Section 2.2: Polynomial Functions

- Polynomial Functions and Basic Graphs
- Guidelines for Graphing Polynomial Functions

Polynomial Functions and Basic Graphs

Polynomials:

A polynomial in a single variable x is the sum of a finite number of terms of the form ax^n, where a is a constant and the exponent n is a whole number. Recall that the set of whole numbers is $\{0, 1, 2, \ldots\}$.

Example of polynomials in x are $3x$, $5x^3 + 8x$, and $4x^2 - 7x + 1$. They can be classified according to the number of terms:

$3x$ is called a monomial (a polynomial with one term).
$5x^3 + 8x$ is called a binomial (a polynomial with two terms).
$4x^2 - 7x + 1$ is called a trinomial (a polynomial with three terms).

A polynomial can contain more than one variable:

$x^2y^3 - 9x$ is a polynomial in x and y. It is a binomial since it has two terms.
$4xz^4 - 2xy^2 + 5xyz$ is a polynomial in x, y, and z. It is a trinomial.

The expressions $3x^{1/2}$ and $2x^{-6} + 7$ are not polynomials since the exponents $1/2$ and -6 are not whole numbers.

Degree of a Polynomial:

The degree of a monomial in a single variable is the exponent that appears on that variable. If the monomial is a constant that is not zero, then its degree is 0. The degree of the monomial 0 is undefined.

$3x^5$ is a monomial of degree 5 since the exponent on x is 5.
$-8x$ is a monomial of degree 1 since the exponent on x is understood to be 1.
15 is a monomial of degree 0.

The degree of a monomial in more than one variable is the sum of the exponents that appear on those variables.

$\frac{1}{3}x^5y^2z$ is a monomial of degree $5+2+1=8$.
$-\sqrt{2}wx^2y^3z^6$ is a monomial of degree $1+2+3+6=12$.

The degree of a polynomial is the degree of that term in the polynomial which has the greatest degree.

$12x^3+5x-9$ is a trinomial of degree 3. The term of greatest degree is $12x^3$.
($5x$ has degree 1 and -9 has degree 0.)
$-8x^3y^5+12xy$ is a binomial of degree 8. The term of greatest degree is $-8x^3y^5$.
($12xy$ has degree 2.)

Example:

Determine whether or not the following expressions are polynomials. For those expressions that are polynomials, state the number of terms and classify them as monomials, binomials, or trinomials (if appropriate). Then state the degree of each polynomial.

(a) $\frac{1}{2}\sqrt[3]{x}-9x^5+23$

(b) $4x^2yz^7-19x^3z+98x+z$

(c) $-\frac{1}{4}x^8+12x^4-3$

(d) $\frac{2}{3}x^7+11$

Solution:

(a) We can write $\frac{1}{2}\sqrt[3]{x} - 9x^5 + 23 = \frac{1}{2}x^{1/3} - 9x^5 + 23$. Since the exponent $1/3$ is not a whole number, the given expression is not a polynomial.

(b) The given expression is a polynomial. The given polynomial has 4 terms.

$4x^2yz^7$ has degree $2+1+7=10$.
$-19x^3z$ has degree $3+1=4$.
$98x$ has degree 1.
z has degree 1.

Thus, the degree of the polynomial is 10.

(c) The given expression is a polynomial. The given polynomial has 3 terms. Thus, it is a trinomial.

$-\frac{1}{4}x^8$ has degree 8.
$12x^4$ has degree 4.
-3 has degree 0.

Thus, the degree of the polynomial is 8.

(d) The given expression is a polynomial. The given polynomial has 2 terms. Thus, it is a binomial.

$\frac{2}{3}x^7$ has degree 7.
11 has degree 0.

Thus, the degree of the polynomial is 7.

Definition of a Polynomial Function:

A function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $a_n \neq 0$ and the exponents on x are whole numbers, is called a polynomial function of degree n.

The number a_n is called the leading coefficient and the term $a_n x^n$ is called the leading term. The number a_0 is called the constant term.

Examples of Polynomial Functions:

$P(x) = 5x + 12$ is a polynomial function of degree 1.

leading term: $5x$
leading coefficient: 5
constant term: 12

$P(x) = x^2 - 6x + 5$ is a polynomial function of degree 2.

leading term: x^2
leading coefficient: 1
constant term: 5

$P(x) = 2x^3 - 7x^2 + 5$ is a polynomial function of degree 3.

leading term: $2x^3$
leading coefficient: 2
constant term: 5

$P(x) = -x^4 + 2x^3 + 6x - 7$ is a polynomial function of degree 4.

leading term: $-x^4$
leading coefficient: -1
constant term: -7

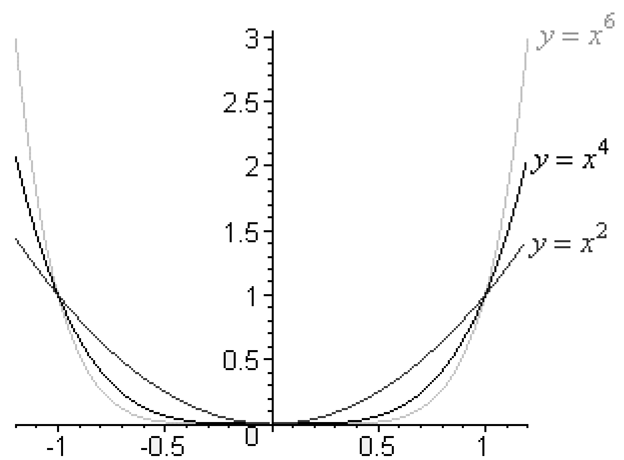
Basic Graphs of Polynomial Functions:

The graph of a polynomial function of the form $P(x) = a_1 x + a_0$ is a line with slope a_1 and y-intercept a_0. Note that if $a_1 = 0$, then the graph is a horizontal line.

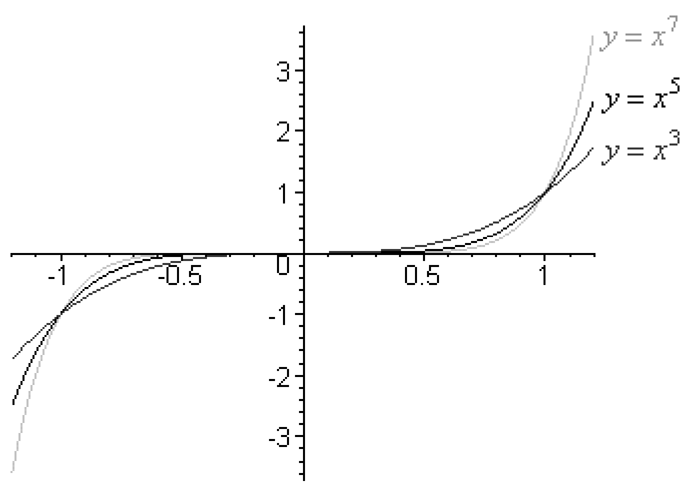
A polynomial function of the form $P(x) = a_2 x^2 + a_1 x + a_0$, with $a_2 \neq 0$, is a quadratic function and its graph is a parabola.

Techniques for graphing these functions are discussed in previous sections.

The graph of a polynomial function of the form $P(x) = x^n$ has the same general shape as the graph of $y = x^2$ when n is an <u>even</u> whole number greater than 2.



The graph of a polynomial function of the form $P(x) = x^n$ has the same general shape as the graph of $y = x^3$ when n is an <u>odd</u> whole number greater than 3.

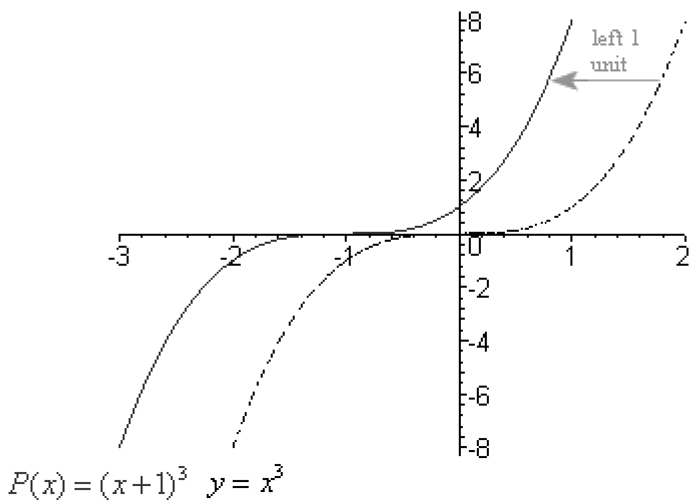


Example:

Sketch the graph of the polynomial function $P(x) = (x+1)^3$ by transforming the graph of the function $y = x^3$.

Solution:

The graph of $y = x^3$ is shifted 1 unit to the left to obtain the graph of $P(x) = (x+1)^3$.

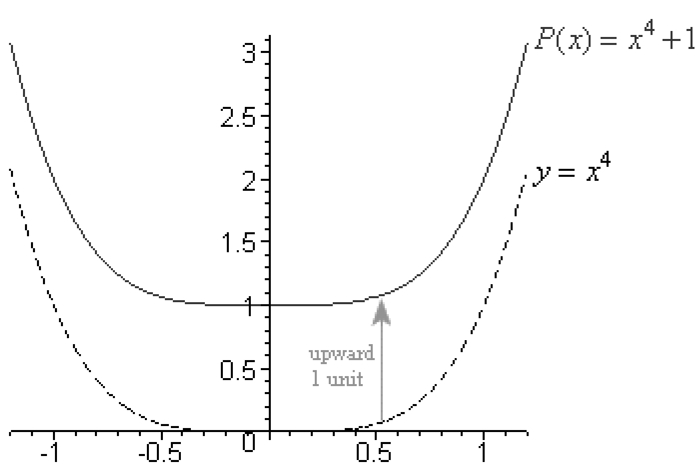


Example:

Sketch the graph of the polynomial function $P(x) = x^4 + 1$ by transforming the graph of the function $y = x^4$.

