Problem 1.

(a) (*i*) domain: the whole plane; range: $[0, \infty)$

(*ii*) domain: $\{(x, y, z) : x^2 + y^2 + z^2 > 1\}$ i.e., the exterior of the unit sphere; range: $(0, \infty)$

(b) (*i*) a family of ellipses: $4x^2 + y^2 = c$

(*ii*) a family of planes: $2x + 3y + 6z = c$

(c) $C(x, y) = 5xy + \dfrac{72}{x} + \dfrac{72}{y}$

Problem 2.

(a) 0 (b) 0 (c) 0 (d) $\dfrac{2\lambda}{1 + \lambda^2}$ (e) No

Problem 3.

(a) $f_{xx} = y^4 e^{xy}$; $f_{yx} = f_{xy} = 3y^2 e^{xy} + xy^3 e^{xy} - \dfrac{1}{y^2}$

(c) $\dfrac{du}{dt} = 2x\,\cos t - 8y\,e^{2t} + 9z^2 = \sin\,2t - 8e^{4t} + 81t^2$

(d) $\dfrac{\partial z}{\partial u} = \left(2e^{2x}\,\ln y\right)(2u) + \left(\dfrac{1}{y}\,e^{2x}\right)(-2)$

$\dfrac{\partial z}{\partial v} = \left(2e^{2x}\,\ln y\right)(-2) + \left(\dfrac{1}{y}\,e^{2x}\right)(2v)$

Problem 4.

(a) (*i*) $\nabla F = (2x + 4y)\,\mathbf{i} + (3z + 4x)\,\mathbf{j} + 3y\,\mathbf{k}$

(*ii*) $-\nabla f(2, 2) = \left(\frac{1}{2} - \frac{\pi}{4}\right)\,\mathbf{i} - \frac{1}{2}\,\mathbf{j}$; rate: $-\|\nabla f(2, 2)\|$

(b) $F'_{\mathbf{u}}(1, 1, -5) = (6\,\mathbf{i} - 11\,\mathbf{j} + 3\,\mathbf{k}) \cdot \left(\frac{1}{2}\,\mathbf{i} + \frac{3}{4}\,\mathbf{j} - \frac{1}{4}\sqrt{3}\,\mathbf{k}\right) = \dfrac{-21 - 3\sqrt{3}}{4}$

(c) tangent plane: $2(x - 3) + 6(y + 1) - 3(z + 2) = 0$

(d) tangent plane: $\left(\frac{1}{2} - \frac{\pi}{4}\right)(x - 2) + \frac{1}{2}(y + 2) - (z + \pi/2) = 0$

normal line: $x = 2 + \left(\frac{1}{2} - \frac{\pi}{4}\right)t$, $y = -2 + \frac{1}{2}\,t$, $z = -\frac{1}{2}\,\pi - t$

Problem 5.

(a) No; $\dfrac{\partial P}{\partial y} = 6x^2y + 3$, $\dfrac{\partial Q}{\partial x} = 6x^2y + 3y$; $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$

(b) Yes, $\dfrac{\partial P}{\partial y} = 2x\,e^y + 4x = \dfrac{\partial Q}{\partial x}$; $f(x,y) = x^2e^y + 2x^2y + \frac{1}{2}\,e^{2x} + \frac{1}{2}\,\sin 2y - y + C$, C any constant.

Problem 6.

(a) $f_x = 2x - 2xy$, $f_y = 4y - x^2$

Solve the system of equations:

$$2x - 2xy = 0$$

$$4y - x^2 = 0$$

Solutions: $(0,0)$, $(2,1)$, $(-2,1)$

(b) $f_{xx} = 2 - 2y$, $f_{xy} = -2x$, $f_{yy} = 4$, $D = AC - B^2$

point	A	B	C	D	result
$(0,0)$	2	0	4	8	loc. min
$(2,1)$	0	-4	4	-16	saddle
$(-2,1)$	0	4	4	-16	saddle

Problem 7.

(a) $\nabla f = (2x-1)\,\mathbf{i} + 4y\,\mathbf{j} = \mathbf{0}$ at $\left(\frac{1}{2}, 0\right) \in D$.

