
Chapter 1

Counting and Measuring

Section 1.1

Systems of Numeration and Additive Systems of Numeration

Counting

The language of mathematics began when human civilization began to develop words and later written symbols to represent quantities. Humans first had to understand the concept of quantity separate from the items counted. The concept of “two” had to be understood as a separate from the two items counted. Once a group of people developed the understanding of the concept of quantity, most languages developed words for counting numbers to use in their every day tasks.

However, the first evidence of counting did not necessarily require words for individual numbers. In fact, some primitive languages have been discovered that only have words for “one”, “two” and “many”. They used some sort of counting object and the principle of one-to-one correspondence to keep track of collections of objects. Societies used sticks, pebbles, notches in bones or marks on clay tablets to represent each item to be counted. A simple tally system was used to account for objects, animals or people in the village. Evidence of these simple physical counters has been found in many different parts of the world.

Counting became much easier when names for the counting numbers were developed in the spoken language. With words a large number of objects could be counted without keeping track of cumbersome counters.

Numerals

The next giant step forward in the development of mathematics was written representations of quantities. Written languages were developed to record more information than one human could physically remember. In the same way written symbols for numbers were developed to record transactions and keep track of larger quantities.

A number is a quantity. A number answers the question: How many? A *numeral* is a written symbol used to represent a number. The distinction between the two concepts is not very large in our modern world, but they are two very different developments in human civilization. The number two is the answer to the question of how many hands most human beings have. The symbols “2”, “II” and “ll” are three different numerals that have been used in various times and places to represent the number two.

The symbol “II” for the number two is a Roman numeral, first used in ancient Rome and widely used in Europe until the 18th century. Even today, Roman numerals are often used in decorative or artistic settings. The numeral “ll” is a tally system numeral for the number two. Many times players keep score in games by counting points with a simple tally system. One line is drawn for each point, so two is represented by “ll”. The numeral “2” is the Hindu-Arabic numeral for the quantity two. It is the most familiar representation for two, yet it is not the only one. The history of how numerals evolved into the modern form is a fascinating story that spans the whole of recorded human history.

Systems of Numeration

A *system of numeration* consists of a set of numerals and a rule for combining them to represent numbers. The first requirement of a system of numeration is a set of symbols. In the Hindu-Arabic system we use today the symbols are 0,1,2,3,4,5,6,7,8,9 and the decimal point “.”. In a simple tally system the numerals are just vertical lines “|”. In ancient Egyptian hieroglyphics, the symbols were pictures representing the numbers 1, 10, 100, 1000, etc. Many different cultures have used many different sets of symbols. Furthermore, even in a single system of numeration, the symbols have also changed over the course of time (and typesetting improvements!).

The second requirement of a system of numeration is the rule for combining symbols to represent numbers. The system of numeration that is used in most of the world today is the Hindu-Arabic system. It is a place value system with base ten. Do the numerals 123 and 231 represent the same numbers? Of course not! Yet without the rules for place value that are the key feature of our system, these two combinations of the same three numerals would be indistinguishable.

There have been four basic types of systems of numeration that different cultures developed in the course of human history. They are

1. Additive Systems
2. Alphabetic Systems
3. Multiplicative Systems
4. Position Value Systems

In this chapter we will look at one or two examples of each type of system of numeration to learn the key features of each system. The goal is to appreciate the sophistication and utility of the modern place value system that is the fundamental building block for the language of mathematics.

Additive Systems of Numeration

In an additive system of numeration symbols are given for groups of ten and powers of ten. The value of a combination of symbols is simple the sum of the values of each symbol used. The order of the symbols does not matter in calculating the value of a group of symbols. The earliest systems that have been found in the archeological record are additive systems.

EGYPTIAN NUMERALS

Ancient Egyptians developed writing in the form of pictograms. Pictograms were symbols for objects and ideas that took the form of simple pictures. They also developed pictograms for numbers. They carved their symbols into the stones of their temples and monuments. When the Greeks came to the Nile Valley, they looked at the symbols carved into the stones of the temples and tombs and called them “*gammata hierogluphika*”. From this term, we get the word hieroglyphics.