Solution:

The graph of $y = x^4$ is shifted 1 unit upward to obtain the graph of $P(x) = x^4 + 1$.



Additional Example 1:

Determine whether or not the following expressions are polynomials. For those expressions that are polynomials, state the number of terms and classify them as monomials, binomials, or trinomials (if appropriate). Then state the degree of each polynomial.

(a) $-5x^4 - 3x^3 + 2x^2 + 11x - 37$

(b) $2xy^6z - 19x^5y + 10$

(c) $\frac{1}{2}x - 9x^{-2} + 23\sqrt{x}$

Solution:

(a) The given expression is a polynomial. The given polynomial has 5 terms.

$-5x^4$ has degree 4.

$-3x^3$ has degree 3.

$2x^2$ has degree 2.

$11x$ has degree 1.

-37 has degree 0.

Thus, the degree of the polynomial is 4.

(b) The given expression is a polynomial. The given polynomial has 3 terms. Thus, it is a trinomial.

$2xy^6z$ has degree $1+6+1=8$.

$-19x^5y$ has degree $5+1=6$.

10 has degree 0.

Thus, the degree of the polynomial is 8.

(c) We can write $\frac{1}{2}x - 9x^{-2} + 23\sqrt{x} = \frac{1}{2}x - 9x^{-2} + 23x^{1/2}$. Since the exponents -2 and $1/2$ are not whole numbers, the given expression is not a polynomial.

Additional Example 2:

For the polynomial function $P(x) = -11x^9 + 7x^5 - 2x + 3$, identify the leading term, the degree, the leading coefficient, and the constant term.

Solution:

leading term:

$$P(x) = \boxed{-11x^9} + 7x^5 - 2x + 3$$

term of greatest degree

The leading term is $-11x^9$, which is the term of greatest degree.

degree:

exponent on the leading term

$$P(x) = -11x^{\boxed{9}} + 7x^5 - 2x + 3$$

The degree is 9, which is the exponent on the leading term.

leading coefficient:

$$P(x) = \boxed{-11}x^9 + 7x^5 - 2x + 3$$

coefficient of the leading term

The leading coefficient is -11, which is the coefficient of the leading term.

constant term:

term not containing the variable x

$$P(x) = -11x^9 + 7x^5 - 2x \boxed{+3}$$

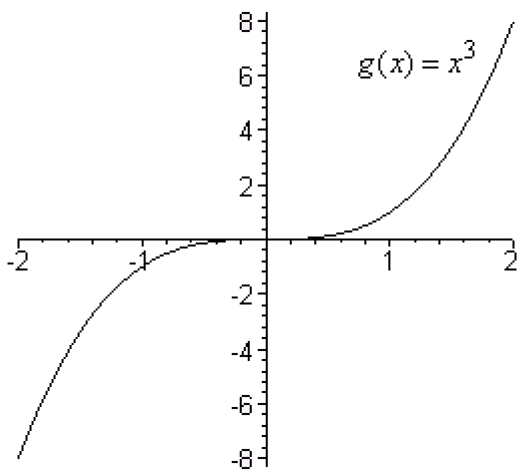
The constant term is 3, which is the term that does not contain the variable x.

Additional Example 3:

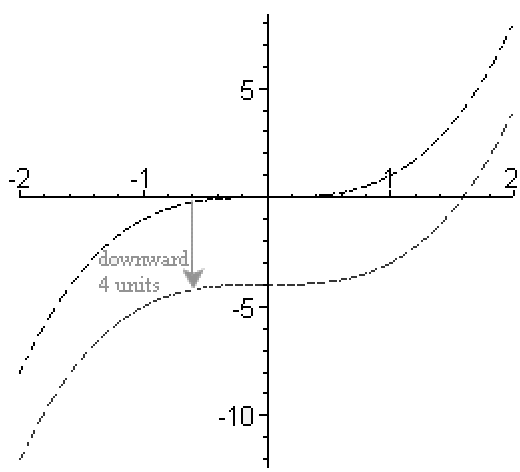
Sketch the graph of the polynomial function $P(x) = x^3 - 4$. Do not plot points, but instead apply transformations to the graph of the polynomial function $g(x) = x^3$.

Solution:

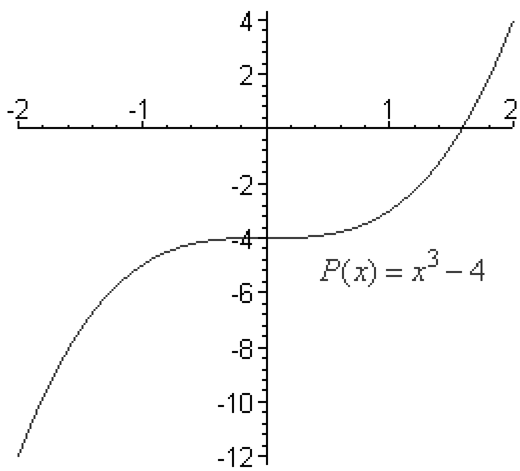
Begin with the graph of $g(x) = x^3$ shown below.



To graph the function $P(x) = x^3 - 4$, shift the graph of g 4 units downward.



The graph of the given function is shown below.

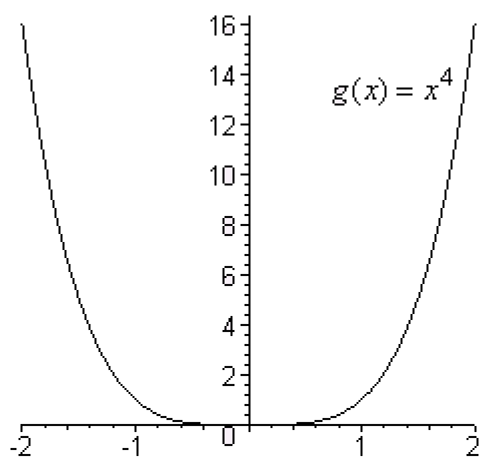


Additional Example 4:

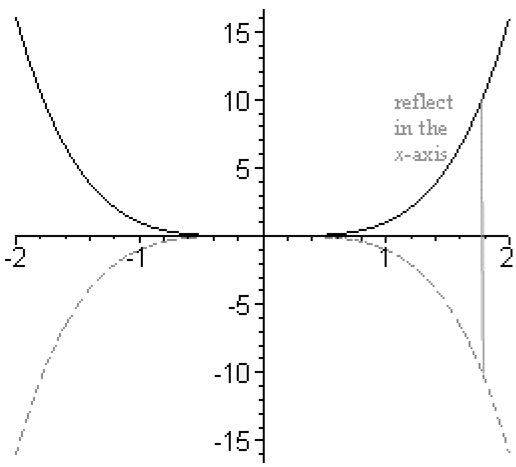
Sketch the graph of the polynomial function $P(x) = -x^4 + 8$. Do not plot points, but instead apply transformations to the graph of the polynomial function $g(x) = x^4$.

Solution:

Begin with the graph of $g(x) = x^4$ shown below.

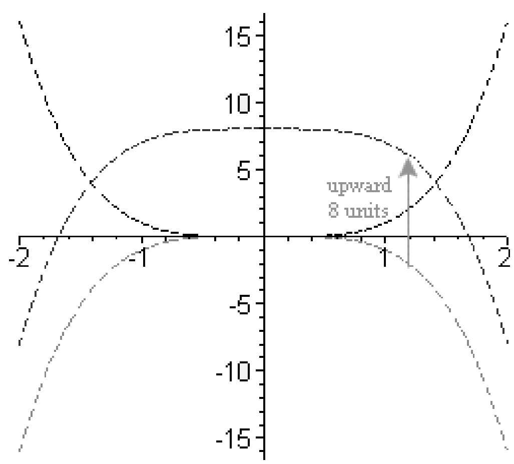


To graph the function $P(x) = -x^4 + 8$, first reflect the graph of g in the x-axis.

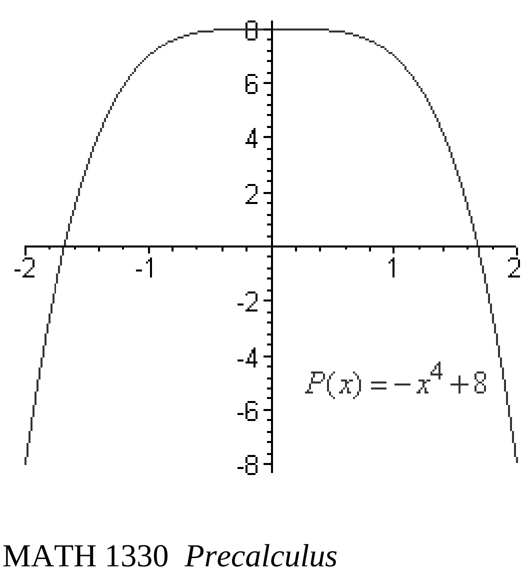


Next shift upward 8 units.

