given function. In this case we can find y simply by integrating:

$$y = \int f(x)\,dx = F(x) + C$$

where F is an antiderivative of f and C is an arbitrary constant. Not only did we find *a* solution of the differential equation, we found a whole family of solutions, each member of which is determined by assigning a specific value to the constant C. In this context, the arbitrary constant is called a *parameter* and the family of solutions is called a *one-parameter family.*

Remark In calculus you learned that not only is each member of the family $y = F(x) + C$ a solution of the differential equation but this family actually represents the set of *all* solutions of the equation; that is, there are no other solutions outside of this family. ■

Example 1. The differential equation

$$y' = 3x^2 - \sin\,2x$$

has the one-parameter family of solutions

$$y(x) = \int \left(3x^2 - \sin\,2x\right)\,dx = x^3 + \tfrac{1}{2}\cos\,2x + C$$

As noted above, this family of solutions represents the set of all solutions of the equation. ■

In a similar manner, if we are given a second order equation of the form

$$y'' = f(x)$$

then we can find y by integrating twice, with each integration step producing an arbitrary constant of integration.

Example 2. If

$$y'' = 6x + 4e^{2x},$$

then

$$y' = \int \left(6x + 4e^{2x}\right)\,dx = 3x^2 + 2e^{2x} + C_1$$

and

$$y = \int \left(3x^2 + 2e^{2x} + C_1\right)\,dx = x^3 + e^{2x} + C_1x + C_2, \quad C_1,\ C_2 \text{ arbitrary constants.}$$

The set of functions

$$y = x^3 + e^{2x} + C_1x + C_2$$

is a *two-parameter family* of solutions of the differential equation

$$y'' = 6x + 4e^{2x}.$$

Again from calculus, we can conclude that this family actually represents the set of all solutions of the differential equation; there are no other solutions. ■

n-PARAMETER FAMILY OF SOLUTIONS The examples given above are very special cases. In general, to find the solutions of an n-th order differential equation we would expect, intuitively, to "integrate" n times, with each integration step producing an arbitrary constant of integration. As a result, we *expect* an n-th order differential equation to have an *n-parameter family of solutions.*

SOLVING A DIFFERENTIAL EQUATION To *solve* an n-th order differential equation means to find an n-parameter family of solutions. It is important to understand that the two n's here are the same. For example, to solve a fourth-order differential equation we need to find a four-parameter family of solutions.

Example 3. Show that $y = Ce^{kx}$ is a one-parameter family of solutions of

$$y' = ky, \quad k \text{ a given constant.} \quad \text{(Equation (1) in Section 1.1)}$$

SOLUTION

$$\begin{aligned} y &= Ce^{kx} \\ y' &= kCe^{kx} \end{aligned}$$

Substituting into the differential equation, we get

$$\begin{aligned} kCe^{kx} &\stackrel{?}{=} k\left(Ce^{kx}\right) \\ kCe^{kx} &= kCe^{kx}. \end{aligned}$$

Thus $y = Ce^{kx}$ is a one-parameter family of solutions. You were shown in calculus that $y = Ce^{kx}$ represents the set of all solutions of the equation. ■

Example 4. Show that $y = C_1x^2 + C_2x + 2x^3$ is a two-parameter family of solutions of

$$x^2y'' - 2xy' + 2y = 4x^3.$$

SOLUTION We calculate the first two derivatives of y and then substitute into the differential equation:

$$\begin{aligned} y &= C_1x^2 + C_2x + 2x^3 \\ y' &= 2C_1x + C_2 + 6x^2, \\ y'' &= 2C_1 + 12x; \end{aligned}$$

$$x^2(2C_1 + 12x) - 2x\left(2C_1x + C_2 + 6x^2\right) + 2\left(C_1x^2 + C_2x + 2x^3\right) \stackrel{?}{=} 4x^3.$$

Simplifying the left-hand side and re-arranging the terms, we get

$$\begin{aligned} C_1\left(2x^2 - 4x^2 + 2x^2\right) + C_2(-2x + 2x) + 12x^3 - 12x^3 + 4x^3 &\stackrel{?}{=} 4x^3 \\ C_1(0) + C_2(0) + 4x^3 &\stackrel{?}{=} 4x^3 \\ 4x^3 &= 4x^3 \end{aligned}$$

Thus, for any two constants C_1, C_2, the function $y = C_1x^2 + C_2x + 2x^3$, is a solution of the differential equation. The set of functions $y = C_1x^2 + C_2x + 2x^3$ is a two-parameter family of solutions of the equation. In Chapter 3 we will see that this two-parameter family represents the set of all solutions of the equation. ■

GENERAL SOLUTION/SINGULAR SOLUTIONS For most of the equations that we will study in this course, an n-parameter family of solutions of a given n-th order equation will represent the set of *all* solutions of the equation. In such cases, the term *general solution* is often used in place of n-parameter family of solutions. Because it less cumbersome, we will use the term “general solution” rather than “n-parameter family of solutions” recognizing that there is possible imprecision in the use of the term; in some cases an n-parameter family of solutions may not be the set of all solutions .

