Section 1.5: Inverse Functions

- Inverses of One-to-One Functions

Inverses of One-to-One Functions

Definition of a One-to-One Function:

A function f is said to be one-to-one provided that the following holds for all x_1 and x_2 in the domain of f:

$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

Example:

Let $f(x) = 3x + 7$. Use the definition to show that f is one-to-one.

Solution:

To show that f is one-to-one, we begin by assuming that $f(x_1) = f(x_2)$. We must then show that $x_1 = x_2$.

Assume that $f(x_1) = f(x_2)$. Then:

$$\begin{aligned} 3x_1 + 7 &= 3x_2 + 7 \\ 3x_1 + 7 - 7 &= 3x_2 + 7 - 7 \\ 3x_1 &= 3x_2 \\ \frac{\cancel{3}x_1}{\cancel{3}} &= \frac{\cancel{3}x_2}{\cancel{3}} \\ x_1 &= x_2 \end{aligned}$$

This shows, by definition, that f is one-to-one.

Example:

Let $f(x) = x^2$. Show that f is not one-to-one.

Solution:

To show that f is not one-to-one, we need to give a specific example to show that the condition $f(x_1) = f(x_2)$ fails to imply that $x_1 = x_2$.

Using $x_1 = -1$ and $x_2 = 1$, we have:

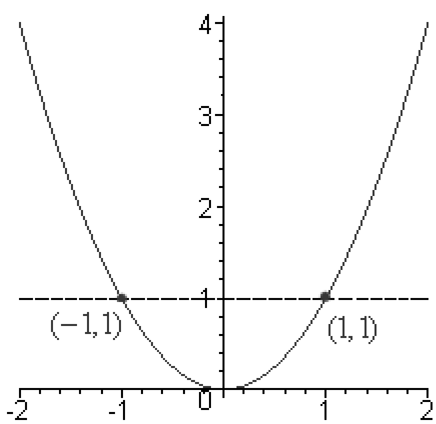
$$f(-1) = (-1)^2 = 1 \qquad \text{and } f(1) = 1^2 = 1.$$

Thus, $f(-1) = f(1)$, but $-1 \neq 1$. This shows that f is not one-to-one.

Horizontal Line Test:

A function is one-to-one if no horizontal line intersects its graph more than once.

To see why the horizontal line test is valid, the figure below shows the graph of the function $f(x) = x^2$. From the example above, we know that f is not one-to-one. Note that the horizontal line shown on the graph intersects the graph in two points: $(-1,1)$ and $(1,1)$. This shows that $f(-1) = f(1)$ even though $-1 \neq 1$.

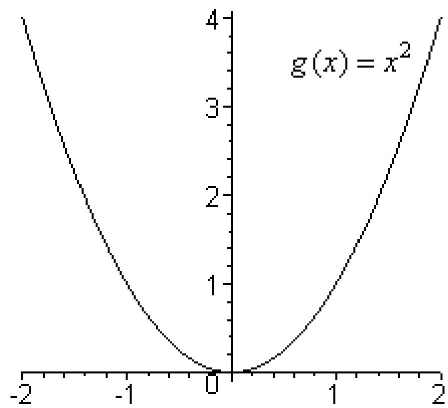


Example:

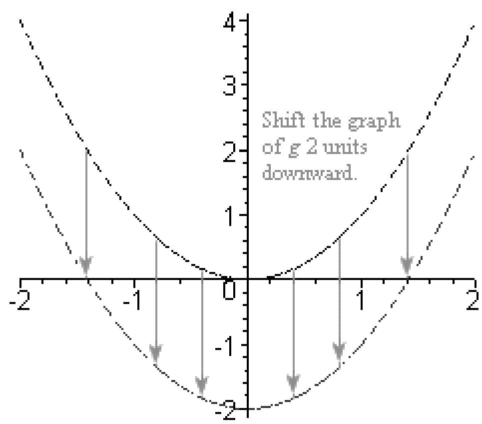
Determine whether the function $f(x) = x^2 - 2$ is one-to-one.