$\left[P(x) = -x^4 + 8\right]$



The graph of the given function is shown below.

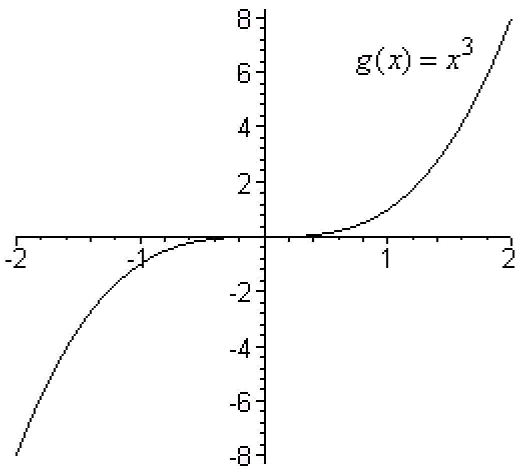


Additional Example 5:

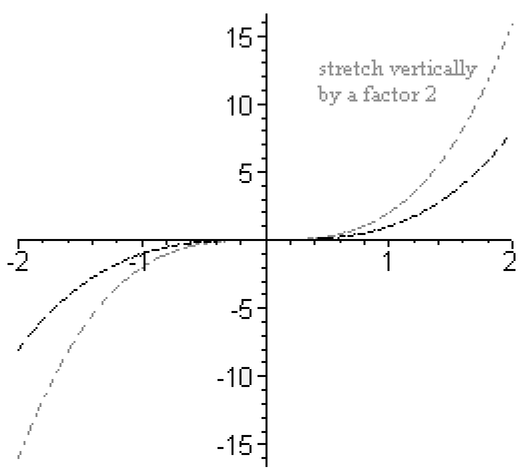
Sketch the graph of the polynomial function $P(x) = 2x^3 - 7$. Do not plot points, but instead apply transformations to the graph of the polynomial function $g(x) = x^3$.

Solution:

Begin with the graph of $g(x) = x^3$ shown below.

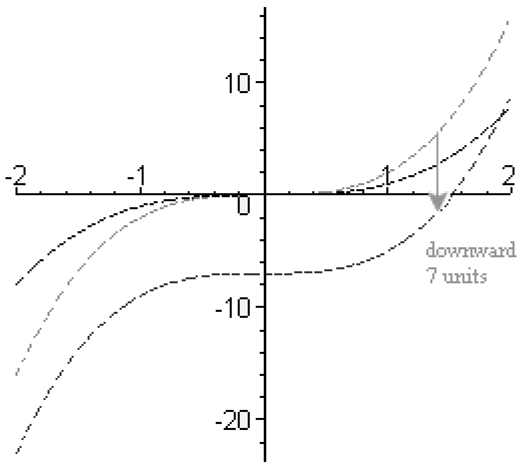


To graph the function $P(x) = 2x^3 - 7$, first stretch the graph of g vertically by a factor of 2.

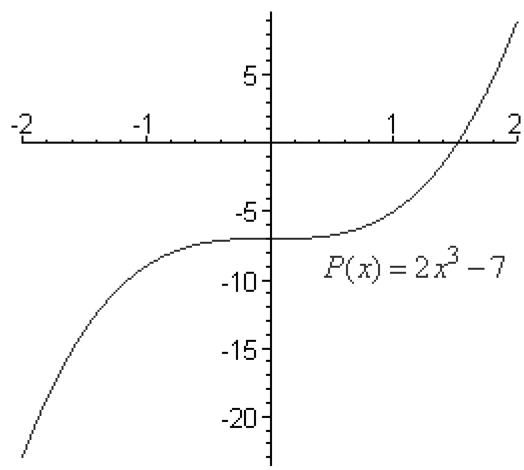


Next shift downward 7 units.

$\left[P(x) = 2x^3 - 7\right]$



The graph of the given function is shown below.



Guidelines for Graphing Polynomial Functions

A Strategy for Graphing Polynomial Functions:

To sketch the graph of a polynomial function $y = P(x)$ of degree three or greater, it is helpful to follow the guidelines given below.

Find the intercepts of the graph.

A zero of P is a number c that satisfies $P(c) = 0$. If c is a zero of P, then $x - c$ is a factor of $P(x)$. Factor the polynomial to find its real zeros. These are the x-intercepts of the graph.

The y-intercept is given by $P(0)$.

Investigate the behavior near the x-intercepts.

If c is a real zero of P and the factor $x - c$ occurs m times in the factorization of P, then the graph will cross the x-axis at c if m is odd and will not cross the x-axis if m is even. Near the x-intercept c, the graph will have the same general shape as the graph of $y = A(x - c)^m$.

Consider the number of turning points of the graph.

A polynomial of degree n can have at most $n - 1$ turning points. A turning point is a point where the graph changes from rising to falling or from falling to rising.

Determine the end behavior of *P*.

The end behavior of P is determined by the degree of P and the sign of the leading coefficient. (See the explanation given below.)

Make a table of values.

Make a table of values to determine other points on the graph.

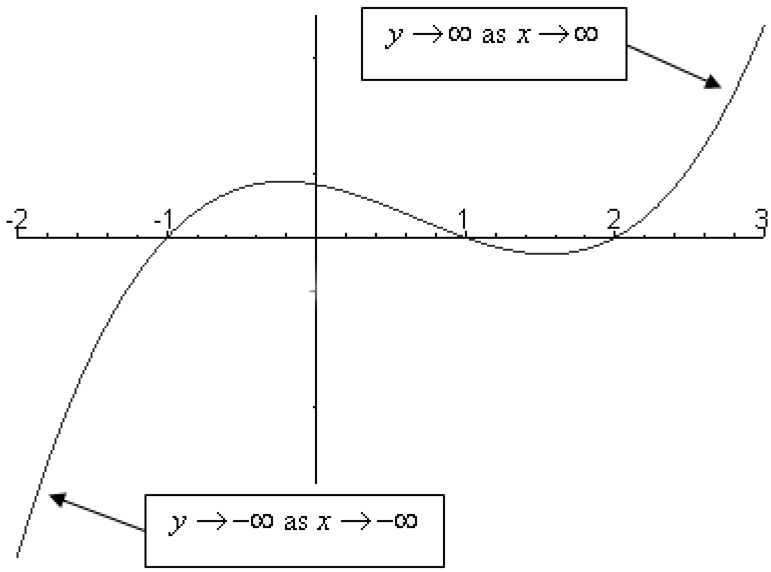
Sketch the graph.

Plot the points determined in the above steps. Connect the points with a smooth curve and make sure that the curve exhibits the behavior determined in the above steps.

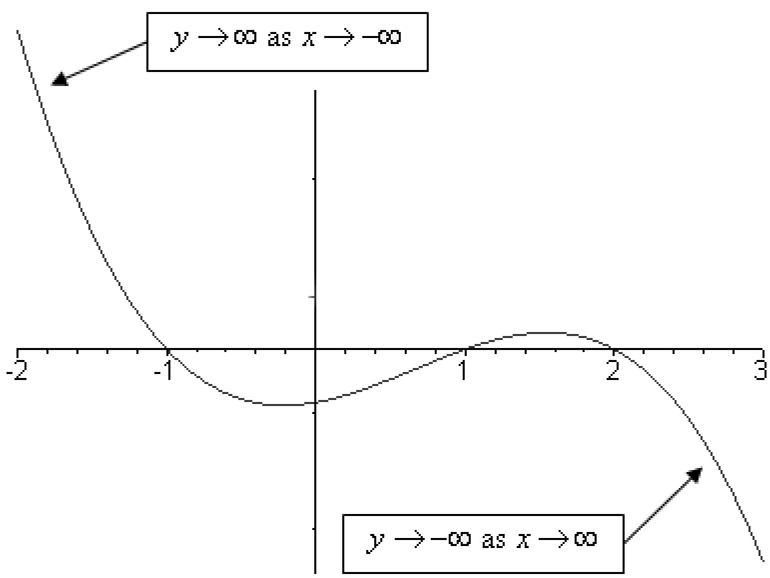
End Behavior of Polynomials:

For the polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, with $a_n \neq 0$, the end behavior is determined by the degree n and the leading coefficient a_n.

