Section 2.3: Rational Functions

- Asymptotes and Holes
- Graphing Rational Functions

Asymptotes and Holes

Definition of a Rational Function:

A rational function r is a function of the form

$$r(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomials.

Definition of a Vertical Asymptote:

A vertical line $x = a$ is called a vertical asymptote of the rational function $y = r(x)$ if

$$y \to \infty \text{ or } y \to -\infty \quad \text{as} \quad x \to a^{+} \text{ or } x \to a^{-}.$$

Definition of a Horizontal Asymptote:

A horizontal line $y = b$ is called a horizontal asymptote of the rational function $y = r(x)$ if

$$y \to b \quad \text{as} \quad x \to \infty \text{ or } x \to -\infty.$$

Finding Vertical Asymptotes, Horizontal Asymptotes, and Holes:

Let $r(x)=\dfrac{P(x)}{Q(x)}$ be a rational function and assume that $P(x)$ and $Q(x)$ have no common factors.

The vertical and horizontal asymptotes of the rational function

$$r(x)=\frac{P(x)}{Q(x)}=\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_1x+b_0}$$

are determined in the following way:

1) The vertical asymptotes are the vertical lines $x=a$, where a is a real zero of the denominator $Q(x)$.

2) If $n<m$, the horizontal asymptote is $y=0$. If $n=m$, the horizontal asymptote is $y=\dfrac{a_n}{b_m}=\dfrac{\text{leading coefficient in the numerator}}{\text{leading coefficient in the denominator}}$. If $n>m$, there is no horizontal asymptote.

Note: The vertical and horizontal asymptotes should be found only after checking to see that the numerator and denominator have no common factors. If the numerator and denominator have common factors, then the graph will have holes in it.

Example:

Identify any holes, vertical asymptotes, and horizontal asymptotes for the rational function $r(x)=\dfrac{x+2}{x^2-4}$.

Solution:

Rewrite the function $r(x)=\dfrac{x+2}{x^2-4}$ by factoring the denominator.

$$r(x)=\frac{x+2}{x^2-4}=\frac{x+2}{(x+2)(x-2)}$$

The domain of the function is $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$.

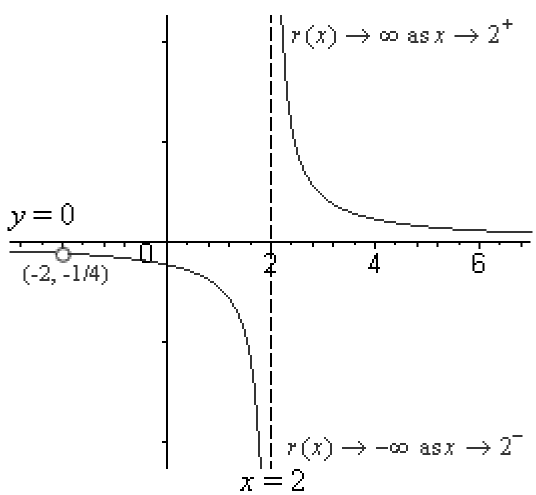
$$r(x) = \frac{x+2}{x^2-4} = \frac{x+2}{(x+2)(x-2)} = \frac{1}{x-2} \text{ for } x \neq -2.$$

The graph of the given function is the graph of $y = \frac{1}{x-2}$ with the point $\left(-2, -\frac{1}{4}\right)$ removed from the graph. Thus, there is a **hole** in the graph at the point $\left(-2, -\frac{1}{4}\right)$.

The zero of $x-2$ is 2. The vertical asymptote is $x = 2$.

The horizontal asymptote is $y = 0$ since the degree of the numerator is less than the degree of the denominator.

The graph is shown below.



Example:

Find the vertical and horizontal asymptotes of the function $r(x) = \frac{3x+1}{x-3}$.

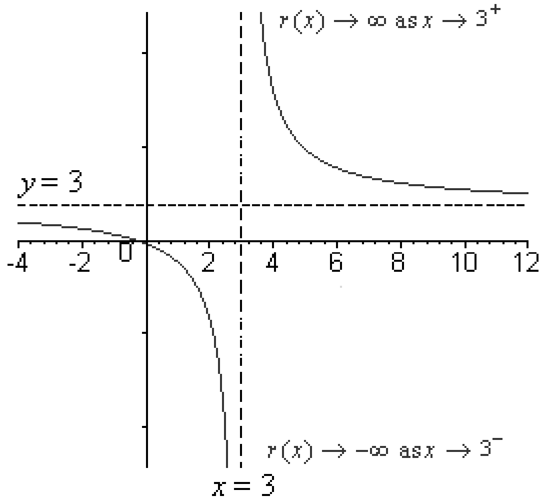
Solution:

The numerator and denominator have no common factors.

The zero of the denominator $x-3$ is 3. The vertical asymptote is $x = 3$.

The degree of the numerator is 1, which is equal to the degree of the denominator. The leading coefficient of the numerator is 3 and the leading coefficient of the denominator is 1. The horizontal asymptote is $y = \frac{3}{1} = 3$.

The graph is shown below.



Example:

Find the vertical and horizontal asymptotes of the function $r(x) = \dfrac{4x}{x^2 - 5x + 4}$.

Solution:

Factor the denominator.

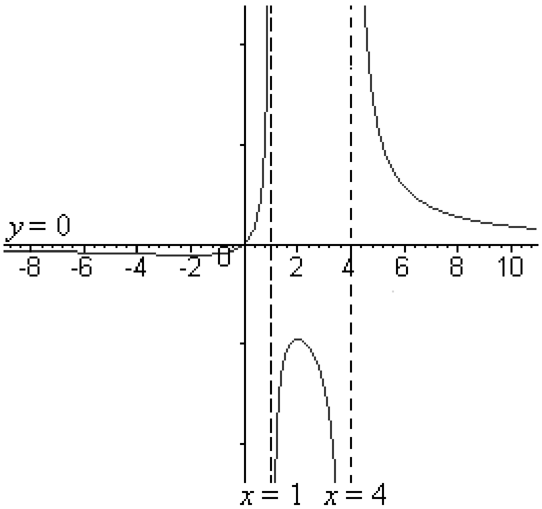
$$x^2 - 5x + 4 = (x-1)(x-4)$$

The numerator and denominator have no common factors.