Solution:

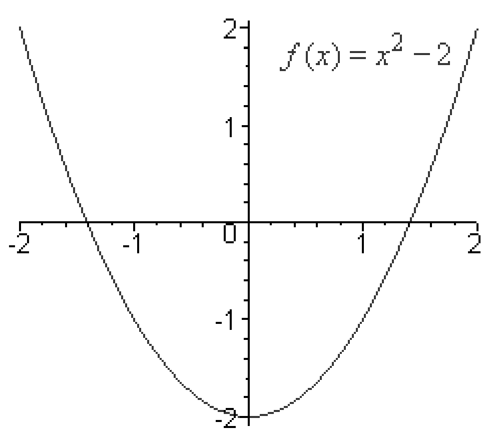
Begin with the graph of $g(x) = x^2$ shown below.



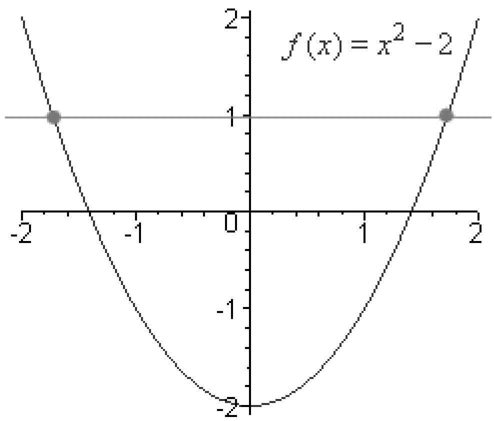
To graph the function $f(x) = x^2 - 2$, shift the graph of g 2 units downward.



The graph of the given function is shown below.



Now use the Horizontal Line Test.



Since we can find a horizontal line that intersects the graph more than once, by the Horizontal Line Test, f is not one-to-one.

The Inverse of a One-to-One Function:

If f is a one-to-one function with domain A and range B, then there is a one-to-one function g, the inverse of f, with domain B and range A such that

$$f(g(x)) = x \text{ for each } x \text{ in } B$$

and

$$g(f(x)) = x \text{ for each } x \text{ in } A.$$

The functions f and g are called inverses of each other.

Example:

Verify that the functions $f(x) = 2x - 5$ and $g(x) = \dfrac{x+5}{2}$ are inverse functions.

Solution:

To verify that the given functions are inverses of each other, we must show that $f(g(x)) = x$ and $g(f(x)) = x$.

$$f(g(x)) = f\left(\frac{x+5}{2}\right) \qquad \left[g(x) = \frac{x+5}{2}\right]$$
$$= 2\left(\frac{x+5}{2}\right) - 5 \qquad [f(x) = 2x-5]$$
$$= x+5-5$$
$$= x$$

$$g(f(x)) = g(2x-5) \qquad [f(x) = 2x-5]$$
$$= \frac{2x-5+5}{2} \qquad \left[g(x) = \frac{x+5}{2}\right]$$
$$= \frac{2x}{2}$$
$$= x$$

A Technique for Finding the Inverse of a One-to-One Function:

If f is a one-to-one function, then its inverse function is denoted by f^{-1}.

Now, suppose that f is a one-to-one function defined by $y = f(x)$. If $f(a) = b$, then $f^{-1}(b) = a$; that is, f^{-1} reverses the correspondence of f.

To find f^{-1}, we can interchange the variables x and y in the equation $y = f(x)$. The equation that results, $x = f(y)$, defines the inverse function. If we can solve this equation for y, we can then express the inverse as the equation $y = f^{-1}(x)$.

Example:

The function $f(x) = 2x+3$ is one-to-one. Find its inverse function.

Solution:

$f(x) = 2x+3$

$y = 2x+3$ Step 1: Write $y = f(x)$.

$x = 2y+3$ Step 2: Interchange x and y.

$x-3 = 2y+3-3$ Step 3: Solve for y.

$x-3 = 2y$

$$\frac{x-3}{2} = \frac{\not{2}y}{\not{2}}$$

$x-3 = y$

$f^{-1}(x) = \dfrac{x-3}{2}$ Step 4: Write $y = f^{-1}(x)$.