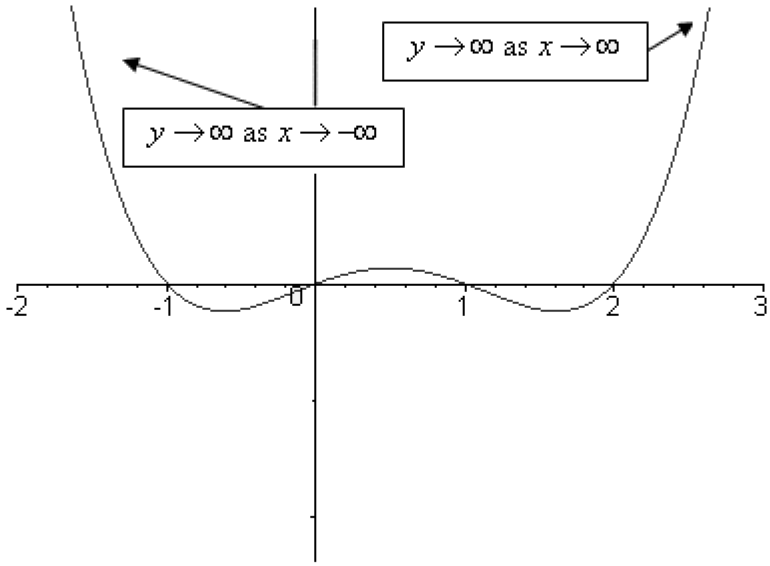
n odd and $a_n > 0$ (odd degree and leading coefficient positive)



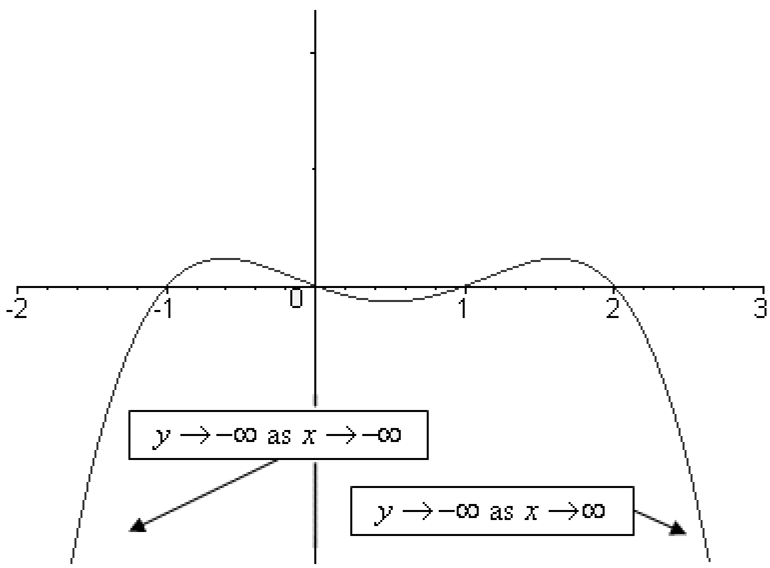
n odd and $a_n < 0$ (odd degree and leading coefficient negative)



n even and $\boldsymbol{a_n > 0}$ (even degree and leading coefficient positive)



n even and $\boldsymbol{a_n < 0}$ (even degree and leading coefficient negative)



Example:

Use the strategy for graphing polynomial functions to sketch the graph of $P(x) = x^3 - x^2 - 2x$.

Solution:

The y-intercept is $P(0) = 0^3 - 0^2 - 2(0) = 0$.

Find the real zeros of the polynomial by solving the equation $P(x)=0$ to find the x-intercepts of the graph.

$$P(x)=0$$
$$x^3-x^2-2x=0$$
$$x(x^2-x-2)=0$$
$$x(x+1)(x-2)=0$$

$x=0$ or $x+1=0$ or $x-2=0$

$x+1-1=0-1$ $\quad$ $x-2+2=0+2$

$x=-1$ $\quad$ $x=2$

The x-intercepts are $0, -1,$ and 2.

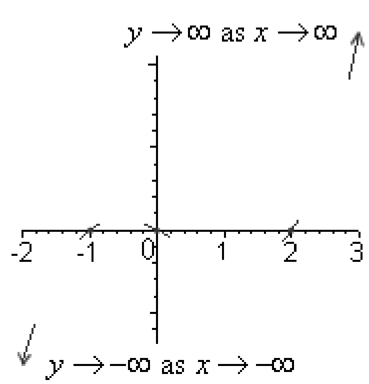
The graph crosses the x-axis at all the x-intercepts since each of the factors x, $x+1$, and $x+2$ in the factorization has an exponent of 1.

Near the x-intercepts 0, -1, and 2, the graph has the same general shape as the graphs of $y=Ax, y=A(x+1)$ and $y=A(x-2)$, respectively.

The given function is a polynomial function of degree three. There can be at most $3-1=2$ turning points.

The degree of the polynomial function is 3, which is odd, and the leading coefficient is 1, which is positive. Thus, the end behavior of the polynomial is:
$y\to-\infty$ as $x\to-\infty$ and $y\to\infty$ as $x\to\infty$.

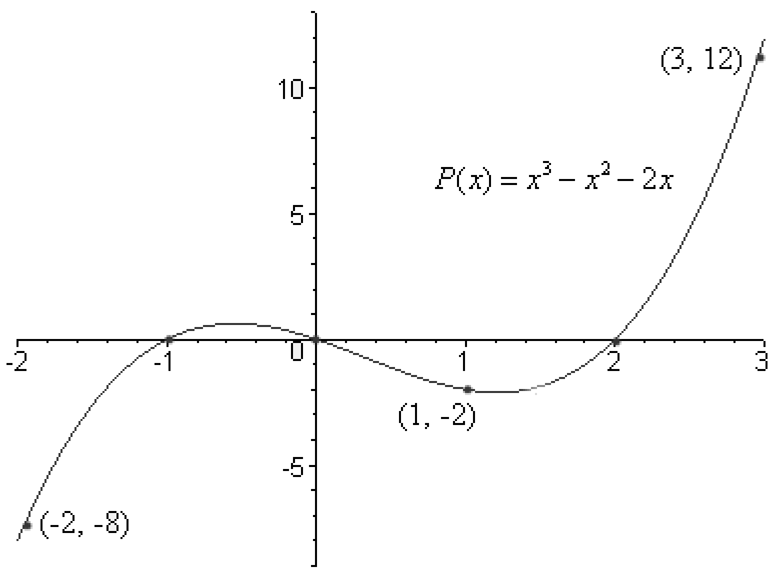
Show the end behavior, the intercepts, and the behavior near the x-intercepts on a graph.



Make a table of values to find additional points on the graph.

x	$P(x)$	$(x, P(x))$
-2	$P(-2)=(-2)^3-(-2)^2-2(-2)=-8$	$(-2,-8)$
1	$P(1)=(1)^3-(1)^2-2(1)=-2$	$(1,-2)$
3	$P(3)=(3)^3-(3)^2-2(3)=12$	$(3,12)$

Sketch the graph. Connect all points with a smooth curve.



Example:

Using the function $P(x)=x^2(x-1)^3(x+1)$,

(f) Find the *x*- and *y*-intercepts.

(g) Sketch the graph of the function. Be sure to show all *x*- and *y*-intercepts, along with the proper behavior at each *x*-intercept, as well as the proper end behavior.

Solution:

(a) The *x*-intercepts of the function occur when $P(x)=0$, so we must solve the equation

$$x^2(x-1)^3(x+1)=0$$

Set each factor equal to zero and solve for x.

Solving $x^2=0$, we see that the graph has an *x*-intercept of 0.

Solving $(x-1)^3=0$, we see that the graph has an *x*-intercept of 1.

Solving $x+1=0$ we see that the graph has an *x*-intercept of -1.

To find the *y*-intercept, find $P(0)$. (In other words, let $x = 0$.)

$$P(x) = 0^2 (0-1)^3 (0+1) = 0(-1)(1) = 0$$

Therefore, the *y*-intercept is 0.

(b) Next, we will determine the degree of the function. Look at the highest power of *x* in each factor (along with its sign). If this polynomial were to be multiplied out, it would be of the form $P(x) = x^6 + ...$ (the rest of the polynomial need not be shown; we are simply determining the end behavior of the graph). Remember that for the graph of any even function, both 'tails' of the function either go up together, or down together. Since there is a positive leading coefficient, we know that the end behavior of this function will look something like this:

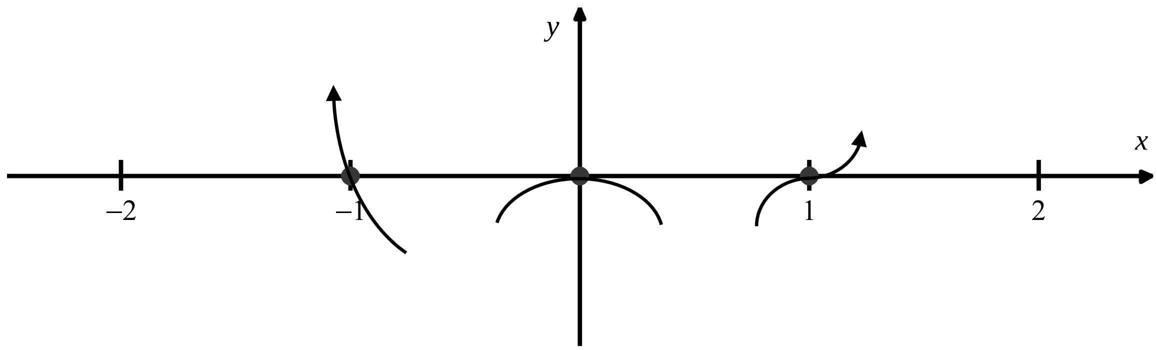
Next, place the *x* and *y*-intercepts on a set of axes, and determine the behavior at each intercept.

The *x*-intercepts of -1, 0, and 1 are shown on the graph below, along with the *y*-intercept of 0. Because the polynomial has degree 6 and the leading coefficient is positive, we know that both 'tails' go upward, as indicated by the arrows on the graph above. We now need to consider the behavior at each *x*-intercept. Let us deal first with the leftmost and rightmost x-intercepts (the ones which are affected by the upward-rising 'tails').