The zeros of the denominator are 1 and 4. The vertical asymptotes are $x = 1$ and $x = 4$.

The degree of the numerator is 1 and the degree of the denominator is 2. Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y = 0$.

The graph is shown below.



The graph of a rational function will not intersect a vertical asymptote, but the graph might pass through a horizontal asymptote.

Example:

The graph of the rational function $f(x) = \dfrac{x^2}{x^2 - x + 2}$ intersects its horizontal asymptote $y = 1$. Find the point of intersection.

Solution:

To find the x-coordinate of the point of intersection, set $f(x) = 1$ and solve the resulting equation.

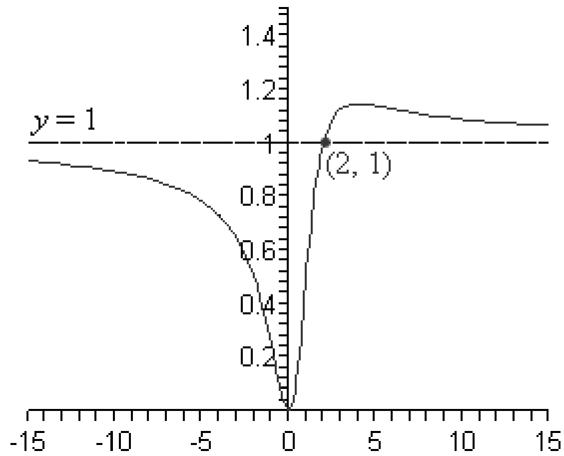
$$\frac{x^2}{x^2 - x + 2} = 1$$

$$x^2 = x^2 - x + 2 \quad \text{Multiply both sides by } x^2 - x + 2.$$

$$0 = -x + 2 \quad \text{Subtract } x^2 \text{ from both sides.}$$

$$x = 2 \quad \text{Add } x \text{ to both sides.}$$

The point of intersection is $(2,1)$. The graph of the function and its horizontal asymptote is shown in the figure below.



Definition of a Slant Asymptote:

Let $r(x) = \dfrac{P(x)}{Q(x)}$ be a rational function where the degree of the numerator is one more than the degree of the denominator.

(Note that r does not have a horizontal asymptote since the degree of the numerator is greater than the degree of the denominator.)

By the Division Algorithm, the rational function can be expressed in the form

$$r(x) = mx + b + \frac{R(x)}{Q(x)}$$

where the degree of R is less than the degree of Q and $m \neq 0$. As $x \to \pm\infty$, $\dfrac{R(x)}{Q(x)} \to 0$ and the graph of r approaches the graph of the line $y = mx + b$. This line is called a slant asymptote.

Example:

Find all holes, vertical asymptotes, horizontal asymptotes, and slant asymptotes, if any, for the rational function $r(x) = \frac{x^2 + x - 2}{x - 3}$.

Solution:

Rewrite the function by factoring the numerator.

$$r(x) = \frac{x^2 + x - 2}{x - 3} = \frac{(x+2)(x-1)}{x-3}$$

The numerator and denominator share no common factors.

Vertical Asymptotes:

The zero of the denominator $x - 3$ is 3.

The vertical asymptotes of r are the lines $x = a$, where a is a real zero of the denominator.

The vertical asymptote is $x = 3$.

Horizontal Asymptotes:

Identify the degree of both numerator and denominator.

The degree of $x^2 + x - 2$ is 2.
The degree of $x - 3$ is 1.

There is no horizontal asymptote since the degree of the numerator is greater than the degree of the denominator.

Slant Asymptotes:

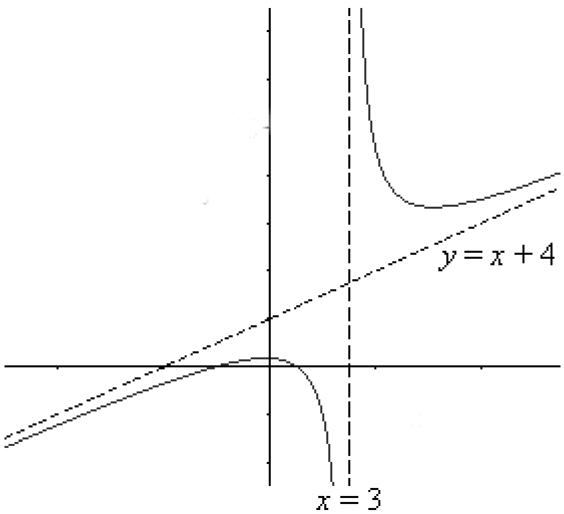
The degree of the numerator is one more than the degree of the denominator. There is a slant asymptote. To find the slant asymptote, divide $x - 3$ into $x^2 + x - 2$ by polynomial long division to obtain:

$$r(x) = x + 4 + \frac{10}{x-3}.$$

The slant asymptote is the line $y = x + 4$.

Note: For a review of polynomial long division, please refer to Appendix A.2: *Dividing Polynomials* .

The graph is shown below.



Additional Example 1:

Find all holes, vertical asymptotes, and horizontal asymptotes, if any, for the rational function $r(x) = \dfrac{2x}{x+5}$.

Solution:

The numerator and denominator have no common factors.

Vertical Asymptotes:

Find all zeros of the denominator.

The zero of the denominator $x + 5$ is -5.

The vertical asymptotes of r are the lines $x = a$, where a is a real zero of the denominator.

The vertical asymptote is $x = -5$.