To verify that $f^{-1}(x) = \dfrac{x-3}{2}$ is the inverse of $f(x) = 2x+3$:

$$f(f^{-1}(x)) = f\left(\frac{x-3}{2}\right) = \not{2}\left(\frac{x-3}{\not{2}}\right)+3 = x-3+3 = x$$

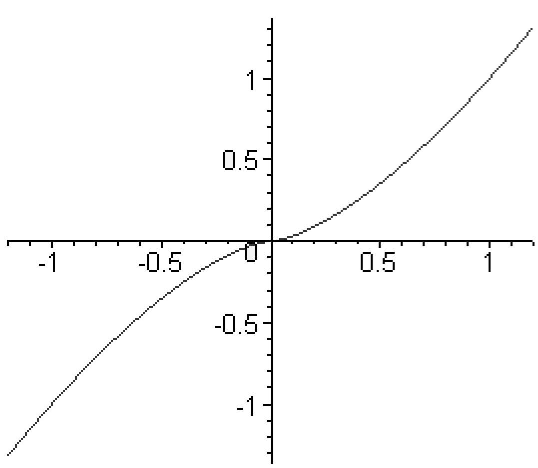
$$f^{-1}(f(x)) = f^{-1}(2x+3) = \frac{2x+3-3}{2} = \frac{\not{2}x}{\not{2}} = x$$

The Graphs of f and f^{-1}:

If the point (a,b) lies on the graph of a one-to one function f, then the point (b,a) lies on the graph of its inverse function f^{-1}. These points are mirror images in the line $y = x$. Thus, to find the graph of f^{-1}, reflect the graph of f in the line $y = x$.

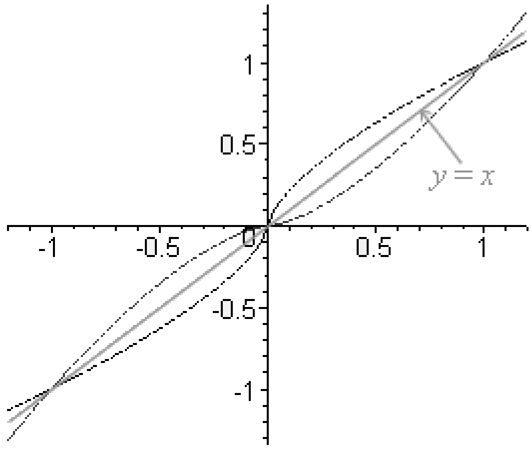
Example:

The graph of a one-to-one function $y = f(x)$ is shown below. Sketch the graph of f^{-1}.

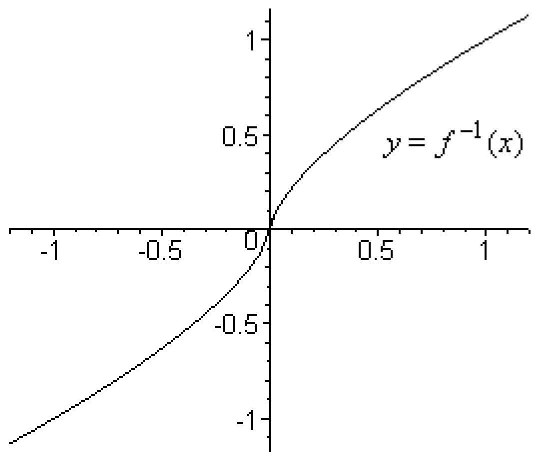


Solution:

The graph of f^{-1} is found by reflecting the graph of f in the line $y = x$.

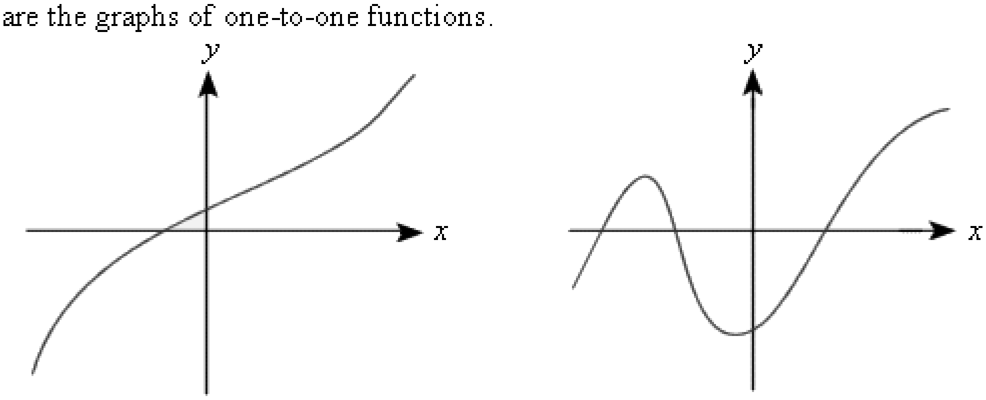


The graph of the inverse function is shown below.



