Section 1.5: Inverse Functions

➢ Inverses of One-to-One Functions

Inverses of One-to-One Functions

Definition of a One-to-One Function:

A function \( f \) is said to be one-to-one provided that the following holds for all \( x_1 \) and \( x_2 \) in the domain of \( f \):

\[ \text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2. \]

Example:

Let \( f(x) = 3x + 7 \). Use the definition to show that \( f \) is one-to-one.

Solution:

To show that \( f \) is one-to-one, we begin by assuming that \( f(x_1) = f(x_2) \). We must then show that \( x_1 = x_2 \).

Assume that \( f(x_1) = f(x_2) \). Then:

\[
\begin{align*}
3x_1 + 7 &= 3x_2 + 7 \\
3x_1 + 7 - 7 &= 3x_2 + 7 - 7 \\
3x_1 &= 3x_2 \\
\frac{3x_1}{3} &= \frac{3x_2}{3} \\
x_1 &= x_2
\end{align*}
\]

This shows, by definition, that \( f \) is one-to-one.
Example:

Let \( f(x) = x^2 \). Show that \( f \) is not one-to-one.

Solution:

To show that \( f \) is not one-to-one, we need to give a specific example to show that the condition \( f(x_1) = f(x_2) \) fails to imply that \( x_1 = x_2 \).

Using \( x_1 = -1 \) and \( x_2 = 1 \), we have:

\[
f(-1) = (-1)^2 = 1 \quad \text{and} \quad f(1) = 1^2 = 1.
\]

Thus, \( f(-1) = f(1) \), but \(-1 \neq 1\). This shows that \( f \) is not one-to-one.

**Horizontal Line Test:**

A function is one-to-one if no horizontal line intersects its graph more than once.

To see why the horizontal line test is valid, the figure below shows the graph of the function \( f(x) = x^2 \). From the example above, we know that \( f \) is not one-to-one. Note that the horizontal line shown on the graph intersects the graph in two points: \((-1, 1)\) and \((1, 1)\). This shows that \( f(-1) = f(1) \) even though \(-1 \neq 1\).

![Graph of f(x) = x^2](image)

Example:

Determine whether the function \( f(x) = x^2 - 2 \) is one-to-one.
Solution:
Begin with the graph of \( g(x) = x^2 \) shown below.

To graph the function \( f(x) = x^2 - 2 \), shift the graph of \( g \) 2 units downward.

The graph of the given function is shown below.
Now use the Horizontal Line Test

\[ f(x) = x^2 - 2 \]

Since we can find a horizontal line that intersects the graph more than once, by the Horizontal Line Test, \( f \) is not one-to-one.

The Inverse of a One-to-One Function:

If \( f \) is a one-to-one function with domain \( A \) and range \( B \), then there is a one-to-one function \( g \), the inverse of \( f \), with domain \( B \) and range \( A \) such that

\[ f(g(x)) = x \text{ for each } x \text{ in } B \]

and

\[ g(f(x)) = x \text{ for each } x \text{ in } A. \]

The functions \( f \) and \( g \) are called inverses of each other.

Example:

Verify that the functions \( f(x) = 2x - 5 \) and \( g(x) = \frac{x+5}{2} \) are inverse functions.

Solution:

To verify that the given functions are inverses of each other, we must show that

\[ f(g(x)) = x \text{ and } g(f(x)) = x. \]
A Technique for Finding the Inverse of a One-to-One Function:

If $f$ is a one-to-one function, then its inverse function is denoted by $f^{-1}$.

Now, suppose that $f$ is a one-to-one function defined by $y = f(x)$. If $f(a) = b$, then $f^{-1}(b) = a$; that is, $f^{-1}$ reverses the correspondence of $f$.

To find $f^{-1}$, we can interchange the variables $x$ and $y$ in the equation $y = f(x)$. The equation that results, $x = f(y)$, defines the inverse function. If we can solve this equation for $y$, we can then express the inverse as the equation $y = f^{-1}(x)$.