Horizontal Asymptotes:

Identify the degree of both numerator and denominator.

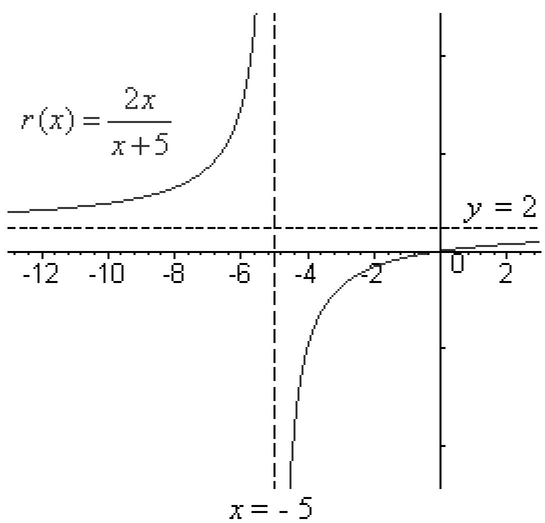
The degree of $2x$ is 1.
The degree of $x+5$ is 1.

The horizontal asymptote is $y = \frac{\text{leading coefficient in numerator}}{\text{leading coefficient in denominator}}$ if degrees in numerator and denominator are the same.

The leading coefficient in the numerator is 2.
The leading coefficient in the denominator is 1.

The horizontal asymptote is $y = \frac{2}{1} = 2$.

The graph is shown below.



Additional Example 2:

Find all holes, vertical asymptotes, and horizontal asymptotes, if any, for the rational function $r(x) = \frac{3x^2}{x^2 - 4}$.

Solution:

Factor the denominator: $x^2 - 4 = (x+2)(x-2)$

The numerator and denominator have no common factors.

Vertical Asymptotes:

The zeros of the denominator are -2 and 2.

The vertical asymptotes of r are the lines $x = a$, where a is a real zero of the denominator.

The vertical asymptotes are $x = -2$ and $x = 2$.

Horizontal Asymptotes:

Identify the degree of both numerator and denominator.

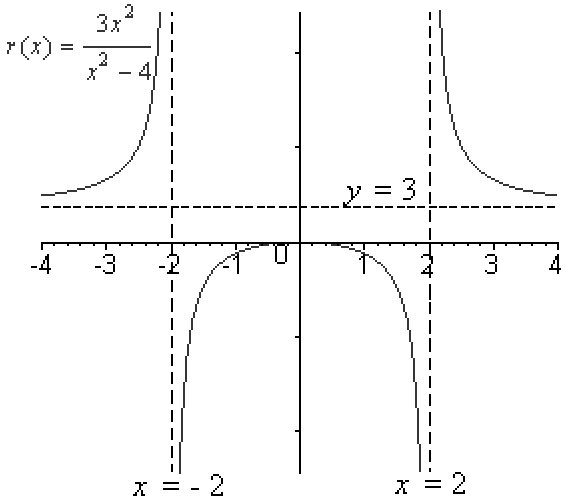
The degree of $3x^2$ is 2.
The degree of $x^2 - 4$ is 2.

The horizontal asymptote is $y = \dfrac{\text{leading coefficient in numerator}}{\text{leading coefficient in denominator}}$ if degrees in numerator and denominator are the same.

The leading coefficient in the numerator is 3.
The leading coefficient in the denominator is 1.

The horizontal asymptote is $y = \dfrac{3}{1} = 3$.

The graph is shown below.



Additional Example 3:

Find all holes, vertical asymptotes, and horizontal asymptotes, if any, for the rational function $r(x) = \dfrac{x-2}{x^2+2x-3}$.

Solution:

Factor the denominator: $x^2 + 2x - 3 = (x+3)(x-1)$

The numerator and denominator have no common factors.

Vertical Asymptotes:

The zeros of the denominator are -3 and 1.

The vertical asymptotes of r are the lines $x = a$, where a is a real zero of the denominator.

The vertical asymptotes are $x = -3$ and $x = 1$.

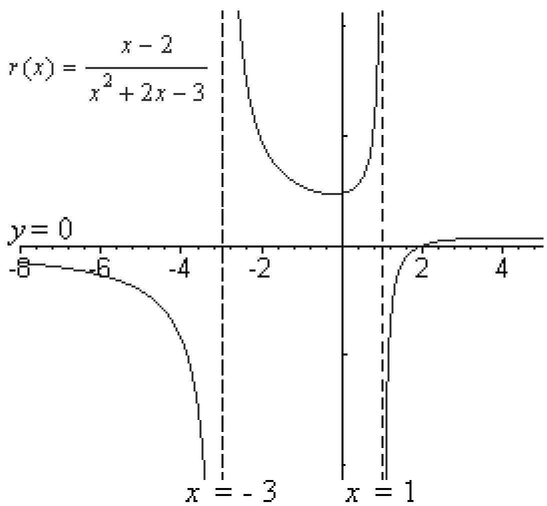
Horizontal Asymptotes:

Identify the degree of both numerator and denominator.

The degree of $x-2$ is 1.
The degree of x^2+2x-3 is 2.

The degree of the numerator is less than the degree of the denominator.

The horizontal asymptote is $y=0$.
The graph is shown below.



Additional Example 4:

Find all holes, vertical asymptotes, horizontal asymptotes, and slant asymptotes, if any, for the rational function $r(x)=\dfrac{x^2+1}{x}$.

Solution:

The numerator and denominator have no common factors.

Vertical Asymptotes:

The zero of the denominator is 0.

The vertical asymptotes of r are the lines $x=a$, where a is a real zero of the denominator.

The vertical asymptote is $x=0$.

Horizontal Asymptotes:

Identify the degree of both numerator and denominator.

The degree of x^2+1 is 2.
The degree of x is 1.

The degree of the numerator is greater than the degree of the denominator.

