Section 1.7
Dimensional Analysis

Dimensional Analysis

Many times it is necessary to convert a measurement made in one unit to an equivalent measurement in another unit. Sometimes this conversion is between two units in the same system of measurement, and sometimes this conversion is between two units in different systems of measurement.

Example 1:
Timothy is pricing wood flooring for his living room. The price of flooring at the home improvement store is given in dollars per square meter. However, Timothy measured the area of his living room in square feet. To calculate the cost of installing wood flooring in his living room, Timothy will need to convert the area measurement of his room to square meters. Then he can multiply the area of his room in square meters times the price of the flooring per square meter to find the cost of the project.

Definition: *Dimensional analysis* is the procedure for converting a measurement in one unit to an equivalent measurement in another unit.

How do we know when two measurements in different units are equivalent? Measure the characteristics of the SAME OBJECT in both units. For example, to determine how many centimeters are equivalent to one foot, measure the length of an object that is one foot long in centimeters. When this measurement is made, the one foot long object measures 30.48 centimeters. Therefore,

$$1 \text{ ft} = 30.48 \text{ cm.}$$

Equivalent Units

Within the US Customary System equivalent units for length, mass and volume are not easily calculated. Some conversions are straightforward because we use them all the time. For example, most people easily know that 12 inches is the same as one foot. Other equivalences are not as easy to remember, like the fact that there are 5280 feet in a mile. Therefore, for most equivalent units in the US Customary System, most people rely on a table of equivalent units, like the one printed below. Two measurements in different units are equivalent if they are on the same row of the conversion chart.

US Customary System Units

1 foot (ft)	12 inches (in)
1 yard (yd)	3 feet
1 mile (mi)	5280 feet

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1 pound (lb)	16 ounces
1 ton (T)	2000 pounds
1 cup (c)	8 fluid ounces
1 pint (pt)	2 cups
1 quart (qt)	2 pints
1 gallon (gal)	4 quarts

The beauty of the metric system is that within the metric system converting a measurement to one in a different unit is simply a matter of moving the decimal point. The procedure was discussed in Section 1.6 and will not be repeated here. The prefix chart is included here for ease of reference in the following discussion.

Metric Prefixes

Prefix	Symbol	Meaning
kilo	k	1000 X base unit
hecto	h	100 X base unit
deka	da	10 X base unit
-	-	Base unit
deci	d	1/10 of base unit
centi	c	1/100 of base unit
milli	m	1/1000 of base unit

Volume is the only fundamental property that requires extra attention when performing dimensional analysis. Volume can be measured in liters (or multiples of liters) or in cubic length units. The conversion between the two units is based on the fact that 1 liter is defined to be the volume of a cube that has sides of length 1 decimeter. The following table lists several equivalent metric volume units of varying sizes.

Metric Volume Conversion Chart

Volume in Cubic Units	Volume in Liters
1 cm^3	1 ml
1 dm^3	1 l
1 m^3	1 kl

Converting between the metric system and the US Customary System is not as simple as multiplying or dividing by powers of 10, so a conversion chart is essential. There are many science textbooks and internet sites that have very exhaustive lists of equivalent units between the US Customary System, the metric system and even other systems used in isolated parts of the world. In this text we will use the following table of conversion factors. It is not exhaustive, but contains enough information to do most of the dimensional analysis required in the lesson and exercises. If a unit is introduced that is

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not directly on the list, its equivalent in either the metric system or the US Customary System will be stated. The later examples will demonstrate how to convert between units that are not directly connected on the table.

Also note that the equivalent measurements in different units are given to four significant digits of precision. More accurate conversion factors exist for some units, but we will not require that degree of precision.

US Customary System and Metric System Conversion Chart

Length	
1 inch	2.54 centimeters
1 foot	30.48 centimeters
1 yard	0.9144 meters
1 mile	1.609 kilometers
Area	
1 in^2	6.4516 cm^2
1 ft^2	0.0929 m^2
1 yd^2	0.8361 m^2
1 mi^2	2.59 km^2
1 acre	0.405 hectare (ha)
Volume	
1 teaspoon (tsp)	4.929 milliliters
1 tablespoon (tbsp)	14.79 milliliters
1 fluid ounce	29.57 milliliters
1 cup	0.2366 liters
1 pint	0.4732 liters
1 quart	0.9464 liters
1 gallon	3.785 liters
1 cubic foot	0.02832 cubic meters
1 cubic yard	0.7646 cubic meters
Weight (Mass)	
1 ounce	28.35 grams
1 pound	0.4536 kilograms
1 ton (T)	0.9 tonne (t)

Unit Fractions

The arithmetic algorithm for converting between units is based on the fact that measurements are given in proportion to a standard. Proportions are often expressed in the form of a fraction.