Example:

The function $f(x) = 2x + 3$ is one-to-one. Find its inverse function.
Solution:

\[ f(x) = 2x + 3 \]

\[ y = 2x + 3 \quad \text{Step 1: Write } y = f(x). \]

\[ x = 2y + 3 \quad \text{Step 2: Interchange } x \text{ and } y. \]

\[ x - 3 = 2y + 3 - 3 \quad \text{Step 3: Solve for } y. \]

\[ x - 3 = 2y \]

\[ \frac{x - 3}{2} = \frac{2y}{2} \]

\[ x - 3 = y \]

\[ f^{-1}(x) = \frac{x - 3}{2} \quad \text{Step 4: Write } y = f^{-1}(x). \]

To verify that \( f^{-1}(x) = \frac{x - 3}{2} \) is the inverse of \( f(x) = 2x + 3 \):

\[ f(f^{-1}(x)) = f\left(\frac{x - 3}{2}\right) = 2\left(\frac{x - 3}{2}\right) + 3 = x - 3 + 3 = x \]

\[ f^{-1}(f(x)) = f^{-1}(2x + 3) = \frac{2x + 3 - 3}{2} = \frac{2x}{2} = x \]

The Graphs of \( f \) and \( f^{-1} \):

If the point \((a, b)\) lies on the graph of a one-to-one function \( f \), then the point \((b, a)\) lies on the graph of its inverse function \( f^{-1} \). These points are mirror images in the line \( y = x \). Thus, to find the graph of \( f^{-1} \), reflect the graph of \( f \) in the line \( y = x \).

Example:

The graph of a one-to-one function \( y = f(x) \) is shown below. Sketch the graph of \( f^{-1} \).
Solution:
The graph of \( f^{-1} \) is found by reflecting the graph of \( f \) in the line \( y = x \).

The graph of the inverse function is shown below.

Additional Example 1:
The graphs of two functions are shown below. State whether or not the graphs are the graphs of one-to-one functions.

(a)  
(b)
Solution:
Use the Horizontal Line Test for the graph in part (a).

It is easy to see (from the 4 example lines shown) that no horizontal line intersects the graph more than once.

The first graph is the graph of a one-to-one function.

Use the Horizontal Line Test for the graph in part (b).

We can find a horizontal line that intersects the graph more than once.

The second graph is not the graph of a one-to-one function.
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Additional Example 2:
The graph of a one-to-one function \( y = f(x) \) is shown below. (a) Give the domain and range of \( f \). (b) Sketch the graph of the inverse function \( f^{-1} \). (c) Give the domain and range of \( f^{-1} \).

Solution:
Part (a):

Find the domain of \( f \) by inspecting its graph.

The domain of \( f \) is \([0, 1]\).
Find the range of $f$ by inspecting its graph.

The range of $f$ is $\left[\frac{1}{2}, 1\right]$.

Part (b):

The graph of $f^{-1}$ can be sketched by reflecting the graph of $f$ in the line $y = x$. 
The graph of $f^{-1}$ is shown below.

\[ y = f^{-1}(x) \]

Part (c):

Find the domain of $f^{-1}$ by inspecting its graph.

The domain of $f^{-1}$ is $\left(\frac{1}{2}, 1\right]$, which is equal to the range of $f$. 
Find the range of $f^{-1}$ by inspecting its graph.

The range of $f^{-1}$ is $[0, 1]$, which is equal to the domain of $f$.

**Additional Example 3:**

Determine whether or not the function $f(x) = (x - 1)^2$ is one-to-one.

**Solution:**

Sketch the graph of $f$ by using transformations. Begin with the graph of $g(x) = x^2$ shown below.
To graph the function \( f(x) = (x - 1)^2 \), shift the graph of \( g \) 1 unit to the right.

The graph of \( f \) is shown below.

Now use the Horizontal Line Test.

We can find a horizontal line that intersects the graph of \( f \) more than once.

\( f \) is not a one-to-one function.
Additional Example 4:

The function \( f(x) = 8x + 3 \) is one-to-one. Find the inverse function \( f^{-1} \).

**Solution:**

Step 1: Write \( y = f(x) \).

\[
y = 8x + 3
\]

Step 2: Interchange \( x \) and \( y \).

\[
x = 8y + 3
\]

Step 3: Solve the equation \( x = 8y + 3 \) for \( y \).

\[
x = 8y + 3
\]
\[
x - 3 = 8y + 3 - 3
\]
\[
x - 3 = 8y
\]
\[
\frac{x - 3}{8} = \frac{8y}{8}
\]
\[
\frac{x - 3}{8} = y
\]

Step 4: Write \( y = f^{-1}(x) \).

\[
\frac{x - 3}{8} = f^{-1}(x)
\]

The inverse function is \( f^{-1}(x) = \frac{x - 3}{8} \).