There is no horizontal asymptote.

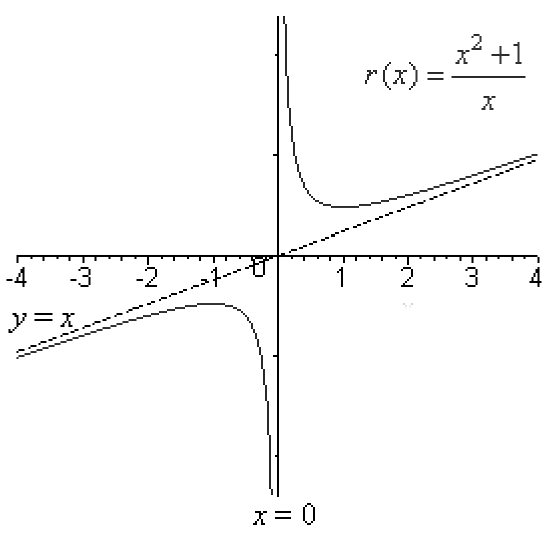
Slant Asymptotes:

The degree of the numerator is one more than the degree of the denominator. There is a slant asymptote. To find the slant asymptote, divide x into x^2+1 to obtain:

$$r(x) = x + \frac{1}{x}.$$

The slant asymptote is the line $y = x$.

The graph is shown below.



Additional Example 5:

Find all holes, vertical asymptotes, and horizontal asymptotes, if any, for the rational function $r(x) = \frac{5(x-1)}{(x-1)(x^2+4)}$.

Solution:

The numerator and denominator share a common factor. Begin by simplifying.

$$r(x) = \frac{5(x-1)}{(x-1)(x^2+4)} = \frac{5}{x^2+4} \text{ for } x \neq 1.$$

The graph of the given function is the graph of $y = \dfrac{5}{x^2+4}$ with the point $(1,1)$ removed. There is a hole in the graph at the point $(1,1)$.

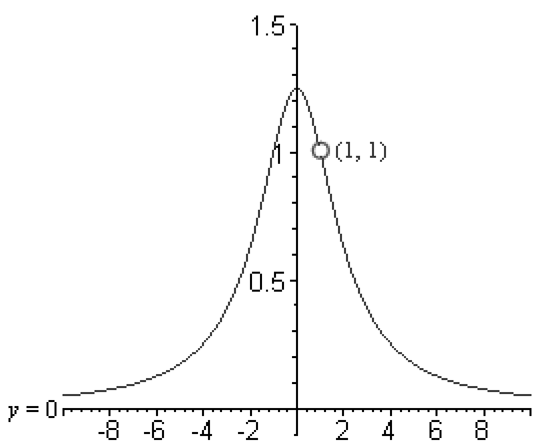
Vertical Asymptotes:

The zeros of x^2+4 are $-2i$ and $2i$. There is no vertical asymptote since these zeros are not real.

Horizontal Asymptotes:

The horizontal asymptote is $y = 0$ since the degree in the numerator is less than the degree of the denominator.

The graph is shown below.



Graphing Rational Functions

A Strategy for Graphing Rational Functions:

To sketch the graph of a rational function $r(x) = \frac{P(x)}{Q(x)}$, it is helpful to follow the guidelines given below.

- Factor the numerator and denominator, if necessary, to check for common factors.
- Find the x- and y-intercepts.
- Find the vertical and horizontal asymptotes.
- If the degree of the numerator is one more than the degree of the denominator, find the slant asymptote.
- Determine the behavior near the vertical asymptotes.
- Find as many additional points as needed to complete the graph.

Example:

Sketch the graph of the rational function $r(x) = \frac{x+6}{x-2}$.

Solution:

Common Factors:

The numerator and denominator share no common factors.

Intercepts:

The x-intercept is -6 since $\frac{x+6}{x-2} = 0$ whenever $x = -6$.

The y-intercept is $r(0) = \frac{0+6}{0-2} = \frac{6}{-2} = -3$.

Asymptotes:

The zero of the denominator $x-2$ is 2. The vertical asymptote is $x=2$.
The degree of the numerator is equal to the degree of the denominator. The leading coefficient in both numerator and denominator is 1. The horizontal asymptote is $y=\frac{1}{1}=1$.

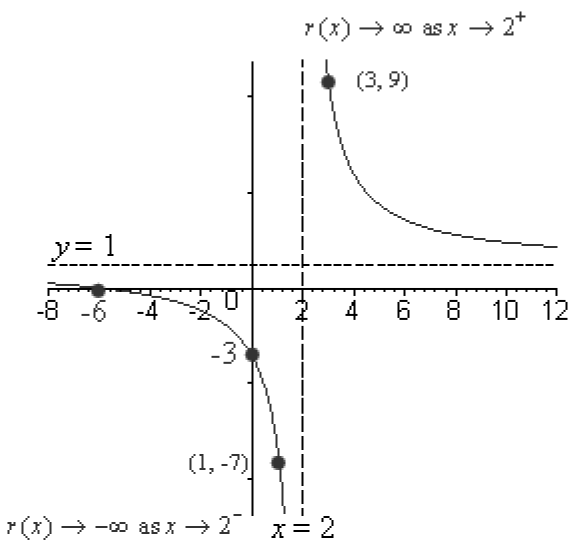
Behavior near vertical asymptote:

As $x \to 2^+$, the sign of $r(x)=\frac{x+6}{x-2}=\frac{+}{+}$. So, $r(x) \to \infty$ as $x \to 2^+$.
As $x \to 2^-$, the sign of $r(x)=\frac{x+6}{x-2}=\frac{+}{-}$. So, $r(x) \to -\infty$ as $x \to 2^-$.

Additional points on the graph:

$r(3)=\frac{3+6}{3-2}=\frac{9}{1}=9$ Point on the graph: $(3,9)$
$r(1)=\frac{1+6}{1-2}=\frac{7}{-1}=-7$ Point on the graph: $(1,-7)$

The graph is shown below.



