Problem 1.

(a) Given the points: $A\,(1,-2,1),\ B\,(3,2,2),\ C\,(-2,1,-5)$.

(i) Prove that A, B and C are the vertices of a right triangle.

(ii) Determine the length of the hypotenuse of the triangle.

(iii) Find an equation for the sphere that has the hypotenuse as a diameter.

(b) Given the vectors: $\mathbf{a} = 2\,\mathbf{i} - \mathbf{k},\ \mathbf{b} = 2\,\mathbf{i} + 3\mathbf{j} - \mathbf{k},\ \mathbf{c} = -\mathbf{i} + 3\,\mathbf{j} - 2\,\mathbf{k}$.

(i) Calculate: $3(2\mathbf{a} - \mathbf{b} + \mathbf{c})$

(ii) Find: $\text{proj}_{\mathbf{b}}\mathbf{c}$.

(iii) Determine a unit vector in the direction of $\mathbf{b} \times \mathbf{a}$.

(iv) Determine the volume of the parallelepiped that has $\mathbf{a}, \mathbf{b},$ and $\mathbf{c}$ as sides.

Problem 2. Given the plane $P:\ 3x - y + 4z = 3$, the line $L:\ \dfrac{x-1}{-2} = \dfrac{y+4}{2} = \dfrac{z+3}{2}$, and the point $A\,(-3,0,5)$.

(a) Determine whether P and L are parallel.

(b) Determine whether A lies on L. If it doesn't, find the distance from A to L.

(c) Determine the parametric equations for the line M that passes through A and is perpendicular to P.

(d) Find the point of intersection of M and P.

Problem 3. Given the planes $P_1:\ x + 4y - z = 10,\ P_2:\ 3x - y + 2z = 4$, and the point $A\,(-2,1,0)$.

(a) Determine an equation for the plane that contains A and is parallel to P_1.

(b) Determine whether A lies on P_2. If it doesn't, find the distance from A to P_2.

(c) Determine whether P_1 and P_2 are parallel. If not, find symmetric equations for the line of intersection of P_1 and P_2.

(d) Determine the cosine of the angle between P_1 and P_2.

Problem 4.

(a) Let $\mathbf{f}(t) = (2t+1)\,\mathbf{i} - (\cos \pi\, t)\,\mathbf{j}$ and $\mathbf{g}(t) = t^2\,\mathbf{i} + 3\,\mathbf{j}$.

(i) Find $\lim\limits_{t \to 3} \mathbf{f}(t)$.

(ii) Calculate $[\mathbf{f}(t)\cdot\mathbf{g}(t)]'$.

(b) The position of an object at time t is given by the vector function

$$\mathbf{r}(t) = \left(e^{-t}\,\sin t\right)\,\mathbf{i} + \left(e^{-t}\,\cos t\right)\,\mathbf{j} + 3t\,\mathbf{k}$$

Determine:

(i) The velocity vector $\mathbf{v}(t)$.

(ii) The speed of the object at time t.

(iii) The acceleration vector $\mathbf{a}(t)$.

(iv) $\lim_{t\to\infty} \mathbf{v}(t)$

Problem 5.

(a) A curve C in the plane is defined by the parametric equations:

$$x(t) = \tfrac{1}{2}\,t^2 + 1, \qquad y(t) = \tfrac{1}{3}\,t^3 - 1.$$

(i) Find the length of C from $t = 0$ to $t = 3$.

(ii) Find the curvature of C at the point where $t = 1$.

(b) The vector function $\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k}$ determines a curve C in space.

(i) Find scalar parametric equations for the line tangent to C at the point where $t = 3$.

(ii) Find the length of C from $t = 0$ to $t = 3$.

(iii) Determine the unit tangent vector $\mathbf{T}$ and the principal normal vector $\mathbf{N}$ at $t = 1$.

(iv) Find an equation for the osculating plane at $t = 1$.

Problem 6. The vector function $\mathbf{r}(t) = (\cos t + t\,\sin t)\,\mathbf{i} + (\sin t - t\,\cos t)\,\mathbf{j} + \frac{1}{2}\sqrt{3}\,t^2\,\mathbf{k}$ determines a curve C in space.

(a) Find the unit tangent vector $\mathbf{T}(t)$ and the principal normal vector $\mathbf{N}(t)$.

(b) Determine the curvature κ of C.

(c) Determine the tangential and normal components of acceleration; express the acceleration vector $\mathbf{a}(t)$ in terms of $\mathbf{T}$ and $\mathbf{N}$.