Additional Example 5:

Determine whether or not the functions \( f(x) = 5x^3 \) and \( g(x) = \sqrt[3]{\frac{x}{5}} \) are inverses of each other.
Solution:
Find \( f \circ g \).

\[
(f \circ g)(x) = f(g(x)) \\
= f\left(\frac{x}{\sqrt[3]{5}}\right) \\
= 5\left(\frac{x}{\sqrt[3]{5}}\right)^3 \\
= 5\left(\frac{x}{5}\right) \\
= x
\]

Find \( g \circ f \).

\[
(g \circ f)(x) = g(f(x)) \\
= g\left(\frac{x^3}{\sqrt{5}}\right) \\
= \frac{\sqrt[3]{5x^3}}{\sqrt{5}} \\
= \frac{3\sqrt[3]{x^3}}{\sqrt{5}} \\
= x
\]

The functions \( f \) and \( g \) are inverses of each other since \( f(g(x)) = x \) and \( g(f(x)) = x \).

Additional Example 6:
Assume that \( f \) and \( g \) are inverse functions. Answer the following.

(a) If \( f(3) = 5 \), find \( g(5) \).  (b) If \( g(-3) = -1 \), find \( f(-1) \).
Solution:

Part (a):

Since $f$ and $g$ are inverse functions, we have $g(f(x)) = x$.

$$g(f(3)) = 3$$

Now substitute the given information that $f(3) = 5$.

$$g(5) = 3$$

Part (b):

Since $f$ and $g$ are inverse functions, we have $f(g(x)) = x$.

$$f(g(-3)) = -3$$

Now substitute the given information that $g(-3) = -1$.

$$f(-1) = -3$$
Determine whether each of the following graphs represents a one-to-one function. Explain your answer.

1. 
   ![Graph 1](image1.png)

2. 
   ![Graph 2](image2.png)

3. 
   ![Graph 3](image3.png)

4. 
   ![Graph 4](image4.png)

5. 
   ![Graph 5](image5.png)

6. 
   ![Graph 6](image6.png)

For each of the following functions, sketch a graph and then determine whether the function is one-to-one.

7. \( f(x) = 2x - 3 \)

8. \( g(x) = x^2 + 5 \)

9. \( h(x) = (x - 2)^3 \)

10. \( f(x) = x^3 - 2 \)

11. \( g(x) = |x| + 4 \)

12. \( h(x) = \frac{1}{x} - 3 \)

13. \( f(x) = -(x - 2)^2 + 1 \)

14. \( g(x) = |x - 6| \)

Answer the following.

15. If a function \( f \) is one-to-one, then the inverse function, \( f^{-1} \), can be graphed by either of the following methods:
   (a) Interchange the ____ and ____ values.
   (b) Reflect the graph of \( f \) over the line \( y = ____. \)

16. The domain of \( f \) is equal to the __________ of \( f^{-1} \), and the range of \( f \) is equal to the __________ of \( f^{-1} \).

A table of values for a one-to-one function \( y = f(x) \) is given. Complete the table for \( y = f^{-1}(x) \).

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<th>( x )</th>
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<td>(5)</td>
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<table>
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<tr>
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<tr>
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</tr>
</tbody>
</table>
Exercise Set 1.5: Inverse Functions

For each of the following graphs:
(a) State the domain and range of \( f \).
(b) Sketch \( f^{-1} \).
(c) State the domain and range of \( f^{-1} \).

19.

20.

21.

22.

Answer the following. Assume that \( f \) is a one-to-one function.