Additional Example 1:

Sketch the graph of the rational function $r(x) = \dfrac{3}{x-1}$.

Solution:
The numerator and denominator share no common factors.

Intercepts:

The y-intercept is $r(0) = \dfrac{3}{0-1} = -3$.

There is no x-intercept since there is no value of x for which $\dfrac{3}{x-1} = 0$.

Asymptotes:

The zero of the denominator $x-1$ is 1. The vertical asymptote is $x = 1$.

Determine the behavior near the vertical asymptote $x = 1$ to the right of 1.

As $x \to 1^{+}$, the sign of $r(x) = \dfrac{3}{x-1} = \dfrac{+}{+}$.
So, $r(x) \to \infty$ as $x \to 1^{+}$.

Determine the behavior near the vertical asymptote $x = 1$ to the left of 1.

As $x \to 1^{-}$, the sign of $r(x) = \dfrac{3}{x-1} = \dfrac{+}{-}$.
So, $r(x) \to -\infty$ as $x \to 1^{-}$.

The degree of 3 is 0.
The degree of $x-1$ is 1.

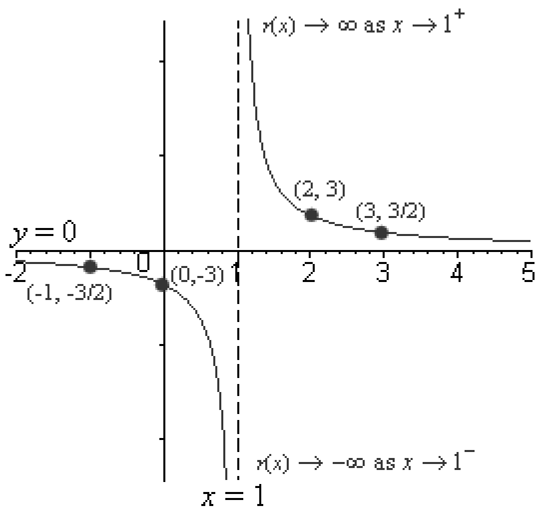
The horizontal asymptote is $y = 0$ since the degree of the numerator is less than the degree of the denominator.

Additional Points:

Make a table of values to find additional points on the graph.

x	$r(x)$	$(x, r(x))$
-1	$r(-1) = \frac{3}{-1-1} = -\frac{3}{2}$	$\left(-1, -\frac{3}{2}\right)$
2	$r(2) = \frac{3}{2-1} = 3$	$(2,3)$
3	$r(3) = \frac{3}{3-1} = \frac{3}{2}$	$\left(3, \frac{3}{2}\right)$

Sketch the graph.



Additional Example 2:

Sketch the graph of the rational function $r(x) = \frac{x-3}{x-1}$.

Solution:
The numerator and denominator share no common factors.

Intercepts:

The y-intercept is $r(0) = \frac{0-3}{0-1} = 3$.

The x-intercept is 3 since $\frac{x-3}{x-1} = 0$ whenever $x - 3 = 0 \Rightarrow x = 3$.

Asymptotes:

The zero of the denominator $x-1$ is 1. The vertical asymptote is $x=1$.

Determine the behavior near the vertical asymptote $x=1$ to the right of 1.

As $x \to 1^+$, the sign of $r(x)=\dfrac{x-3}{x-1}=\dfrac{-}{+}$.

So, $r(x) \to -\infty$ as $x \to 1^+$.

Determine the behavior near the vertical asymptote $x=1$ to the left of 1.

As $x \to 1^-$, the sign of $r(x)=\dfrac{x-3}{x-1}=\dfrac{-}{-}$.

So, $r(x) \to \infty$ as $x \to 1^-$.

The degree of $x-3$ is 1.

The degree of $x-1$ is 1.

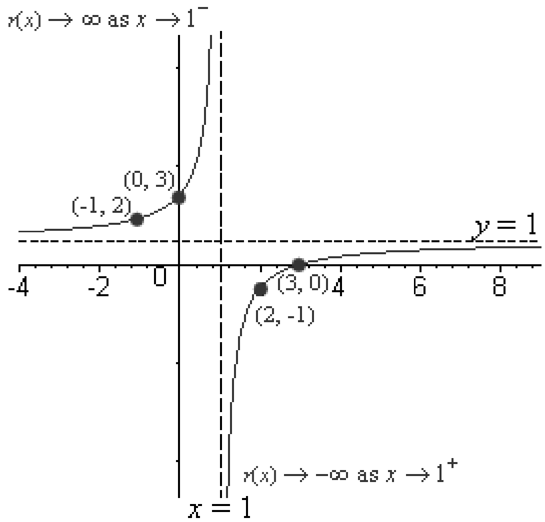
The horizontal asymptote is $y=\dfrac{\text{leading coefficient in numerator}}{\text{leading coefficient in denominator}}=\dfrac{1}{1}=1$.

Additional Points:

Make a table of values to find additional points on the graph.

x	$r(x)$	$(x, r(x))$
-1	$r(-1)=\dfrac{-1-3}{-1-1}=2$	$(-1,2)$
2	$r(2)=\dfrac{2-3}{2-1}=-1$	$(2,-1)$

Sketch the graph.



Additional Example 3:

Sketch the graph of the rational function $r(x) = \dfrac{1-2x}{x+1}$.

Solution:
The numerator and denominator share no common factors.

Intercepts:

The y-intercept is $r(0) = \dfrac{1-2\cdot 0}{0+1} = 1$.

The x-intercept is $\dfrac{1}{2}$ since $\dfrac{1-2x}{x+1} = 0$ whenever $1-2x = 0 \Rightarrow x = \dfrac{1}{2}$.

Asymptotes:

The zero of the denominator $x+1$ is -1. The vertical asymptote is $x = -1$.