Additional Example 1:

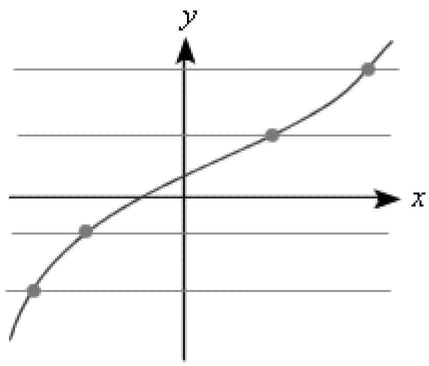
The graphs of two functions are shown below. State whether or not the graphs are the graphs of one-to-one functions.



(a) (b)

Solution:

Use the Horizontal Line Test for the graph in part (a).

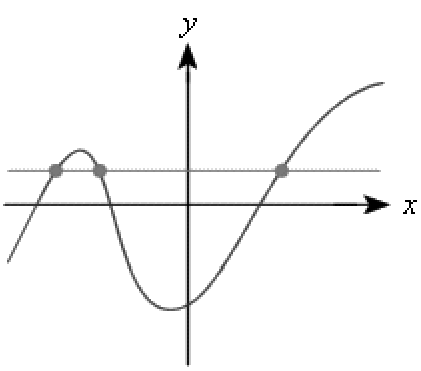


(a)

It is easy to see (from the 4 example lines shown) that no horizontal line intersects the graph more than once.

The first graph is the graph of a one-to-one function.

Use the Horizontal Line Test for the graph in part (b).



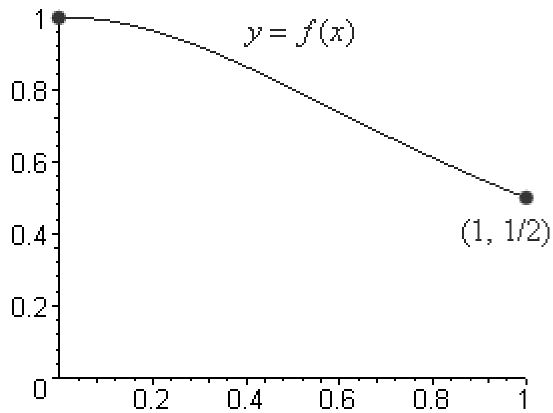
(b)

We can find a horizontal line that intersects the graph more than once.

The second graph is not the graph of a one-to-one function.

Additional Example 2:

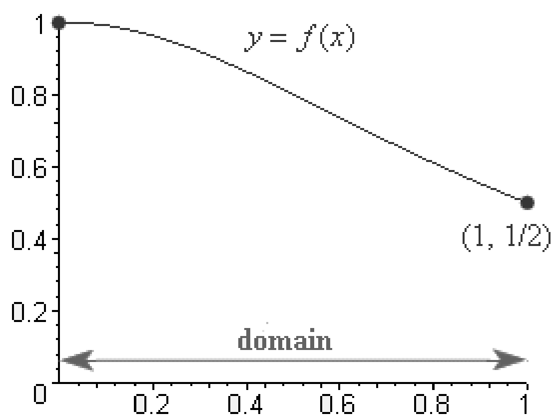
The graph of a one-to-one function $y = f(x)$ is shown below. (a) Give the domain and range of f. (b) Sketch the graph of the inverse function f^{-1}. (c) Give the domain and range of f^{-1}.



Solution:

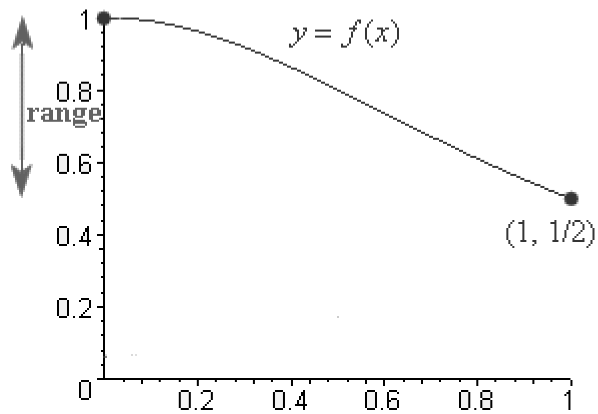
Part (a):

Find the domain of f by inspecting its graph.