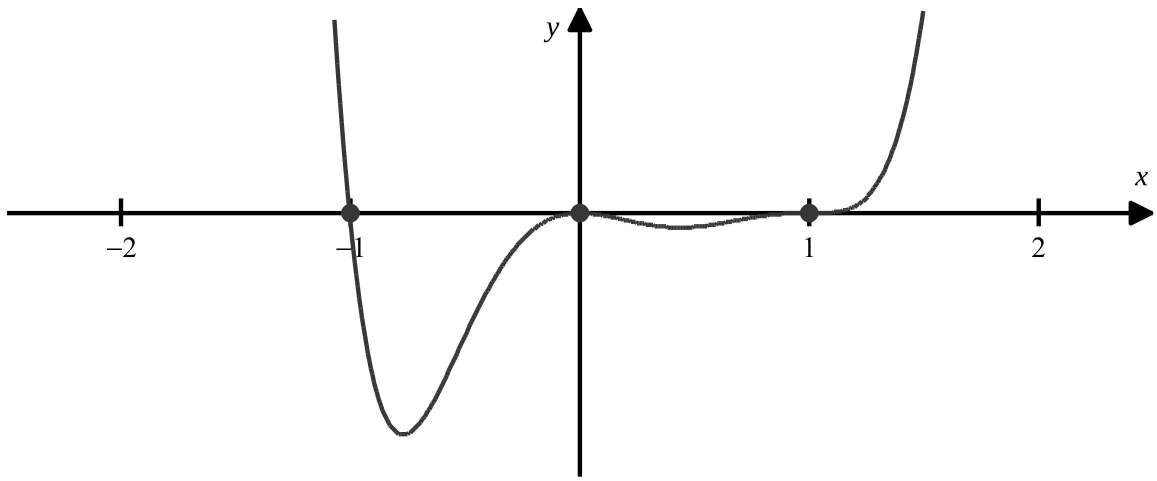
The behavior at $x = -1$ resembles the behavior of $y = x + 1$. We know that $y = x + 1$ is a line, but since we are drawing a polynomial, the behavior at this intercept will have some curvature in its shape.

The behavior at $x = 1$ resembles the behavior of $y = (x-1)^3$, so this portion of the graph resembles the behavior of a cubic graph. We know that it goes upward from left to right rather than downward from left to right, because of the end behavior which has already been discussed. (Both 'tails' need to point upward.).

The behavior at $x = 0$ resembles the behavior of $y = x^2$, so this portion of the graph will resemble the shape of a parabola. It needs to connect to the other parts of the graph (and the nly x-intercepts are -1, 0, and 1). Therefore, this parabola-type shape at $x = 0$ needs to point downward, as shown below.



We then connect the remainder of the graph with a smooth curve. The actual graph is shown below.



Note that aside from plotting points, we do not yet have the tools to know the exact shape of the graph in Figure 3. We would not know the exact value, for example, of the relative minimum which is shown in the graph above between $x=-1$ and $x=0$. It is important that you show the basic features shown in Figure 2 (end behavior, intercepts, and the general behavior at each intercept), and then quickly sketch the rest of the graph with a smooth curve.

Example:

Using the function $P(x)=-(x-4)(x-2)^7(x+1)^{12}$,

(a) Find the *x*- and *y*-intercepts.

(b) Sketch the graph of the function. Be sure to show all *x*- and *y*-intercepts, along with the proper behavior at each *x*-intercept, as well as the proper end behavior.

Solution:

(a) The *x*-intercepts of the function occur when $P(x)=0$, so we must solve the equation

$$-(x-4)(x-2)^7(x+1)^{12}=0$$

Set each factor equal to zero and solve for x.

Solving $x-4=0$, we see that the graph has an *x*-intercept of 4.

Solving $(x-2)^7=0$, we see that the graph has an *x*-intercept of 2.

Solving $(x+1)^{12}=0$, we see that the graph has an *x*-intercept of -1.

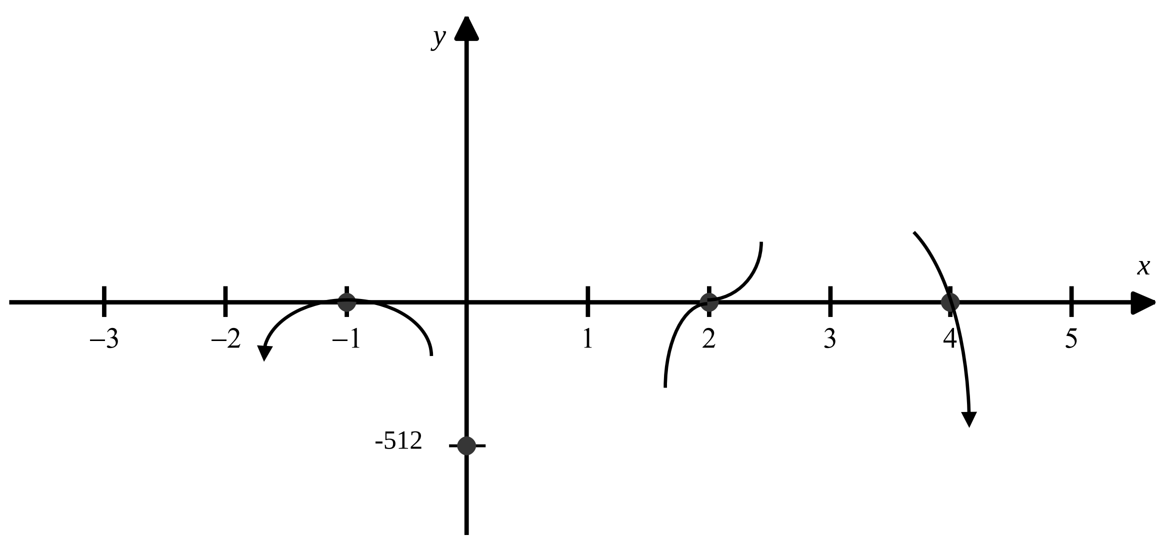
To find the *y*-intercept, find $P(0)$. (In other words, let $x=0$.)

$$P(x)=-(0-4)(0-2)^7(0+1)^{12}=-(-4)(-2)^7(1)^{12}=-(-4)(-128)(1)=-512$$

Therefore, the *y*-intercept is 512.

(b) Next, we will determine the degree of the function. Look at the highest power of *x* in each factor (along with its sign). If this polynomial were to be multiplied out, it would be of the form $P(x)=-x^{20}+...$ (the rest of the polynomial need not be shown; we are simply determining the end behavior of the graph). Remember that for the graph of any even function, both 'tails' of the function either go up together, or down together. Since there is a negative leading coefficient, we know that the end behavior of this function will look something like this:

Next, place the *x* and *y*-intercepts on a set of axes, and determine the behavior at each intercept.



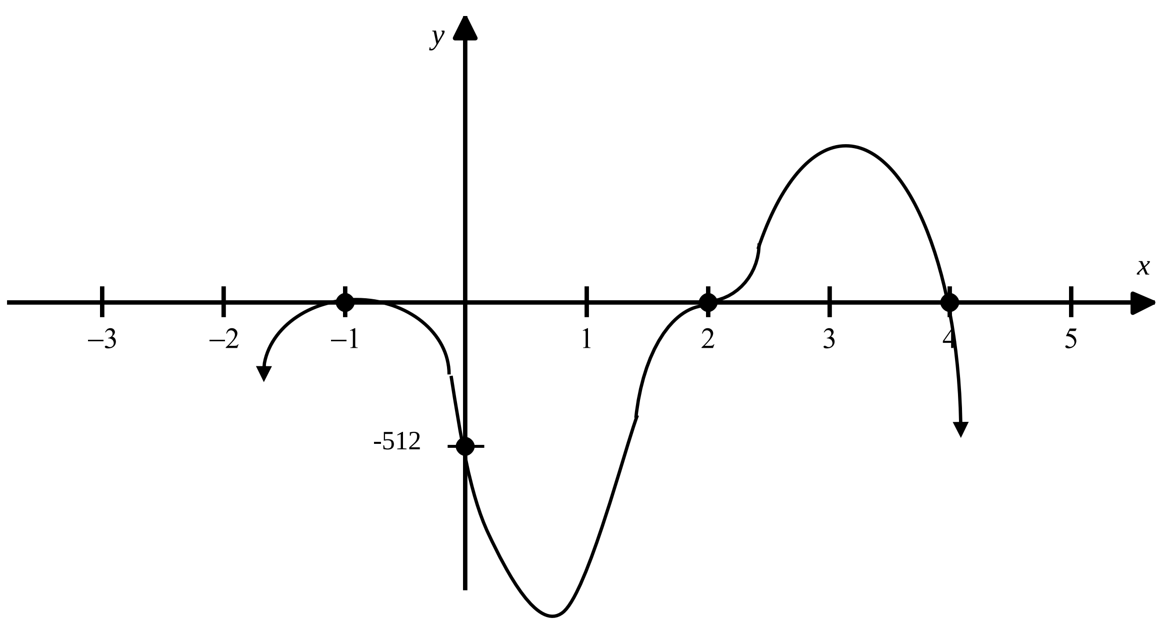
The *x*-intercepts of -1, 2, and 4 are shown on the graph, along with the *y*-intercept of -512. Because the polynomial has degree 20 and the leading coefficient is negative, we know that both 'tails' go downward, as indicated by the arrows on the graph above. We now need to consider the behavior at each *x*-intercept.

The behavior at $x=-1$ resembles the behavior of $y=\left(x+1\right)^{12}$. Notice the even exponent; this portion of the graph will resemble the shape of a parabola (though it is even more 'flattened out' near the *x*-axis than what is shown in the graph above). Notice that this parabola-type shape points downward because of the end behavior which has already been discussed.