23. If \( f(4) = 5 \), find \( f^{-1}(5) \).
24. If \( f(6) = -2 \), find \( f^{-1}(-2) \).
25. If \( f^{-1}(-3) = 7 \), find \( f(7) \).
26. If \( f^{-1}(6) = -8 \), find \( f(-8) \).
27. If \( f(3) = 9 \) and \( f(9) = 5 \), find \( f^{-1}(9) \).
28. If \( f(5) = -4 \) and \( f(2) = 5 \), find \( f^{-1}(5) \).
29. If \( f(-4) = 2 \), find \( f\left(f^{-1}(2)\right) \).
30. If \( f^{-1}(-5) = 3 \), find \( f^{-1}\left(f(3)\right) \).

Answer the following. Assume that \( f \) and \( g \) are defined for all real numbers.

31. If \( f \) and \( g \) are inverse functions, \( f(-2)=3 \) and \( f(4) = -2 \), find \( g(-2) \).
32. If \( f \) and \( g \) are inverse functions, \( f(7)=10 \) and \( f(10) = -1 \), find \( g(10) \).
33. If \( f \) and \( g \) are inverse functions, \( f(5)=8 \) and \( f(9) = 3 \), find \( g\left(f(3)\right) \).
34. If \( f \) and \( g \) are inverse functions, \( f(-1)=6 \) and \( f(7) = 8 \), find \( f\left(g(6)\right) \).

For each of the following functions, write an equation for the inverse function \( y = f^{-1}(x) \).

35. \( f(x) = 5x - 3 \)
36. \( f(x) = -4x + 7 \)
37. \( f(x) = \frac{3 - 2x}{8} \)
38. \( f(x) = \frac{6x - 5}{4} \)
Exercise Set 1.5: Inverse Functions

39. \( f(x) = x^2 + 1 \), where \( x \geq 0 \)

40. \( f(x) = 5 - x^2 \), where \( x \geq 0 \)

41. \( f(x) = 4x^3 - 7 \)

42. \( f(x) = 2x^3 + 1 \)

43. \( f(x) = \frac{3}{x + 2} \)

44. \( f(x) = \frac{5}{7 - x} \)

45. \( f(x) = \frac{2x + 3}{x - 4} \)

46. \( f(x) = \frac{3 - 8x}{x + 5} \)

47. \( f(x) = \sqrt{7 - 2x} \)

48. \( f(x) = 2 + \sqrt{6x + 5} \)

Use the Property of Inverse Functions to determine whether each of the following pairs of functions are inverses of each other. Explain your answer.

49. \( f(x) = 4x - 1 \); \( g(x) = \frac{1}{4}x + 1 \)

50. \( f(x) = 2 + 3x \); \( g(x) = \frac{x - 2}{3} \)

51. \( f(x) = \frac{4 - x}{5} \); \( g(x) = 4 - 5x \)

52. \( f(x) = 2x + 5 \); \( g(x) = \frac{1}{2x + 5} \)

53. \( f(x) = x^3 - 2 \); \( g(x) = \sqrt[3]{x + 2} \)

54. \( f(x) = 2\sqrt{x} - 7 \); \( g(x) = (x + 7)^\frac{1}{2} \)

55. \( f(x) = \frac{5}{x} \); \( g(x) = \frac{5}{x} \)

56. \( f(x) = x^2 + 9 \), where \( x \geq 0 \);
\[ g(x) = \sqrt{x - 9} \]

57. If \( f(x) \) is a function that represents the amount of revenue (in dollars) by selling \( x \) tickets, then what does \( f^{-1}(500) \) represent?

58. If \( f(x) \) is a function that represents the area of a circle with radius \( x \), then what does \( f^{-1}(80) \) represent?

A function is said to be one-to-one provided that the following holds for all \( x_1 \) and \( x_2 \) in the domain of \( f \):

If \( f(x_1) = f(x_2) \), then \( x_1 = x_2 \).

Use the above definition to determine whether or not the following functions are one-to-one. If \( f \) is not one-to-one, then give a specific example showing that the condition \( f(x_1) = f(x_2) \) fails to imply that \( x_1 = x_2 \).

59. \( f(x) = 5x - 3 \)

60. \( f(x) = x^3 + 5 \)

61. \( f(x) = \sqrt[3]{x} - 4 \)

62. \( f(x) = |x| - 4 \)

63. \( f(x) = |x - 4| \)

64. \( f(x) = \frac{1}{x} + 4 \)

65. \( f(x) = x^2 + 3 \)

66. \( f(x) = (x + 3)^2 \)

Answer the following.