The domain of f is $[0,1]$.

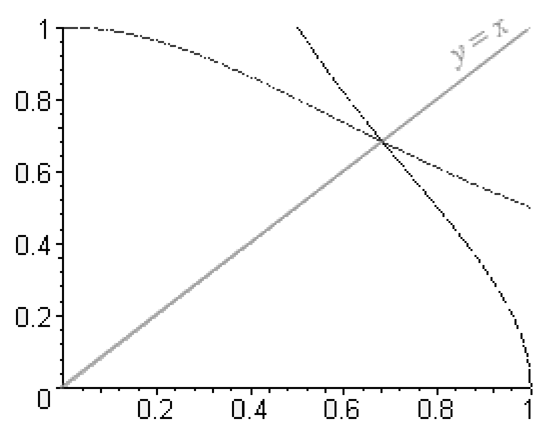
Find the range of f by inspecting its graph.



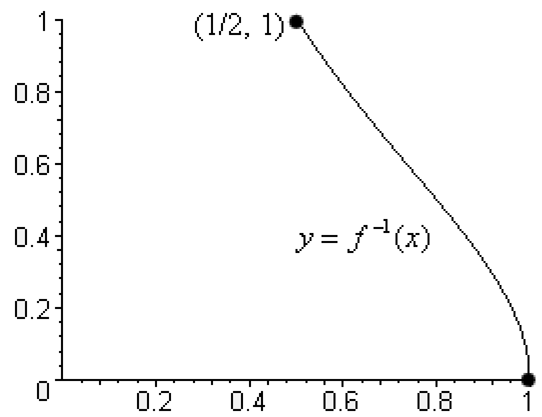
The range of f is $\left[\frac{1}{2}, 1\right]$.

Part (b):

The graph of f^{-1} can be sketched by reflecting the graph of f in the line $y = x$.

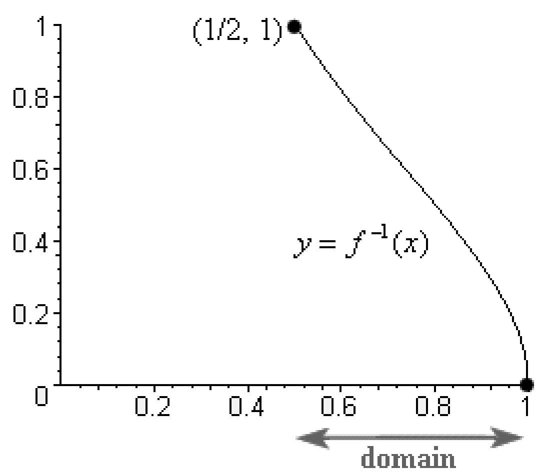


The graph of f^{-1} is shown below.



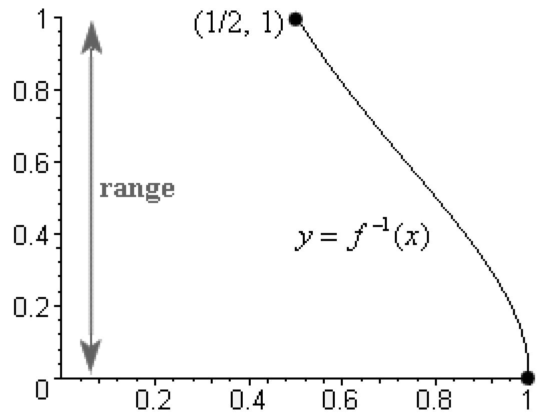
Part (c):

Find the domain of f^{-1} by inspecting its graph.



The domain of f^{-1} is $\left[\frac{1}{2},1\right]$, which is equal to the range of f.

Find the range of f^{-1} by inspecting its graph.