The behavior at $x=2$ resembles the behavior of $y=\left(x-2\right)^{7}$. Notice the odd exponent; this resembles the behavior of a cubic graph (and since the exponent is higher, is more 'flattened' near the *x*-axis than a cubic graph). We know that it goes upward from left to right rather than downward, because the graph needs to pass through the *y*-intercept $\left(0,-512\right)$ before passing through the point $\left(2,0\right)$.

The behavior at $x=4$ resembles that of $y=(x-4)$. This portion of the graph goes through the point $\left(4,0\right)$ as would the line $y=x-4$. We know from our previous discussion that the end behavior of the polynomial must point downward (as indicated by the arrow on the right side of the graph). Since the function is a polynomial (and not a line), we see a slight curvature as the graph passes through $x=4$.

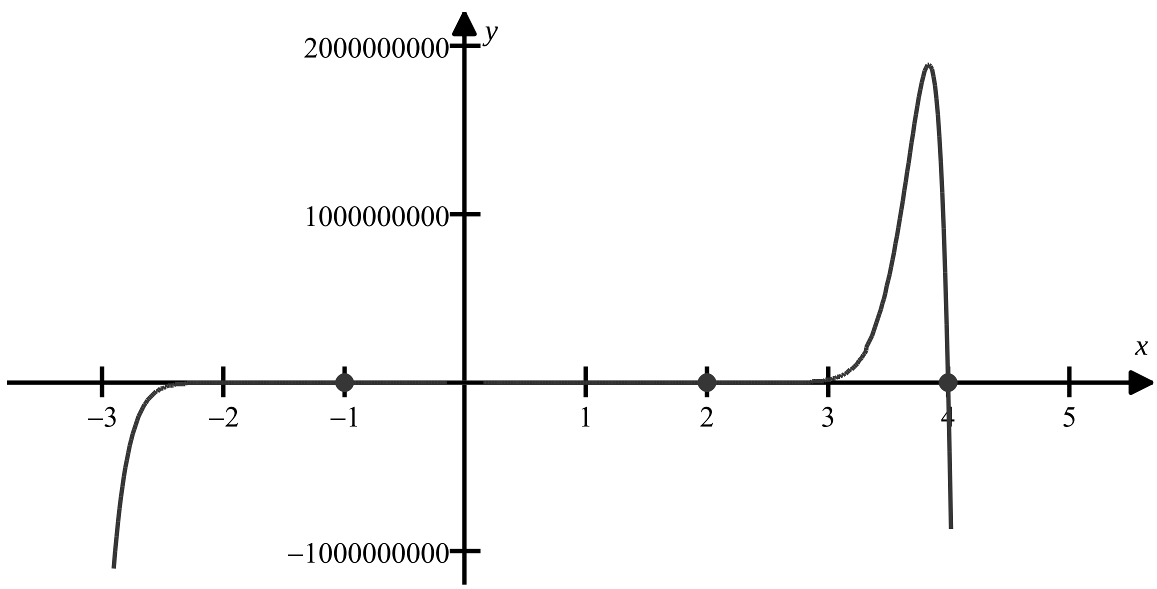
Based on the analysis above, a rough sketch of $P\left(x\right)=-\left(x-4\right)\left(x-2\right)^{7}\left(x+1\right)^{12}$ is shown below.



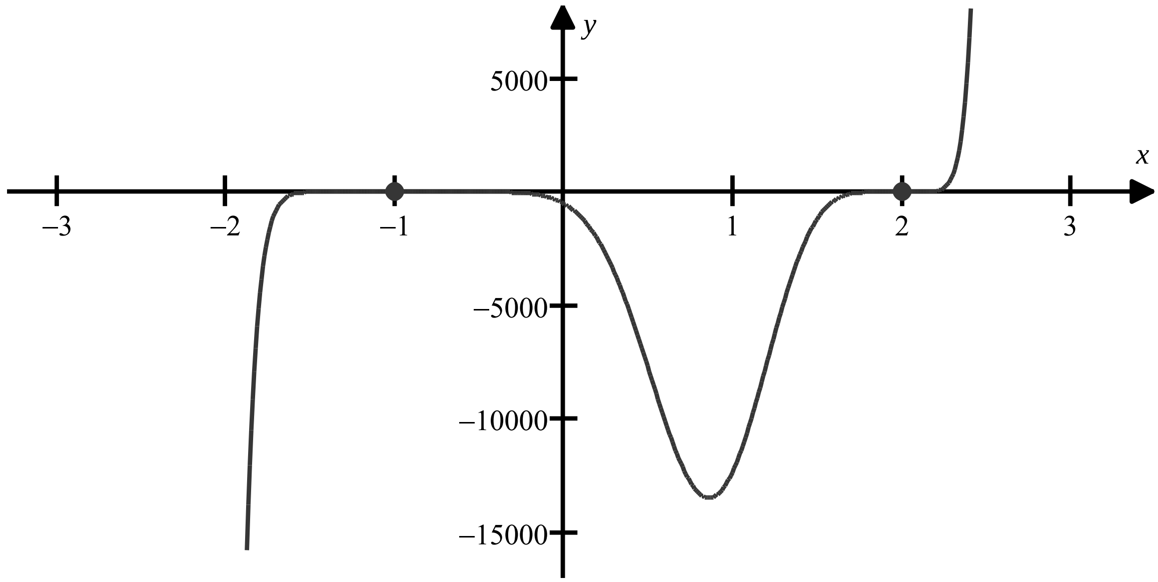
Note:
The graph above is only a rough sketch which gives an idea of the behavior of the graph, and is sufficient for the purpose of this course. In reality, the behavior near $x = -1$ and $x = 2$ is more 'flattened' (very close to the *x*-axis). Moreover, this graph has *y*-values of a very large magnitude because of the large exponents in the polynomial function.

The analysis shown below is beyond the scope of the Math 1330 course, but is included to show you what the graph of the above function really looks like.

We could try to make the graph more accurate by plugging values into the function, but we would quickly realize that a true picture of the graph would be difficult to even illustrate on this page. For example, $f(1) = \text{-}12{,}288$ and $f(3) = 16{,}777{,}216$ -- and these do not even represent the lowest and highest points in those regions of the graph! (Methods of finding the minimum and maximum values are learned in Calculus.) If we scale the graph to show the true *y*-values, the *y*-intercept of -512 will appear to be at the origin, because the scale on the graph will be so large.



A closeup is shown below to show the actual behavior of the graph between $x = -3$ and $x = 3$. The graph below does not show the portion of the graph which shoots high up and comes down through the point (4, 0).



At this point, these more detailed graphs can only be obtained with a graphing calculator or with graphing software. It should be noted again that **the first rough sketch was sufficient for the purposes of this course**.

Additional Example 1:

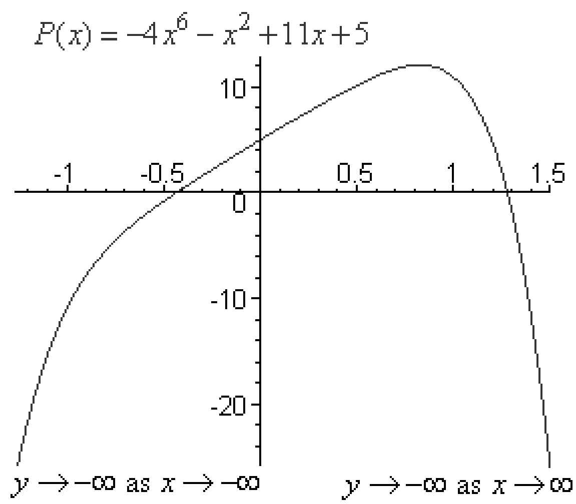
Describe the end behavior of the polynomial function $P(x) = -4x^6 - x^2 + 11x + 5$.

Solution:

The degree is 6 and the leading coefficient is -4.

Since the degree is even and the leading coefficient is negative, $y \to -\infty$ as $x \to -\infty$ and $y \to -\infty$ as $x \to \infty$.

The graph is shown below.



Additional Example 2:

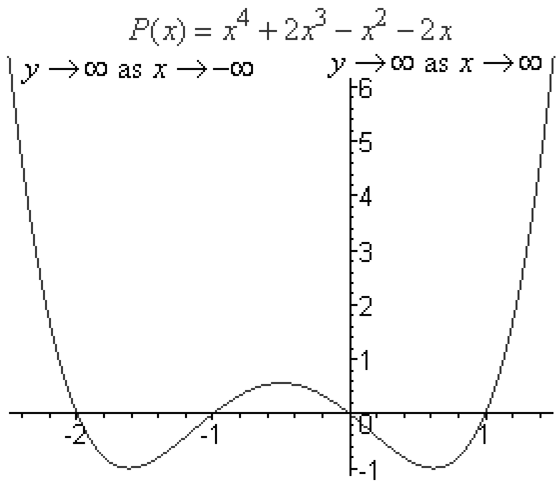
Describe the end behavior of the polynomial function $P(x) = x^4 + 2x^3 - x^2 - 2x$.

Solution:

The degree is 4 and the leading coefficient is 1.

Since the degree is even and the leading coefficient is positive, $y \to \infty$ as $x \to -\infty$ and $y \to \infty$ as $x \to \infty$.

The graph is shown below.



Additional Example 3:

Find the intercepts of the graph of the polynomial function $P(x) = x^3 - 3x^2 + 2x$.

