Combining Transformations

Suppose that you want to graph the function $f(x) = 3\sqrt{x+2} - 7$. We can quickly identify from the function that the 'base' function is $g(x) = \sqrt{x}$, and that there has been a vertical stretch with a factor of 3, a shift left of 2 units, and a downward shift of 7 units. If you are graphing this function, does the order matter when you perform the transformations? For example, can you shift down, then do the vertical stretch, then shift left? Or should you first shift left, then shift down, and then perform the vertical stretch? We could come up with many different possibilities for the order of transformations for this problem. In this particular example, the order does matter, and we could get an incorrect graph if we perform certain operations out of order. (There are other cases where the order does not matter, depending on which transformations are used.) It is worth spending some time analyzing the order of transformations – which can be done algebraically, without any trial-and-error in graphing.

First, remember the rules for transformations of functions.
(These are not listed in any recommended order; they are just listed for review.)

RULES FOR TRANSFORMATIONS OF FUNCTIONS

If $f(x)$ is the original function, $a > 0$ and $c > 0$:

Function	**Transformation of the graph of $f(x)$**
$f(x)+c$	Shift $f(x)$ upward *c* units
$f(x)-c$	Shift $f(x)$ downward *c* units
$f(x+c)$	Shift $f(x)$ to the left *c* units
$f(x-c)$	Shift $f(x)$ to the right c units
$-f(x)$	Reflect $f(x)$ in the *x*-axis
$f(-x)$	Reflect $f(x)$ in the *y*-axis
$a \cdot f(x),\ a > 1$	Stretch $f(x)$ vertically by a factor of *a*.
$a \cdot f(x),\ 0 < a < 1$	Shrink $f(x)$ vertically by a factor of *a*.
$f(ax),\ a > 1$	Shrink $f(x)$ horizontally by a factor of $\frac{1}{a}$.
$f(ax),\ 0 < a < 1$	Stretch $f(x)$ horizontally by a factor of $\frac{1}{a}$.

Let us look at Examples 1 through 6 below, and we will then look for a pattern as to when the order of transformations matters.

Example Problem 1: Start with the function $f(x)=\sqrt{x}$, and write the function which results from the given transformations. Then decide if the results from parts (a) and (b) are equivalent.

(a) Shift upward 7 units, then right 2 units.

(b) Shift right 2 units, then upward 7 units.

(c) Do parts (a) and (b) yield the same function? (You should be able to tell without graphing.)

SOLUTION

(a) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=\sqrt{x}+7 \quad \rightarrow \quad h(x)=\sqrt{x-2}+7$

Up 7   Right 2

(b) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=\sqrt{x-2} \quad \rightarrow \quad h(x)=\sqrt{x-2}+7$

Right 2   Up 7

(c) Yes, parts (a) and (b) yield the same function.

Example Problem 2: Start with the function $f(x)=\sqrt{x}$, and write the function which results from the given transformations. Then decide if the results from parts (a) and (b) are equivalent.

(a) Stretch vertically by a factor of 2, then shift downward 5 units.

(b) Shift downward 5 units, then stretch vertically by a factor of 2.

(c) Do parts (a) and (b) yield the same function? (You should be able to tell without graphing.)

SOLUTION

(a) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=2\sqrt{x} \quad \rightarrow \quad h(x)=2\sqrt{x}-5$

Stretch vertically by a factor of 2   Down 5

(b) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=\sqrt{x}\ -5 \quad \rightarrow \quad h(x)=2\left(\sqrt{x}-5\right)$

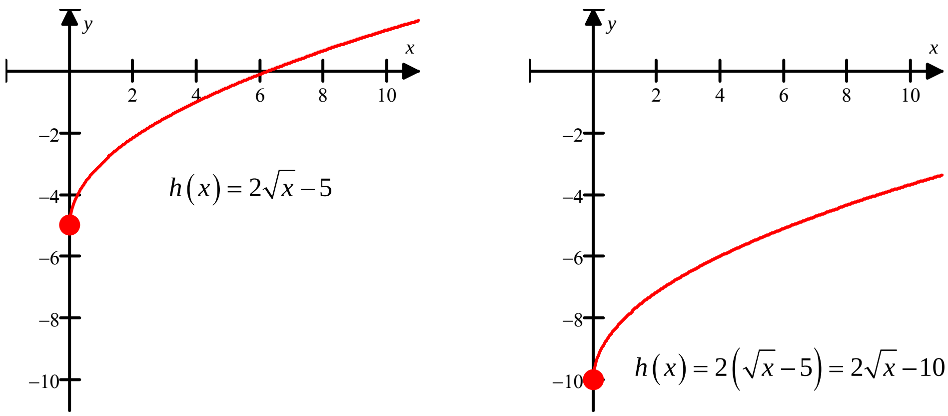
Down 5

Stretch vertically
by a factor of 2

Note: In part (b), $h(x)$ can also be written as $h(x)=2\sqrt{x}-10$.