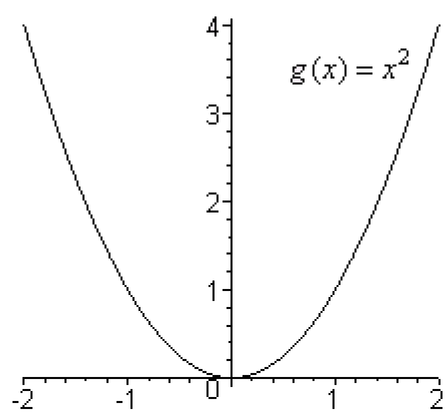
The range of f^{-1} is $[0,1]$, which is equal to the domain of f.

Additional Example 3:

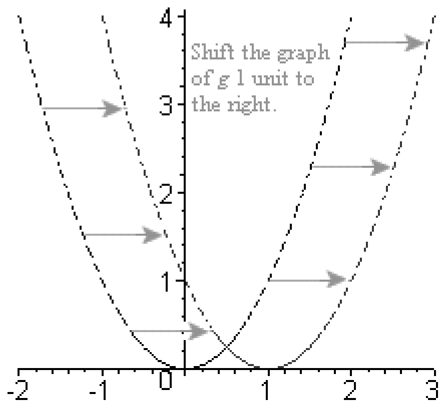
Determine whether or not the function $f(x) = (x-1)^2$ is one-to-one.

Solution:

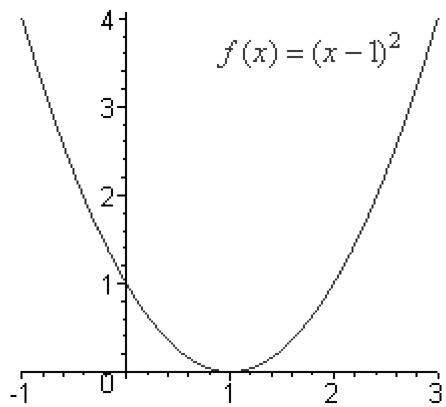
Sketch the graph of f by using transformations. Begin with the graph of $g(x) = x^2$ shown below.



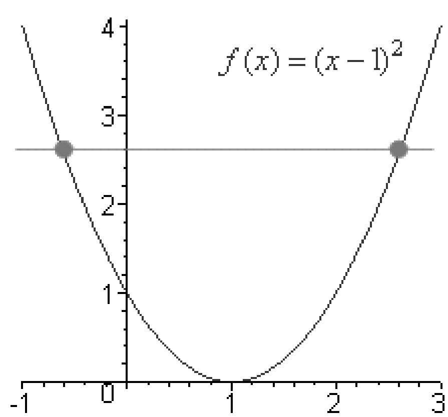
To graph the function $f(x) = (x-1)^2$, shift the graph of g 1 unit to the right.



The graph of f is shown below.



Now use the Horizontal Line Test.



We can find a horizontal line that intersects the graph of f more than once.

f is not a one-to-one function.

Additional Example 4:

The function $f(x) = 8x + 3$ is one-to-one. Find the inverse function f^{-1}.

Solution:

Step 1: Write $y = f(x)$.

$y = 8x + 3$

Step 2: Interchange x and y.

$x = 8y + 3$

Step 3: Solve the equation $x = 8y + 3$ for y.

$$x = 8y + 3$$
$$x - 3 = 8y + 3 - 3$$
$$x - 3 = 8y$$
$$\frac{x-3}{8} = \frac{\cancel{8}y}{\cancel{8}}$$
$$\frac{x-3}{8} = y$$

Step 4: Write $y = f^{-1}(x)$.

$$\frac{x-3}{8} = f^{-1}(x)$$

The inverse function is $f^{-1}(x) = \frac{x-3}{8}$.

Additional Example 5:

Determine whether or not the functions $f(x) = 5x^3$ and $g(x) = \sqrt[3]{\frac{x}{5}}$ are inverses of each other.

Solution:

Find $f \circ g$.

$$\begin{aligned}(f \circ g)(x) &= f\left(g(x)\right)\\ &= f\left(\sqrt[3]{\frac{x}{5}}\right)\\ &= 5\left(\sqrt[3]{\frac{x}{5}}\right)^3\\ &= 5\left(\frac{x}{5}\right)\\ &= x\end{aligned}$$

Find $g \circ f$.

$$\begin{aligned}(g \circ f)(x) &= g\left(f(x)\right)\\ &= g\left(5x^3\right)\\ &= \sqrt[3]{\frac{5x^3}{5}}\\ &= \sqrt[3]{x^3}\\ &= x\end{aligned}$$

The functions f and g are inverses of each other since $f\left(g(x)\right) = x$ and $g\left(f(x)\right) = x$.