Egyptian hieroglyphics have been discovered that date to 3000 BCE. The combination of the dry climate of Egypt and the permanence of stone has preserved many records of Egyptian writing. This rich source of information is where our knowledge of ancient Egypt is derived. The hieroglyphics included symbols for numbers as well as words. The Egyptians had symbols for powers of ten from one up to one million. The symbols were pictures that suggested common objects. The symbol for one was a simple stroke or line. The symbol for ten was “ \cap ”, and the exact origin of this symbol is unclear. Some texts refer to it as a heel bone, possibly reflecting “ten toes”. Some refer to it as a horseshoe or a cattle yoke. We will call it a heel bone in this text.

The symbols used in ancient Egypt are given in the following table.

Systems Used in Ancient Egypt:

Number	Picture	Symbol
1	Stroke	
10	Heel bone	∩
100	Coil of rope	Ⓒ
1000	Lotus Flower	⬇
10,000	Pointing Finger	☞
100,000	Tadpole	🐸
1,000,000	Astonished man	🙄

The Egyptian system of numeration was an additive system. To write the number 342, three coils of rope, four heel bone symbols and two lines would be written.

Example 1: Write the number 5242 in Egyptian numerals.

Solution:

5242 is $5000 + 200 + 40 + 2$.



Example 2: Write the number 1003 in Egyptian numerals.

Solution:

1003 is $1000 + 3$

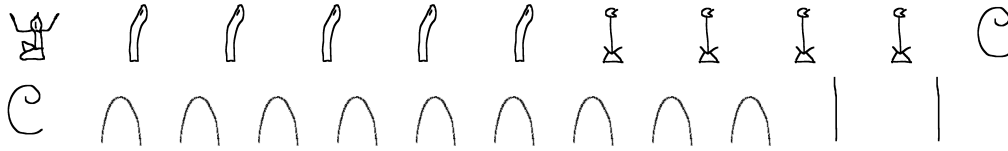


Section 1.1

Example 3: Write the number 1,054,292 in Egyptian numerals.

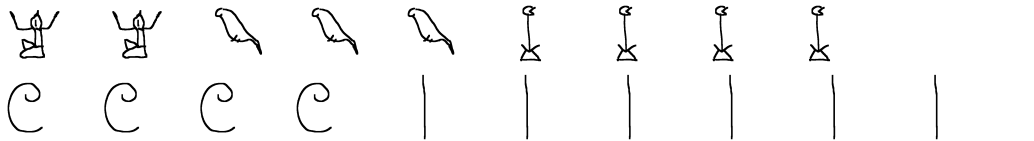
Solution:

1,054,292 is $1,000,000 + 50,000 + 4000 + 200 + 90 + 2$



Notice that there is no zero in the Egyptian system. Also notice how many symbols are needed!!

Example 4: Write the Egyptian numeral



in Hindu-Arabic numerals.

Solution:

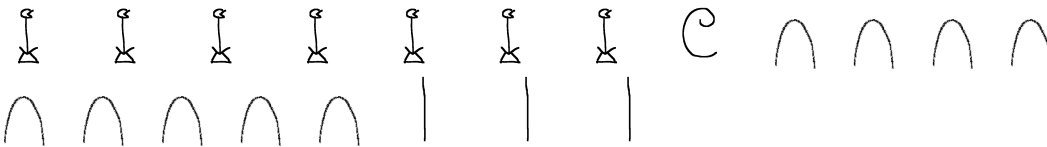
First, count the number of symbols of each type:

Lines - 7	Heel bones - 0	Coils of Rope - 4
Lotus Flowers - 4	Pointing Fingers - 0	Tadpoles - 3
Astonished Man - 2		

Then, write this quantity in Hindu-Arabic numerals.

$$2000000 + 400000 + 3000 + 400 + 0 + 7 = 2,304,407$$

Example 5: Write the following Egyptian numeral in Hindu-Arabic numerals.



Solution:

First, count the number of symbols of each type:

Lines - 3	Heel bones - 9	Coils of Rope - 1
Lotus Flowers - 7	Pointing Fingers - 0	Tadpoles - 0
Astonished Man - 0		

Section 1.1

Then express this number in Hindu-Arabic numerals.

$$7000 + 100 + 90 + 3 = 7193$$

The simplicity of the Egyptian system made it easy to work with. Notice that it is based on powers of 10. Also, addition and subtraction were easy to perform in the system by simply adding or subtracting symbols and regrouping. The biggest disadvantage of the Egyptian system was the number of symbols required to write numbers like 999. Another challenge was multiplying in the Egyptian system.