Solution:

To find the y-intercept, let $x = 0$ to find $P(0)$.

$$P(x) = x^3 - 3x^2 + 2x$$
$$P(0) = 0^3 - 3(0)^2 + 2(0) = 0$$

Find the real zeros of the polynomial by solving the equation $P(x) = 0$ to find the x-intercepts of the graph.

$$P(x) = 0$$
$$x^3 - 3x^2 + 2x = 0$$
$$x\left(x^2 - 3x + 2\right) = 0$$
$$x(x-1)(x-2) = 0$$

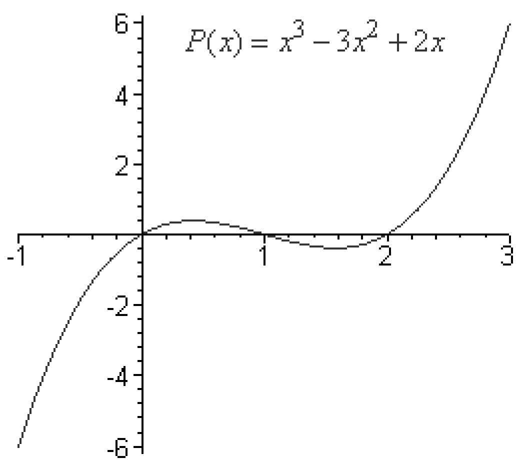
$x = 0$ or $x - 1 = 0$ or $x - 2 = 0$

$x - 1 + 1 = 0 + 1$ $\quad$ $x - 2 + 2 = 0 + 2$

$x = 1$ $\quad$ $x = 2$

The y-intercept is 0. The x-intercepts are 0, 1, and 2.

The graph is shown below.



Additional Example 4:

Sketch the graph of the polynomial function $P(x) = -x^3 + x^2 + 20x$.

Solution:

To find the y-intercept, let $x = 0$ to find $P(0)$.

$$P(x) = -x^3 + x^2 + 20x$$
$$P(0) = -(0)^3 + 0^2 + 20(0) = 0$$

Find the real zeros of the polynomial by solving the equation $P(x) = 0$ to find the x-intercepts of the graph.

$$P(x) = 0$$
$$-x^3 + x^2 + 20x = 0$$
$$-x\left(x^2 - x - 20\right) = 0$$
$$-x(x-5)(x+4) = 0$$

$-x = 0$ or $x - 5 = 0$ or $x + 4 = 0$

$x = 0$ $\quad$ $x - 5 + 5 = 0 + 5$ $\quad$ $x + 4 - 4 = 0 - 4$

$x = 5$ $\quad$ $x = -4$

The x-intercepts are $0, 5,$ and -4. The graph crosses the x-axis at each of the x-intercepts since the exponent on each of the factors in the factorization is 1.

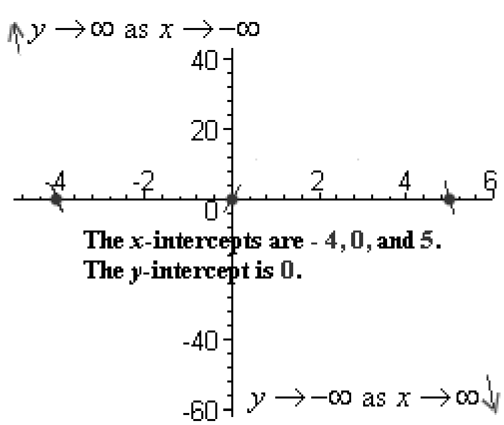
Near the x-intercepts 0, 5, and -4, the graph has the same general shape as the graphs of $y = Ax$, $y = A(x-5)$, and $y = A(x+4)$, respectively.

Determine the number of turning points and describe the end behavior by noting that the degree is 3 (odd) and the leading coefficient is -1 (negative).

There can be at most $3 - 1 = 2$ turning points.

$y \to \infty$ as $x \to -\infty$
$y \to -\infty$ as $x \to \infty$

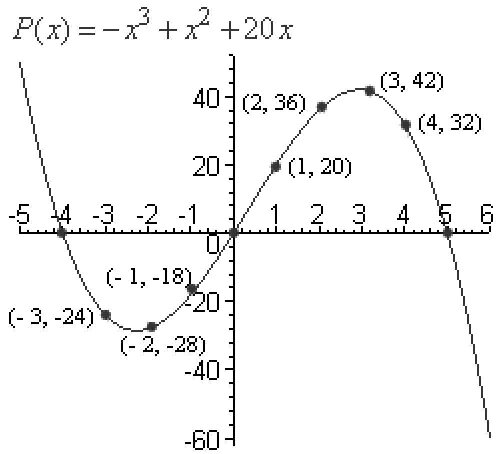
Show the end behavior, the intercepts, and the behavior at the x-intercepts on a graph.



Make a table of values to find additional points on the graph.

x	$P(x)$	$(x, P(x))$
-3	$P(-3) = -(-3)^3 + (-3)^2 + 20(-3) = -24$	$(-3, -24)$
-2	$P(-2) = -(-2)^3 + (-2)^2 + 20(-2) = -28$	$(-2, -28)$
-1	$P(-1) = -(-1)^3 + (-1)^2 + 20(-1) = -18$	$(-1, -18)$
1	$P(1) = -(1)^3 + (1)^2 + 20(1) = 20$	$(1, 20)$
2	$P(2) = -(2)^3 + (2)^2 + 20(2) = 36$	$(2, 36)$
3	$P(3) = -(3)^3 + (3)^2 + 20(3) = 42$	$(3, 42)$
4	$P(4) = -(4)^3 + (4)^2 + 20(4) = 32$	$(4, 32)$

Sketch the graph. Connect all points with a smooth curve.



Additional Example 5:

Sketch the graph of the polynomial function $P(x) = (x+2)^2(x-2)^2$.

Solution:

To find the y-intercept, let $x = 0$ to find $P(0)$.

$$P(x) = (x+2)^2(x-2)^2$$
$$P(0) = (0+2)^2(0-2)^2 = 16$$

The polynomial is given in factored form. The real zeros of the polynomial are the x-intercepts of the graph. Since $x+2$ and $x-2$ appear in the factorization of the polynomial, -2 and 2 are the x-intercepts.

The graph does not cross the x-axis at each of the x-intercepts since the exponent on the factors $x+2$ and $x-2$ in the factorization is 2 (even).

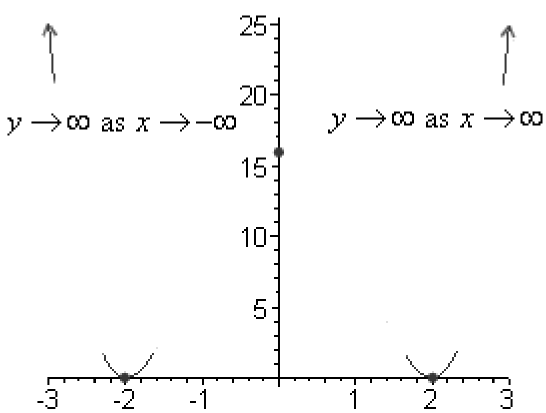
Near the x-intercepts -2 and 2 the graph has the same general shape as the graphs of $y = A(x+2)^2$ and $y = A(x-2)^2$, respectively.

Determine the number of turning points and describe the end behavior by noting that the degree is 4 (even) and the leading coefficient is 1 (positive).

There can be at most $4-1=3$ turning points.

$y \to \infty$ as $x \to -\infty$
$y \to \infty$ as $x \to \infty$

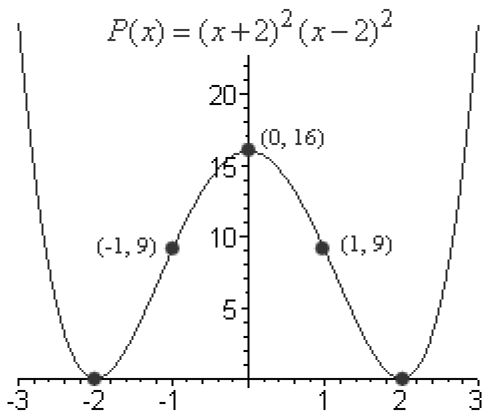
Show the end behavior, the intercepts, and the behavior at the x-intercepts on a graph.



Make a table of values to find additional points on the graph.

x	$P(x)$	$(x, P(x))$
-1	$P(-1) = (-1+2)^2(-1-2)^2 = 9$	$(-1, 9)$
1	$P(1) = (1+2)^2(1-2)^2 = 9$	$(1, 9)$

Sketch the graph. Connect all points with a smooth curve.



Exercise Set 2.2: Polynomial Functions

Answer the following.

(a) **State whether or not each of the following expressions is a polynomial. (Yes or No.)**

(b) **If the answer to part (a) is yes, then state the degree of the polynomial.**

(c) **If the answer to part (a) is yes, then classify the polynomial as a monomial, binomial, trinomial, or none of these. (Polynomials of four or more terms are not generally given specific names.)**

1. $4+3x^3$

2. $6x^5+3x^3+\dfrac{8}{x}$

3. $3x-5$

4. $2x^3+4x^2-7x-4$

5. $\dfrac{5x^3-6x^2+7}{x^2-4x+5}$

6. 8

7. $\frac{7}{2}x^2-\frac{5}{3}x+9$

8. $\dfrac{7}{x^3}+\dfrac{5}{x^2}-\dfrac{3}{x}-2$

9. $3^{-1}x^4-7^{-1}x+2$

10. $-9x^{1/4}+2x^{1/3}-4x^{1/2}$

11. $\left| x^2-3x+1 \right|$

12. $-\frac{3}{2}x^6$

13. $\dfrac{6x^3+8x^2}{x}$

14. $-3+5x^2+6x^4-3x^9$

15. $3a^3b^4-2a^2b^2$

16. $-4x^5y^{-2}-3x^{-4}y^9$

17. $4x^5y^3+\dfrac{3}{xy^2}$

18. $\frac{2}{5}x^2y^9z+3xy-\frac{1}{4}x^3y^4z^2$

19. $-4xyz^3-\frac{2}{5}y^7-\frac{3}{7}x^4y^3z^2$

20. $-a^7+2a^3b^5+b^6-3a^2b^4$

Answer True or False.