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Definition: A **unit fraction** is a fraction (proportion written in fraction form) in which the numerator and the denominator of the fraction are equivalent measurements in different units. Examples include:

$$\frac{1\,g}{1000\,mg}, \quad \frac{12\,in}{1\,ft}, \quad \frac{2.54\,cm}{1\,in}$$

In arithmetic with rational numbers, multiplying by a unit (1 or $\frac{a}{a}$) does not change the values of a fraction. Similarly, multiplying a measurement by a unit fraction does not change the measurement, only the unit. Also just like rational fractions, common units (factors) in the numerator and denominator of a multiplication problem cancel. The units that do not cancel are the units of the answer to the multiplication.

Example 2: Convert 8.5 inches to centimeters using a unit fraction.

Solution:

The US Customary System and Metric System conversion chart shows that 1 inch is equivalent to 2.54 centimeters. Therefore, the unit fraction that we need has 1 in one place and 2.54 cm in the other. Which should be in the numerator and which in the denominator?

$$\frac{1\,in}{2.54\,cm} \quad \text{or} \quad \frac{2.54\,cm}{1\,in}\,?$$

The answer is the unit fraction with inches in the denominator, since multiplying 8.5 inches times the unit fraction with inches in the denominator will have inches on both top and bottom. The inches unit will cancel and the remaining unit is centimeters.

$$8.5\,in\left(\frac{2.54\,cm}{1\,in}\right) = 8.5\,\cancel{in}\left(\frac{2.54\,cm}{1\,\cancel{in}}\right)$$

The final answer is obtained by multiplying the numerical values and keeping the centimeter units that did not cancel.

$$8.5\,in\left(\frac{2.54\,cm}{1\,in}\right) = 8.5(2.54)\,cm = 21.59\,cm$$

Converting Units of Length, Mass and Volume

Converting length, mass and volume units within the US Customary System, and between the US Customary System and the metric system are accomplished by multiplying the given measurement by one (or more) unit fractions. The procedure is:

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1. Determine if there is an equivalent measurement in the conversion charts for the two units in question. If so, the unit fraction made from that equivalence is the multiplication factor needed.
2. If there is not an equivalent measurement listed in the charts with the two units in question, create a chain of equivalences linking the two units. For example, if you want to convert from inches to meters, use the facts that 1 inch is 2.54 centimeters and that the 100 centimeters equals 1 meter.
3. Write the multiplication problem using the unit fractions needed. The unit fraction should be written so that the corresponding units cancel from the top and the bottom of the multiplication problem, and the desired unit is on the top of a fraction.
4. Multiply and divide the numerical values in the measurement and unit fraction using normal real number multiplication and division rules.
5. Cancel the common units and express the numerical answer in terms of the new units

Example 3: Convert 5 quarts to liters.

Solution:

The conversion chart shows 1 quart = 0.95 liters. Thus

$$5 \text{ quarts} = 5 \text{ quarts} \left(\frac{0.9464\, l}{1 \text{ quart}} \right) = 4.732\, l$$

Example 4: Convert 364 g to pounds.

Solution:

The conversion chart shows that 1 pound = 0.4536 kg and the metric prefixes chart shows that 1 kg = 1000 g. Using these two equivalences:

$$364 \text{ g} = 364 \text{ g} \left(\frac{1 \text{ kg}}{1000 \text{ g}} \right) \left(\frac{1 \text{ lb}}{0.4536 \text{ kg}} \right) = 0.802 \text{ lb}$$

Example 5: Convert 0.15 liters to cups.

Solution:

The conversion chart shows 1 cup = 0.2366 *l*. So,

$$0.15\, l = 0.15\, l \left(\frac{1 \text{ cup}}{0.2366\, l} \right) = 0.634 \text{ cup}$$

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Example 6: Convert 0.5 km to feet.

Solution:

The conversion chart above does not have an equivalence between kilometers and feet directly, but it does show that 1 mile = 1.609 km and 1 mile = 5280 feet. Using the unit fractions from these two equivalences,

$$0.5\,\text{km} = 0.5\,\text{km}\left(\frac{1\,\text{mile}}{1.609\,\text{km}}\right)\left(\frac{5280\,\text{ft}}{1\,\text{mile}}\right) = 1641\,\text{ft}$$

Converting Area and Volume

Volume can be measured in volume units, or in cubic length units. Area is measured in square length units. When performing dimensional analysis in square or cubic length units, the unit fractions must be squared or cubed appropriately so that the units still cancel. This may be accomplished by writing one unit fraction raised to a power, or by multiplying by the unit fraction as many times as needed to cancel the units appropriately.

Example 7: Convert $50.6\,\text{in}^3$ to cubic centimeters.

Solution:

The equivalence needed is that 1 in = 2.54 cm. The unit fraction is written 3 times so that the cubic inches will cancel.

$$50.6\,\text{in}^3 = \frac{50.6\,\text{in}^3}{1}\left(\frac{2.54\,\text{cm}}{1\,\text{in}}\right)\left(\frac{2.54\,\text{cm}}{1\,\text{in}}\right)\left(\frac{2.54\,\text{cm}}{1\,\text{in}}\right) = 829\,\text{cm}^3.$$

Example 8: Convert $0.096\,\text{m}^3$ to cubic decimeters.