Determine the behavior near the vertical asymptote $x = -1$ to the right of -1.

As $x \to -1^+$, the sign of $r(x) = \dfrac{1-2x}{x+1} = \dfrac{+}{+}$.
So, $r(x) \to \infty$ as $x \to -1^+$.

Determine the behavior near the vertical asymptote $x = -1$ to the left of -1.

As $x \to -1^-$, the sign of $r(x) = \dfrac{1-2x}{x+1} = \dfrac{+}{-}$.

So, $r(x) \to -\infty$ as $x \to -1^-$.

The degree of $1-2x$ is 1.

The degree of $x+1$ is 1.

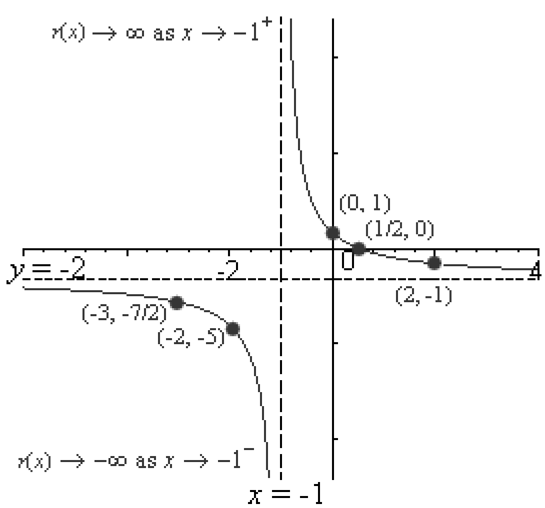
The horizontal asymptote is $y = \dfrac{\text{leading coefficient in numerator}}{\text{leading coefficient in denominator}} = \dfrac{-2}{1} = -2.$

Additional Points:

Make a table of values to find additional points on the graph.

x	$r(x)$	$(x, r(x))$
-3	$r(-3) = \dfrac{1-2(-3)}{-3+1} = -\dfrac{7}{2}$	$\left(-3, -\dfrac{7}{2}\right)$
-2	$r(-2) = \dfrac{1-2(-2)}{-2+1} = -5$	$(-2, -5)$
2	$r(2) = \dfrac{1-2(2)}{2+1} = -1$	$(2, -1)$

Sketch the graph.



Additional Example 4:

Sketch the graph of the rational function $r(x) = \dfrac{1}{(x+1)(x-1)}$.

Solution:

The numerator and denominator share no common factors.

Intercepts:

The y-intercept is $r(0) = \dfrac{1}{(0+1)(0-1)} = -1$.

There is no x-intercept since there is no value of x for which

$\dfrac{1}{(x+1)(x-1)} = 0$.

Asymptotes:

The zeros of the denominator $(x+1)(x-1)$ are -1 and 1. The vertical asymptotes are $x = -1$ and $x = 1$.

Determine the behavior near the vertical asymptote $x = 1$ to the right of 1.

As $x \to 1^+$, the sign of $r(x) = \dfrac{1}{(x+1)(x-1)} = \dfrac{+}{(+)(+)} = \dfrac{+}{+}$.

So, $r(x) \to \infty$ as $x \to 1^+$.

Determine the behavior near the vertical asymptote $x = 1$ to the left of 1.

As $x \to 1^-$, the sign of $r(x) = \dfrac{1}{(x+1)(x-1)} = \dfrac{+}{(+)(-)} = \dfrac{+}{-}$.

So, $r(x) \to -\infty$ as $x \to 1^-$.

Determine the behavior near the vertical asymptote $x = -1$ to the right of -1.

As $x \to -1^+$, the sign of $r(x) = \dfrac{1}{(x+1)(x-1)} = \dfrac{+}{(+)(-)} = \dfrac{+}{-}$.

So, $r(x) \to -\infty$ as $x \to -1^+$.

Determine the behavior near the vertical asymptote $x=-1$ to the left of -1.

As $x \to -1^-$, the sign of $r(x) = \dfrac{1}{(x+1)(x-1)} = \dfrac{+}{(-)(-)} = \dfrac{+}{+}$.

So, $r(x) \to \infty$ as $x \to -1^-$.

The degree of 1 is 0.

The degree of $(x+1)(x-1)$ is 2.

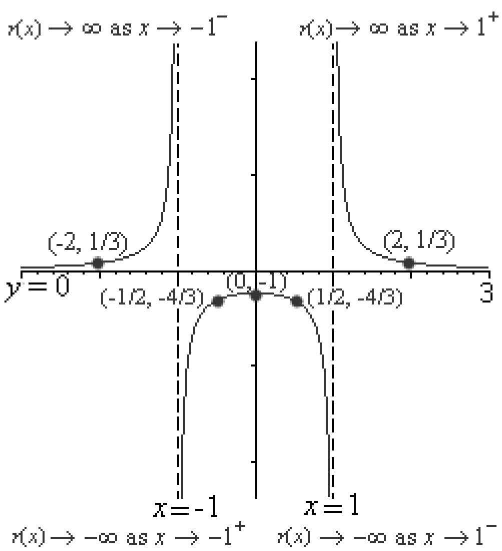
The horizontal asymptote is $y=0$ since the degree of the numerator is less than the degree of the denominator.

Additional Points:

Make a table of values to find additional points on the graph.

x	$r(x)$	$(x, r(x))$
-2	$r(-2) = \dfrac{1}{(-2+1)(-2-1)} = \dfrac{1}{3}$	$\left(-2, \dfrac{1}{3}\right)$
$-\dfrac{1}{2}$	$r\left(-\dfrac{1}{2}\right) = \dfrac{1}{\left(-\dfrac{1}{2}+1\right)\left(-\dfrac{1}{2}-1\right)} = -\dfrac{4}{3}$	$\left(-\dfrac{1}{2}, -\dfrac{4}{3}\right)$
$\dfrac{1}{2}$	$r\left(\dfrac{1}{2}\right) = \dfrac{1}{\left(\dfrac{1}{2}+1\right)\left(\dfrac{1}{2}-1\right)} = -\dfrac{4}{3}$	$\left(\dfrac{1}{2}, -\dfrac{4}{3}\right)$
2	$r(2) = \dfrac{1}{(2+1)(2-1)} = \dfrac{1}{3}$	$\left(2, \dfrac{1}{3}\right)$

Sketch the graph.