ROMAN NUMERALS

The second additive system we will consider is the Roman system of numeration. This system was used in Rome and the Roman Empire and continued to be used in Europe until the 16th century. The symbols changed over the centuries, but the system in roughly the form we are familiar with was in place by the first century BCE.

The Romans modified the additive system to include a subtraction rule for writing fours and nines. In order for this subtraction rule to work, the system also required that the order of the symbols matters. They wrote numerals from left to right, with the numerals representing the largest number first. The convention was to add the symbols as they decrease in order. However, if a symbol for 1, 10 or 100 was placed before that for the next two larger symbols, it was subtracted instead of added. This allows 4 to be written as 5-1 and 9 to become 10-1, reducing the number of symbols needed.

The numerals in the Roman system are given in the following table.

Roman	I	V	X	L	C	D	M
Hindu-Arabic	1	5	10	50	100	500	1000

The first ten numbers written as Roman numerals are:

Hindu-Arabic	1	2	3	4	5	6	7	8	9	10
Roman	I	II	III	IV	V	VI	VII	VIII	IX	X

The numbers 11-99 were written with the ten's digits first, one for each group of ten, and then the single digit number. The subtraction rule made 40= 50-10 and 90=100-10. Thus 44 would be represented as XLIV in Roman numerals. The number 99 is written as XCIX in Roman numerals. The symbols for 400 = CD and 900 = CM follow the same pattern.

Beyond 3000, the general convention was to place a horizontal bar over the top of a numeral or set of numerals to multiply the value by 1000.

Example 6: Write the number 3,846 in Roman numerals.

Solution:

$$3846 = 3000 + 800 + 40 + 6 = 3000 + (500 + 300) + (50 - 10) + (5 + 1)$$

Thus, 3846 = MMMDCCCXLVI.

Example 7: Write the number 7,439 in Roman numerals.

Solution:

$$7,439 = 7000 + 400 + 30 + 9 = (7 * 1000) + (500 - 100) + 30 + (5 + 3)$$

Thus, 7439 = $\overline{VII}CDXXXIX$

Example 8: Write the Roman number *MMCMCCXLVII* as a Hindu-Arabic number.

Solution:

Divide the Roman number into separate powers of ten, starting from the left. Then translate each set of symbols into its corresponding Hindu-Arabic number.

$$\begin{aligned} MMCMXLVII &= MM \quad CM \quad XL \quad VII \\ &= (2 \times 1000) + (1000 - 100) + (50 - 10) + (5 + 2) \\ &= 2947 \end{aligned}$$

Notice that the “CM” is the hundreds digit in Hindu-Arabic, even though the M stands for 1000. The “C” before the “M” represents subtraction, and the answer will be 900.

Example 9: Write the Roman number $\overline{CXII}DXCIII$ as a Hindu-Arabic number.

Solution:

The first symbols *CXII* have a bar over them, which means multiply by 1000. The rest of the symbols are not under the bar, so their values are not changed. Separate the symbols so that each group represents the symbols for one power of ten.

$$\begin{aligned} \overline{CXII}DXCIII &= \overline{C} \quad \overline{X} \quad \overline{II} \quad D \quad XC \quad III \\ &= (100 \times 1000) + (10 \times 1000) + (2 \times 1000) + 500 + (100 - 10) + 3 \\ &= 112,593 \end{aligned}$$

The advantage of the Roman system over the Egyptian system is that it required less symbols to write large numbers like 999. However, the subtraction rule for fours and nines makes performing simple calculations in Roman numerals more difficult, not less.

Section 1.1

Although the Roman system was not well suited to mathematical manipulations, its influence can still be felt today. In many “decorative” applications Roman numerals are still being used. We can find Roman numerals in the labeling of clocks and dating movies. Many inscriptions on monuments and buildings throughout Europe, and even some in the United States, use Roman numerals to represent calendar dates. Therefore, knowing how to interpret an ancient Roman numeral is important, even in our modern world.