Solution:

The equivalence needed is that 10 dm = 1 m. This time we will write the unit fraction raised to the third power instead of writing it out 3 times. The meaning is exactly the same. Note that this means that any numerical values in the unit fraction must be cubed to perform the multiplication properly.

$$0.096\,\text{m}^3 = \frac{0.096\,\text{m}^3}{1}\left(\frac{10\,\text{dm}}{1\,\text{m}}\right)^3 = 96\,\text{dm}^3$$

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The conversion of a volume or area measurement may also be accomplished by converting each dimensional measurement into the appropriate units, and then using a formula to find the area or volume.

Example 9:
A rectangular table is 15 inches wide and 30 inches long. Find the area of the table in square meters.

Solution:

The dimensions of the table are first converted to meters, and then the area is calculated in square meters. The dimensions of the table are:

$$\frac{15\,\text{in}}{1}\left(\frac{2.54\,\text{cm}}{1\,\text{in}}\right)\left(\frac{1\,\text{m}}{100\,\text{cm}}\right) = 0.381\,\text{m} \quad \text{and} \quad \frac{30\,\text{in}}{1}\left(\frac{2.54\,\text{cm}}{1\,\text{in}}\right)\left(\frac{1\,\text{m}}{100\,\text{cm}}\right) = 0.762\,\text{m}$$

The area of the table is equal to the length times the width, so
$\text{Area} = (0.381\,\text{m})(0.762\,\text{m}) = 0.290\,\text{m}^2$.

Sometimes, a combination of conversions is required to complete the dimensional analysis.

Example 10:
A cylindrical bucket has a height of 0.5 meters and a radius of 25 cm. How many liters of water will it hold? How many gallons of water will it hold?

Solution:

The first question asks for the volume in liters. The metric volume conversion chart shows that one liter is equivalent to one cubic decimeter. So, if the dimensions of the bucket are converted to decimeters, the volume of the cylinder formula will give the volume in cubic decimeters (liters). Therefore, the first step is to find the dimensions of the bucket in decimeters.

Height = 0.5 m = 5 dm (since meters are one power of ten larger than decimeters, move the decimal point one place to the right)

Radius = 25 cm = 2.5 dm (since centimeters are one power of ten smaller than decimeters, move the decimal point one place to the left)

Volume – the formula for a cylinder is $V = \pi r^2 h$, where r is the radius and h is the height. Plugging in the numbers:

$V = \pi(2.5)^2(5)\,\text{dm}^3 = 98.2\,\text{dm}^3 = 98.2\,l$

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The bucket will hold 98 liters of water.

Next, convert liters to gallons:

$$98.2\,l = \frac{98.2\,l}{1}\left(\frac{1\,\text{gal}}{3.785\,l}\right) = 25.9\,\text{gal}$$

The bucket will hold 25.9 gallons of water.

Converting Velocity and Cost Per Unit

The conversion charts in this section are given for basic length, volume and mass units. Often there are other measurements that we need to convert from the metric system to the US Customary System, or convert to different units within either system.

One common measurement that requires conversion is that of velocity. Velocity is expressed as length per unit of time. In the US Customary System velocity is often expressed in miles per hour or feet per second. In the metric system, velocity is often given in meters per second or kilometers per hour. The same techniques of multiplying by unit fractions may be used to convert velocity from one unit to an equivalent measurement in another unit.

Example 11:
The speed limit on a country road is posted as 40 miles per hour. Bob's car is a brand new German auto with a speedometer measuring his velocity in kilometers per hour. Convert 40 mph to kilometers per hour so that Bob knows how fast he can legally go on this road.

Solution:

The equivalent measurements needed are that 1 mile = 1.6. The multiplication is:

$$\frac{40\,\text{miles}}{\text{hr}} = \frac{40\,\text{miles}}{\text{hr}}\left(\frac{1.609\,\text{km}}{1\,\text{mile}}\right) = \frac{64.36\,\text{km}}{\text{hr}}$$

The speed limit on the country road is 64.36 kilometers per hour.

Another common application of dimensional analysis is the conversion of price per unit to an equivalent price in a different unit. The example at the beginning of this section of pricing flooring at the home improvement store is a classic example. Again the technique of multiplying by unit fractions is used in this situation successfully.

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Example 12:
The cost of carpeting at one home improvement store is quoted as $2.25 per square foot. The same carpet at a different store is priced at $30 per square yard. Which is the better price?

Solution:

To answer the question, convert $2.25 per square foot into its equivalent price per square yard and compare. The unit fraction we need is the equivalence between 1 yard and 3 feet. The multiplication becomes:

$$\frac{\$2.25}{1\,\text{ft}^2}\left(\frac{3\,\text{ft}}{1\,\text{yard}}\right)^2 = \frac{\$2.25}{1\,\text{ft}^2}\left(\frac{9\,\text{ft}^2}{1\,\text{yd}^2}\right) = \frac{\$20.25}{1\,\text{yd}^2}$$

The price of the carpet at the first store is $20.25 per square yard, clearly less expensive than the price at the second store.