Additional Example 5:

Sketch the graph of the rational function $r(x) = \dfrac{x^2+2}{x-1}$.

Solution:

The numerator and denominator share no common factors.

Intercepts:

The y-intercept is $r(0) = \dfrac{0^2+2}{0-1} = -2$.

There is no x-intercept since there is no real number x for which $\dfrac{x^2+2}{x-1} = 0$.

Asymptotes:

The zero of the denominator $x-1$ is 1. The vertical asymptote is $x = 1$.

Determine the behavior near the vertical asymptote $x = 1$ to the right of 1.

As $x \to 1^+$, the sign of $r(x) = \dfrac{x^2+2}{x-1} = \dfrac{+}{+}$.

So, $r(x) \to \infty$ as $x \to 1^+$.

Determine the behavior near the vertical asymptote $x = 1$ to the left of 1.

As $x \to 1^-$, the sign of $r(x) = \dfrac{x^2+2}{x-1} = \dfrac{+}{-}$.

So, $r(x) \to -\infty$ as $x \to 1^-$.

The degree of the numerator is 2.
The degree of the denominator is 1.

There is no horizontal asymptote since the degree of the numerator is greater than the degree of the denominator.

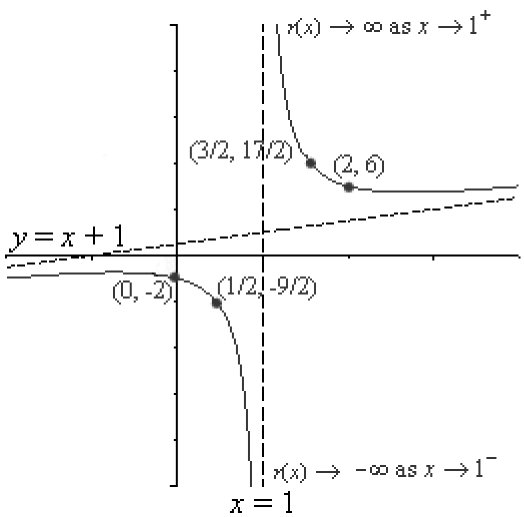
Since the degree in the numerator is one more than the degree of the denominator, there is a slant asymptote. Divide the denominator into the numerator by polynomial long division to obtain $r(x) = x + 1 + \dfrac{3}{x-1}$. The slant asymptote is the line $y = x + 1$.

Additional Points:

Make a table of values to find additional points on the graph.

x	$r(x)$	$(x, r(x))$
1/2	$r(1/2) = \frac{(1/2)^2 + 2}{1/2 - 1} = -9/2$	$(1/2, -9/2)$
3/2	$r(3/2) = \frac{(3/2)^2 + 2}{3/2 - 1} = 17/2$	$(3/2, 17/2)$
2	$r(2) = \frac{(2)^2 + 2}{2 - 1} = 6$	$(2, 6)$

Sketch the graph.



Exercise Set 2.3: Rational Functions

Recall from Section 1.2 that an even function is symmetric with respect to the *y*-axis, and an odd function is symmetric with respect to the origin. This can sometimes save time in graphing rational functions. If a function is even or odd, then half of the function can be graphed, and the rest can be graphed using symmetry.

Determine if the functions below are even, odd, or neither.

1. $f(x)=\frac{5}{x}$

2. $f(x)=-\frac{3}{x-1}$

3. $f(x)=\frac{4}{x^2-9}$

4. $f(x)=\frac{9x^2-1}{x^4}$

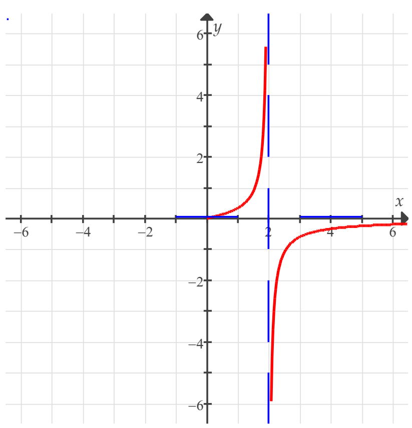
5. $f(x)=\frac{x-1}{x^2-4}$

6. $f(x)=\frac{7}{x^3}$

In each of the graphs below, only half of the graph is given. Sketch the remainder of the graph, given that the function is:

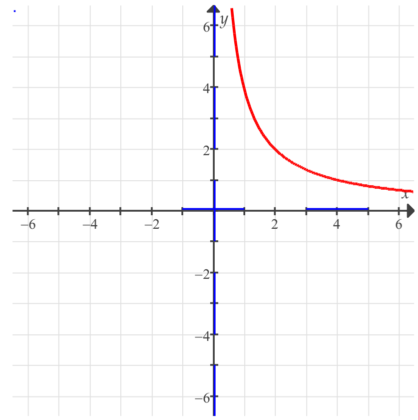
(a) **Even**
(b) **Odd**

7.



(Notice the asymptotes at $x=2$ and $y=0$.)

8.

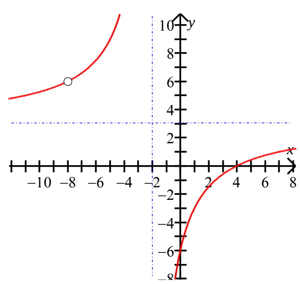


(Notice the asymptotes at $x=0$ and $y=0$.)