Additional Example 6:

Assume that f and g are inverse functions. Answer the following.

(a) If $f(3) = 5$, find $g(5)$; (b) If $g(-3) = -1$, find $f(-1)$.

Solution:

Part (a):

Since f and g are inverse functions, we have $g\left(f(x)\right) = x$.

$g\left(f(3)\right) = 3$

Now substitute the given information that $f(3) = 5$.

$g\left(5\right) = 3$

Part (b):

Since f and g are inverse functions, we have $f\left(g(x)\right) = x$.

$f\left(g(-3)\right) = -3$

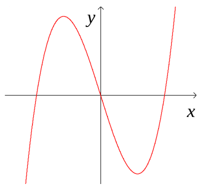
Now substitute the given information that $g(-3) = -1$.

$f(-1) = -3$

Exercise Set 1.5: Inverse Functions

Determine whether each of the following graphs represents a one-to-one function. Explain your answer.

1.



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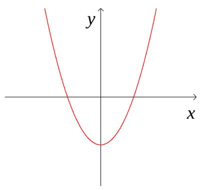

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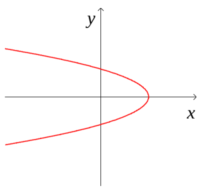

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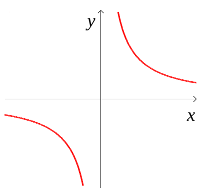

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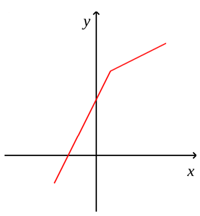
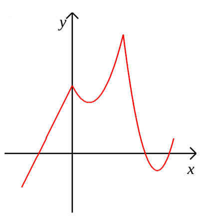


6.



For each of the following functions, sketch a graph and then determine whether the function is one-to-one.

7. $f(x)=2x-3$

8. $g(x)=x^2+5$

9. $h(x)=(x-2)^3$

10. $f(x)=x^3-2$

11. $g(x)=|x|+4$

12. $h(x)=\frac{1}{x}-3$

13. $f(x)=-(x-2)^2+1$

14. $g(x)=|x-6|$

Answer the following.

15. If a function f is one-to-one, then the inverse function, f^{-1}, can be graphed by either of the following methods:

(a) Interchange the ____ and ____ values.

(b) Reflect the graph of f over the line $y=$ ____.

16. The domain of f is equal to the __________ of f^{-1}, and the range of f is equal to the __________ of f^{-1}.

A table of values for a one-to-one function $y=f(x)$ is given. Complete the table for $y=f^{-1}(x)$.

17.

x	$f(x)$
3	–4
2	7
–4	5
5	0
0	3

x	$f^{-1}(x)$
–4	
	2
5	
0	

18.

x	$f(x)$
5	9
4	5
6	–3
–8	2
2	6

x	$f^{-1}(x)$
	5
5	
	6
	–8

Exercise Set 1.5: Inverse Functions

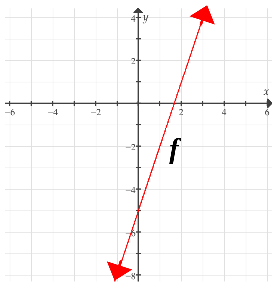
For each of the following graphs:

(a) State the domain and range of f .

(b) Sketch f^{-1} .

(c) State the domain and range of f^{-1} .

19.



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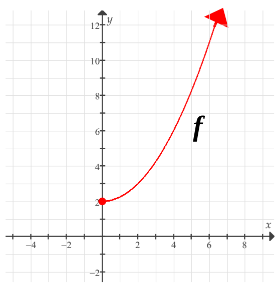

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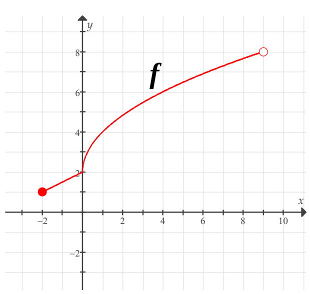

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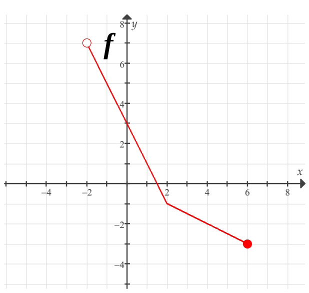

Answer the following. Assume that f is a one-to-one function.