Solutions of an n-th order differential equation which are not included in an n-parameter family of solutions are called *singular solutions.*

Example 5. Consider the differential equation

$$y' = 4x(y-1)^{1/2}.$$

$y = (x^2 + C)^2 + 1$ is a one-parameter family of solutions (verify this). (In Section 2.2 you will learn how to solve this equation.) Also, it is easy to see that the constant function $y \equiv 1$ is a solution of the equation:

$$y \equiv 1 \quad \text{implies} \quad y' \equiv 0.$$

and

$$0 = 4x(1-1)^{1/2} = 0;$$

the equation is satisfied. This solution is not included in the general solution because there is no number that you can assign to C that will produce the solution $y \equiv 1$; $y \equiv 1$ is a singular solution. ■

Additional examples of differential equations having singular solutions are given in the Exercises 1.3.

PARTICULAR SOLUTION If specific values are assigned to the arbitrary constants in the general solution of a differential equation, then the resulting solution is called a *particular solution* of the equation.

Example 6. (a) $y = Ce^{5x}$ is the general solution of the first order differential equation $y' = 5y$ (see Example 3); $y = 200e^{5x}$ is a particular solution of the equation.

(b) $y = C_1x^2 + C_2x + 2x^3$ is the general solution of the second order differential equation

$$x^2y'' - 2xy' + 2y = 4x^3.$$

Setting $C_1 = 2$ and $C_2 = 3$, we get the particular solution $y = 2x^2 + 3x + 2x^3$. ■

THE DIFFERENTIAL EQUATION OF AN n-PARAMETER FAMILY If we are given an n-parameter family of curves, then we can regard the family as the general solution of an n^{th}-order differential equation and attempt to find the equation. The equation that we search for, called the *differential equation of the family*, should be free of the parameters (arbitrary constants), and its order should equal the number of parameters. The general strategy for finding the differential equation of a given n-parameter family is to differentiate the equation n times. This will produce a system of $n+1$ equations which can be used to eliminate the parameters.

Example 7. Given the one-parameter family $y^2 = Cx^3 + 3$. Find the differential equation of the family.

SOLUTION Since we have a one-parameter family, we are looking for a first order equation. Differentiating the given equation, we obtain

$$2yy' = 3Cx^2$$

We can solve this equation for C to get $C = \dfrac{2yy'}{3x^2}$. Substituting this "value" of C into the given equation, we get

$$y^2 = \left(\frac{2yy'}{3x^2}\right)x^3 + 3 = \frac{2xyy'}{3} + 3 \quad \text{which simplifies to} \quad y' = \frac{3y^2 - 9}{2xy}.$$

This is the differential equation of the given one-parameter family. ■

Example 8. Find the differential equation of the two-parameter family

$$y = \frac{C_1}{x} + C_2.$$

SOLUTION We are looking for a second order differential equation. Differentiating twice, we obtain the equations

$$y' = -\frac{C_1}{x^2} \qquad \text{and} \qquad y'' = \frac{2C_1}{x^3}$$

Solving each of these equations for C_1, we get

$$C_1 = -x^2 y' \qquad \text{and} \qquad C_1 = \tfrac{1}{2}\, x^3 y''.$$

Therefore

$$\tfrac{1}{2}\, x^3 y'' = -x^2 y' \qquad \text{which simplifies to} \qquad xy'' + 2y' = 0.$$

This is the differential equation of the given family.

Example 9. Find the differential equation of the two-parameter family

$$y = C_1 \cos 2x + C_2 \sin 2x$$

SOLUTION We differentiate twice:

$$\begin{aligned} y &= C_1 \cos 2x + C_2 \sin 2x \\ y' &= -2C_1 \sin 2x + 2C_2 \cos 2x \\ y'' &= -4C_1 \cos 2x - 4C_2 \sin 2x \end{aligned}$$

Multiplying the first equation by 4 and adding it to the third equation, we get

$$y'' + 4y = 0.$$

This is the differential equation of the given family. ■

Remark These examples illustrate that there is no "general method" for finding the differential equation of a given n-parameter family of functions. You can only follow the strategy and try to find some way to eliminate the parameters from the system of equations.
■