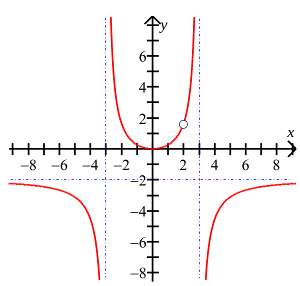
For each of the following graphs:

(j) **Identify the location of any hole(s) (i.e. removable discontinuities)**
(k) **Identify any *x*-intercept(s)**
(l) **Identify any *y*-intercept(s)**
(m) **Identify any vertical asymptote(s)**
(n) **Identify any horizontal asymptote(s)**

9.



10.



Exercise Set 2.3: Rational Functions

For each of the following rational functions:

(a) Find the domain of the function
(b) Identify the location of any hole(s) (i.e. removable discontinuities)
(c) Identify any *x*-intercept(s)
(d) Identify any *y*-intercept(s)
(e) Identify any vertical asymptote(s)
(f) Identify any horizontal asymptote(s)
(g) Identify any slant asymptote(s)
(h) Sketch the graph of the function. Be sure to include all of the above features on your graph.

11. $f(x)=\dfrac{3}{x-5}$

12. $f(x)=\dfrac{-4}{x+7}$

13. $f(x)=\dfrac{2x+3}{x}$

14. $f(x)=\dfrac{9-4x}{x}$

15. $f(x)=\dfrac{x-6}{x+3}$

16. $f(x)=\dfrac{x+5}{x-2}$

17. $f(x)=\dfrac{-4x+8}{2x+3}$

18. $f(x)=\dfrac{3x+6}{2x-1}$

19. $f(x)=\dfrac{(x-2)(x+3)}{(x-2)(x-4)}$

20. $f(x)=\dfrac{(x+3)(6-x)}{(x-2)(x+3)}$

21. $f(x)=\dfrac{x^2+x-20}{x-4}$

22. $f(x)=\dfrac{x^2-3x-10}{x-5}$

23. $f(x)=\dfrac{4x^3}{x^2-1}$

24. $f(x)=\dfrac{x^3}{2x^2-18}$

25. $f(x)=\dfrac{-(3x+5)(x-2)}{x(x-2)}$

26. $f(x)=\dfrac{-(x+4)(5x-7)}{(x-3)(x+4)}$

27. $f(x)=\dfrac{2x^2-18}{x^2+4x+3}$

28. $f(x)=\dfrac{8x^2-16x}{5x^2-20}$

29. $\dfrac{16-x^4}{2x^3}$

30. $f(x)=\dfrac{x^3-2x^2-x+2}{x^2-4}$

31. $f(x)=\dfrac{8}{x^2-4}$

32. $f(x)=\dfrac{-12}{x^2+x-6}$

33. $f(x)=\dfrac{6x-6}{x^2-x-12}$

34. $f(x)=\dfrac{-8x-16}{x^2+2x-15}$

35. $f(x)=\dfrac{(x-3)(x+2)(x-4)}{(x+1)(x-4)(x-2)}$

36. $f(x)=\dfrac{2x^3+10x^2}{x^3+5x^2-9x-45}$

37. $f(x)=\dfrac{x(x-5)(x+1)(x-3)}{(x+1)(x-3)}$

38. $f(x)=-\dfrac{(x-4)(x-3)(x-2)(x+1)}{(x-4)(x-2)}$

39. $f(x)=\dfrac{x^3+2x^2-9x-18}{x^2}$

40. $f(x)=\dfrac{x^4-10x^2+9}{x^3}$

Exercise Set 2.3: Rational Functions

Answer the following.

41. In the function $f(x)=\dfrac{x^2-5x+3}{3x^2+2x-3}$

(a) Use the quadratic formula to find the x-intercepts of the function, and then use a calculator to round these answers to the nearest tenth.

(b) Use the quadratic formula to find the vertical asymptotes of the function, and then use a calculator to round these answers to the nearest tenth.

42. In the function $f(x)=\dfrac{2x^2+7x-1}{x^2-6x+4}$

(a) Use the quadratic formula to find the x-intercepts of the function, and then use a calculator to round these answers to the nearest tenth.

(b) Use the quadratic formula to find the vertical asymptotes of the function, and then use a calculator to round these answers to the nearest tenth.

The graph of a rational function never intersects a vertical asymptote, but at times the graph intersects a horizontal asymptote. For each function $f(x)$ below,

(a) **Find the equation for the horizontal asymptote of the function.**

(b) **Find the x-value where $f(x)$ intersects the horizontal asymptote.**

(c) **Find the point of intersection of $f(x)$ and the horizontal asymptote.**

43. $f(x)=\dfrac{x^2+2x+3}{x^2-x-3}$

44. $f(x)=\dfrac{x^2+4x-2}{x^2-x-7}$

45. $f(x)=\dfrac{x^2+2x+3}{2x^2+6x-1}$

46. $f(x)=\dfrac{3x^2+5x-1}{x^2-x-3}$

47. $f(x)=\dfrac{4x^2+12x+9}{x^2-x+7}$

48. $f(x)=\dfrac{x^2+5x-1}{5x^2-10x-3}$

Answer the following.

49. The function $f(x)=\dfrac{6x-6}{x^2-x-12}$ was graphed in Exercise 33.

(a) Find the point of intersection of $f(x)$ and the horizontal asymptote.

(b) Sketch the graph of $f(x)$ as directed in Exercise 33, but also label the intersection of $f(x)$ and the horizontal asymptote.

50. The function $f(x)=\dfrac{-8x-16}{x^2+2x-15}$ was graphed in Exercise 34.

(a) Find the point of intersection of $f(x)$ and the horizontal asymptote.

(b) Sketch the graph of $f(x)$ as directed in Exercise 34, but also label the intersection of $f(x)$ and the horizontal asymptote.