23. If $f(4)=5$, find $f^{-1}(5)$.

24. If $f(6)=-2$, find $f^{-1}(-2)$.

25. If $f^{-1}(-3)=7$, find $f(7)$.

26. If $f^{-1}(6)=-8$, find $f(-8)$.

27. If $f(3)=9$ and $f(9)=5$, find $f^{-1}(9)$.

28. If $f(5)=-4$ and $f(2)=5$, find $f^{-1}(5)$.

29. If $f(-4)=2$, find $f\left(f^{-1}(2)\right)$.

30. If $f^{-1}(-5)=3$, find $f^{-1}\left(f(3)\right)$.

Answer the following. Assume that f and g are defined for all real numbers.

31. If f and g are inverse functions, $f(-2)=3$ and $f(4)=-2$, find $g(-2)$.

32. If f and g are inverse functions, $f(7)=10$ and $f(10)=-1$, find $g(10)$.

33. If f and g are inverse functions, $f(5)=8$ and $f(9)=3$, find $g\left(f(3)\right)$.

34. If f and g are inverse functions, $f(-1)=6$ and $f(7)=8$, find $f\left(g(6)\right)$.

For each of the following functions, write an equation for the inverse function $y=f^{-1}(x)$.

35. $f(x)=5x-3$

36. $f(x)=-4x+7$

37. $f(x)=\dfrac{3-2x}{8}$

38. $f(x)=\dfrac{6x-5}{4}$

39. $f(x)=x^2+1$, where $x\geq 0$

40. $f(x)=5-x^2$, where $x\geq 0$

41. $f(x)=4x^3-7$

42. $f(x)=2x^3+1$

43. $f(x)=\dfrac{3}{x+2}$

44. $f(x)=\dfrac{5}{7-x}$

45. $f(x)=\dfrac{2x+3}{x-4}$

46. $f(x)=\dfrac{3-8x}{x+5}$

47. $f(x)=\sqrt{7-2x}$

48. $f(x)=2+\sqrt{6x+5}$

Use the Property of Inverse Functions to determine whether each of the following pairs of functions are inverses of each other. Explain your answer.

49. $f(x)=4x-1$; $g(x)=\frac{1}{4}x+1$

50. $f(x)=2+3x$; $g(x)=\dfrac{x-2}{3}$

51. $f(x)=\dfrac{4-x}{5}$; $g(x)=4-5x$

52. $f(x)=2x+5$; $g(x)=\dfrac{1}{2x+5}$

53. $f(x)=x^3-2$; $g(x)=\sqrt[3]{x+2}$

54. $f(x)=\sqrt[5]{x}-7$; $g(x)=(x+7)^5$

55. $f(x)=\dfrac{5}{x}$; $g(x)=\dfrac{5}{x}$

56. $f(x)=x^2+9$, where $x\geq 0$;
$g(x)=\sqrt{x-9}$

Answer the following.

57. If $f(x)$ is a function that represents the amount of revenue (in dollars) by selling *x* tickets, then what does $f^{-1}(500)$ represent?

58. If $f(x)$ is a function that represents the area of a circle with radius *x*, then what does $f^{-1}(80)$ represent?

A function is said to be one-to-one provided that the following holds for all x_1 and x_2 in the domain of f:

If $f(x_1)=f(x_2)$, then $x_1=x_2$.

Use the above definition to determine whether or not the following functions are one-to-one. If f is not one-to-one, then give a specific example showing that the condition $f(x_1)=f(x_2)$ fails to imply that $x_1=x_2$.

59. $f(x)=5x-3$

60. $f(x)=x^3+5$

61. $f(x)=\sqrt{x}-4$

62. $f(x)=|x|-4$

63. $f(x)=|x-4|$

64. $f(x)=\dfrac{1}{x}+4$

65. $f(x)=x^2+3$

66. $f(x)=(x+3)^2$