The boundary of D is given by: $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le 2\pi$.

f on the boundary is given by:

$$f(\mathbf{r}(t)) = g(t) = \cos^2 t + 2\,\sin^2 t - \cos t = \sin^2 t - \cos t + 1, \quad 0 \le t \le 2\pi.$$

$$g'(t) = 2\,\sin t\,\cos t + \sin t; \quad g'(t) = 0 \implies \sin t = 0 \text{ or } \cos t = -\frac{1}{2} \implies t = \pi,\ t = 2\pi/3,\ t = 4\pi/3.$$

The corresponding points are: $(-1,0)$, $\left(-1/2, \sqrt{3}/2\right)$, $\left(-1/2, -\sqrt{3}/2\right)$, plus $(1,0)$ (from the endpoints 0 and 2π)

$f(\frac{1}{2}, 0) = -\frac{1}{4}$ (abs. min.), $f(1,0) = 0$, $f(-1,0) = 2$,
$f(-1/2, \sqrt{3}/2) = f(-1/2, -\sqrt{3}/2) = \frac{9}{4}$ (abs. max.)

(b) $\nabla f = (2-2x)\,\mathbf{i} + (2-2y)\,\mathbf{j} = \mathbf{0}$ at $(1,1) \in D$; $f(1,1) = 4$.

The boundary of D consists of the three sides of the triangle:

$$C_1: \quad 0 \le x \le 9: \quad \mathbf{r}(t) = t\,\mathbf{i}, \quad 0 \le t \le 9,$$

$$C_2: \quad x + y = 9: \quad \mathbf{r}(t) = t\,\mathbf{i} + (9-t)\,\mathbf{j}, \quad 0 \le t \le 9,$$

$$C_3: \quad 0 \le y \le 9: \quad \mathbf{r}(t) = t\,\mathbf{j}, \quad 0 \le t \le 9.$$

On C_1 : $f(\mathbf{r}(t)) = g(t) = 2 + 2t - t^2$; $g'(t) = 2 - 2t$; $g'(t) = 0 \implies t = 1$

$$g(0) = f(0,0) = 2, \quad g(1) = f(1,0) = 3, \quad g(9) = f(9,0) = -61$$

On C_2 : $f(\mathbf{r}(t)) = g(t) = -2t^2 + 18t - 61$; $g'(t) = 0 \implies t = \frac{9}{2}$

$$g(0) = f(0,9) = -61, \quad g(9/2) = f\left(\tfrac{9}{2}, \tfrac{9}{2}\right) = -\frac{41}{2}, \quad g(9) = f(9,0) = -61$$

On C_3 : $f(\mathbf{r}(t)) = g(t) = 2 + 2t - t^2$; $g'(t) = 2 - 2t$; $g'(t) = 0 \implies t = 1$

$$g(0) = f(0,0) = 2, \quad g(1) = f(0,1) = 3, \quad g(9) = f(0,9) = -61$$

The absolute max of f is: $f(1,1) = 4$; the absolute min is: $f(9,0) = f(0,9) = -61$.

Problem 8.

(a) Let the dimensions of the box be: length — x, width — y, height — z.

Maximize the volume: $V = xyz$ subject to the constraint: $g(x,y,z) = 2x + 2y + z - 108 = 0$.

$$\nabla V = yx\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}; \qquad \nabla g = 2\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$$

$$\nabla V = \lambda \nabla g \implies yz = 2\lambda,\ xz = 2\lambda,\ xy = \lambda$$

Solve the system of equations:

$$yz = 2\lambda$$

$$xz = 2\lambda$$

$$xy = \lambda$$

$$2x + 2y + z = 108 \text{ (constraint equation)}$$

The solution is: $x = 18$, $y = 18$, $z = 36$. The dimensions that will maximize the volume of the box are: $18 \times 18 \times 36$; the maximum volume is: $V = 11,664$ cubic inches or 6.75 cubic feet.

(b) Let the dimensions of the box be: length – x, width – y, height – z.

The cost of construction is: $C(x,y,z) = 4(xy) + 3(2xz) + 3(2yz) = 4xy + 6xz + 6yz$

Minimize the cost: $C = 4xy + 6xz + 6yz$ subject to the constraint: $V(x,y,z) = xyz - 12 = 0$.

$$\nabla C = (4y + 6z)\,\mathbf{i} + (4x + 6z)\,\mathbf{j} + (6x + 6y)\,\mathbf{k}; \qquad \nabla V = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$$

$$\nabla C = \lambda \nabla V \implies 4y + 6z = \lambda yz,\ 4x + 6z = \lambda xz,\ 6x + 6y = \lambda xy$$

Solve the system of equations:

$$\begin{aligned} 4y + 6z &= \lambda yz \\ 4x + 6z &= \lambda xz \\ 6x + 6y &= \lambda xy \\ xyz &= 12 \text{ (constraint equation)} \end{aligned}$$

The solution is: $x = \sqrt[3]{18},\ y = \sqrt[3]{18},\ z = \sqrt[3]{16/3}$. The dimensions that will minimize the construction cost of the box are: $\sqrt[3]{18} \times \sqrt[3]{18} \times \sqrt[3]{16/3}$ feet.