21. **(a)** $7x-2x^3$ is a trinomial.
(b) $7x-2x^3$ is a third degree polynomial.
(c) $7x-2x^3$ is a binomial.
(d) $7x-2x^3$ is a first degree polynomial

22. **(a)** $x^2-4x+7x^3$ is a second degree polynomial.
(b) $x^2-4x+7x^3$ is a binomial.
(c) $x^2-4x+7x^3$ is a third degree polynomial.
(d) $x^2-4x+7x^3$ is a trinomial.

23. **(a)** $3x^7-2x^4y^6-3y^8$ is a tenth degree polynomial.
(b) $3x^7-2x^4y^6-3y^8$ is a binomial.
(c) $3x^7-2x^4y^6-3y^8$ is an eighth degree polynomial.
(d) $3x^7-2x^4y^6-3y^8$ is a trinomial.

24. **(a)** $-3a^4b^5$ is a fifth degree polynomial.
(b) $-3a^4b^5$ is a trinomial.
(c) $-3a^4b^5$ is a ninth degree polynomial.
(d) $-3a^4b^5$ is a monomial.

Sketch a graph of each of the following functions.

25. $P(x)=x^3$

26. $P(x)=x^4$

27. $P(x)=x^6$

28. $P(x)=x^5$

29. $P(x)=x^n$, where n is odd and $n>0$.

30. $P(x)=x^n$, where n is even and $n>0$.

Answer the following.

31. The graph of $P(x)=(x-1)(x-2)^3(x+4)^2$ has x-intercepts at $x=1, x=2,$ and $x=-4$.

(a) At and immediately surrounding the point $x=2$, the graph resembles the graph of what familiar function? (Choose one.)

$y=x$ $\quad$ $y=x^2$ $\quad$ $y=x^3$

Continued on the next page…

(b) At and immediately surrounding the point $x = -4$, the graph resembles the graph of what familiar function? (Choose one.)

$y = x$ $\qquad$ $y = x^2$ $\qquad$ $y = x^3$

(c) If $P(x)$ were to be multiplied out completely, the leading term of the polynomial would be: (Choose one; do not actually multiply out the polynomial.)

$x^3;\ -x^3;\ x^4;\ -x^4;\ x^5;\ -x^5;\ x^6;\ -x^6$

32. The graph of $Q(x) = -(x+3)^2(x-5)^3$ has x-intercepts at $x = -3$ and $x = 5$.

(a) At and immediately surrounding the point $x = -3$, the graph resembles the graph of what familiar function? (Choose one.)

$y = x$ $\qquad$ $y = x^2$ $\qquad$ $y = x^3$

(b) At and immediately surrounding the point $x = 5$, the graph resembles the graph of what familiar function? (Choose one.)

$y = x$ $\qquad$ $y = x^2$ $\qquad$ $y = x^3$

(c) If $P(x)$ were to be multiplied out completely, the leading term of the polynomial would be: (Choose one; do not actually multiply out the polynomial.)

$x^3;\ -x^3;\ x^4;\ -x^4;\ x^5;\ -x^5;\ x^6;\ -x^6$

Match each of the polynomial functions below with its graph. (The graphs are shown in the next column.)

33. $P(x) = (x-2)(x+1)(x+4)$

34. $Q(x) = -(x+2)(x-1)(x-4)$

35. $R(x) = -(x-2)(x+1)^2(x+4)^2$

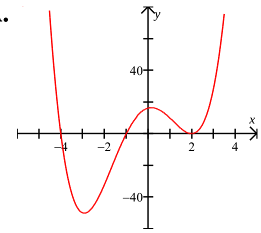
36. $S(x) = (x-2)^2(x+1)(x+4)$

37. $U(x) = (x+2)^2(x-1)^3(x-4)$

38. $V(x) = -(x+2)^3(x-1)^3(x-4)^2$

Choices for 33-38:

A.



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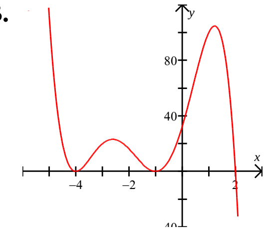

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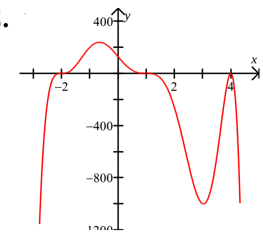

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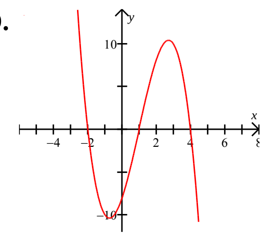

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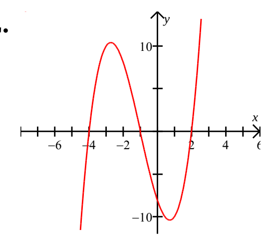

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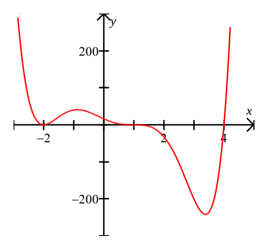

For each of the functions below:

(h) **Find the x- and y-intercepts.**

(i) **Sketch the graph of the function. Be sure to show all x- and y-intercepts, along with the proper behavior at each x-intercept, as well as the proper end behavior.**

39. $P(x) = (x-5)(x+3)$

40. $P(x) = -(x-3)(x+1)$

41. $P(x) = -(x-6)^2$

42. $P(x) = (x+3)^2$

43. $P(x) = (x-5)(x+2)(x+6)$

44. $P(x) = 3x(x-4)(x-7)$

45. $P(x) = -\frac{1}{2}(x-4)(x-1)(x+3)$

46. $P(x) = -(x+6)(x-2)(x-5)$

Exercise Set 2.2: Polynomial Functions

47. $P(x)=(x+2)^2(x-4)$

48. $P(x)=(5-x)(x+3)^2$

49. $P(x)=(3x-2)(x+4)(x-5)(x+1)$

50. $P(x)=-\frac{1}{3}(x+5)(x+1)(x+3)(x-2)$

51. $P(x)=x(x+2)(4-x)(x+6)$

52. $P(x)=(x-1)(x-3)(x+2)(x+5)$

53. $P(x)=(x-3)^2(x+4)^2$

54. $P(x)=-x(2x-5)^3$

55. $P(x)=(x+5)^3(x-4)$

56. $P(x)=x^2(x-6)^2$

57. $P(x)=(x+3)^2(x-4)^3$

58. $P(x)=-2x(3-x)^3(x+1)$

59. $P(x)=-x(x-2)^2(x+3)^2(x-4)$

60. $P(x)=(x-5)^3(x-2)^2(x+1)$

61. $P(x)=x^8(x-1)^6(x+1)^7$

62. $P(x)=-x^3(x+1)^4(x-1)^7$

63. $P(x)=x^3-6x^2+8x$

64. $P(x)=x^3-2x^2-15x$

65. $P(x)=25x-x^3$

66. $P(x)=-3x^3-5x^2+2x$

67. $P(x)=-x^4+x^3+12x^2$

68. $P(x)=x^4-16x^2$

69. $P(x)=x^5-9x^3$

70. $P(x)=-x^5-3x^4+18x^3$

71. $P(x)=x^3+4x^2-x-4$

72. $P(x)=x^3-5x^2-4x+20$

73. $P(x)=x^4-13x^2+36$

74. $P(x)=x^4-17x^2+16$

Polynomial functions can be classified according to their degree, as shown below. (Linear and quadratic functions have been covered in previous sections.)

Degree	Name
0 or 1	Linear
2	Quadratic
3	Cubic
4	Quartic
5	Quintic
n	n^{th} degree polynomial

Answer the following.

75. Write the equation of the cubic polynomial $P(x)$ that satisfies the following conditions: $P(-4)=P(1)=P(3)=0$, and $P(0)=-6$.

76. Write an equation for a cubic polynomial $P(x)$ with leading coefficient -1 whose graph passes through the point $(2, 8)$ and is tangent to the x-axis at the origin.

77. Write the equation of the quartic polynomial with y-intercept 12 whose graph is tangent to the x-axis at $(-2, 0)$ and $(1, 0)$.

78. Write the equation of the sixth degree polynomial with y-intercept -3 whose graph is tangent to the x-axis at $(-2, 0)$, $(-1, 0)$, and $(3, 0)$.

Use transformations (the concepts of shifting, reflecting, stretching, and shrinking) to sketch each of the following graphs.

79. $P(x)=x^3+5$

80. $P(x)=-x^3-2$

81. $P(x)=-(x-2)^3+4$

82. $P(x)=(x+5)^3-1$

83. $P(x)=2x^4-3$

84. $P(x)=-(x-2)^4+5$

85. $P(x)=-(x+1)^5-4$

86. $P(x)=(x+3)^5+2$