(c) No, parts (a) and (b) do not yield the same function, since $2\sqrt{x}-5 \neq 2\sqrt{x}-10$.

Both graphs are shown below to emphasize the difference in the final results (but we can see that the above functions are different without graphing the functions).



Example Problem 3: Start with the function $f(x)=\sqrt{x}$, and write the function which results from the given transformations. Then decide if the results from parts (a) and (b) are equivalent.

(a) Reflect in the *y*-axis, then shift upward 6 units.

(b) Shift upward 6 units, then reflect in the *y*-axis.

(c) Do parts (a) and (b) yield the same function? (You should be able to tell without graphing.)

SOLUTION

(a) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=\sqrt{-x} \quad \rightarrow \quad h(x)=\sqrt{-x}\ +6$

Reflect in the *y*-axis

Up 6

(b) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=\sqrt{x}\ +6 \quad \rightarrow \quad h(x)=\sqrt{-x}\ +6$

Up 6 — Reflect in the *y*-axis

(c) Yes, parts (a) and (b) yield the same function.

Example Problem 4: Start with the function $f(x)=\sqrt{x}$, and write the function which results from the given transformations. Then decide if the results from parts (a) and (b) are equivalent.

(a) Reflect in the *y*-axis, then shift left 2 units.

(b) Shift left 2 units, then reflect in the *y*-axis.

(c) Do parts (a) and (b) yield the same function? (You should be able to tell without graphing.)

SOLUTION

(a) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=\sqrt{-x} \quad \rightarrow \quad h(x)=\sqrt{-(x+2)}$

Reflect in the *y*-axis — Left 2

Note: In part (a), $h(x)$ can also be written as $h(x)=\sqrt{-x-2}$.

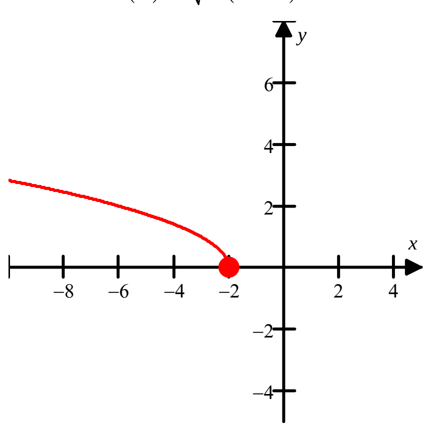
(b) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=\sqrt{x+2} \quad \rightarrow \quad h(x)=\sqrt{-x+2}$

Left 2 — Reflect in the *y*-axis

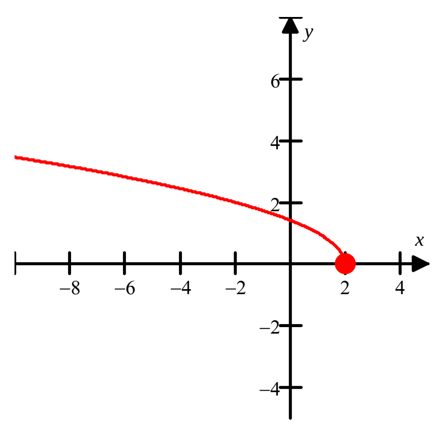
(c) No, parts (a) and (b) do not yield the same function, since $\sqrt{-x-2} \neq \sqrt{-x+2}$.

Both graphs are shown below to emphasize the difference in the final results (but we can see that the above functions are different without graphing the functions).

Part (a): $h(x)=\sqrt{-(x+2)}=\sqrt{-x-2}$



Part (b): $h(x)=\sqrt{-x+2}$



Example Problem 5: Start with the function $f(x)=\sqrt{x}$, and write the function which results from the given transformations. Then decide if the results from parts (a) and (b) are equivalent.

(a) Reflect in the *x*-axis, then shift upward 4 units.

(b) Shift upward 4 units, then reflect in the *x*-axis.

(c) Do parts (a) and (b) yield the same function? (You should be able to tell without graphing.)

SOLUTION

(a) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=-\sqrt{x} \quad \rightarrow \quad h(x)=-\sqrt{x}\ +4$

Reflect in the *x*-axis — Up 4

(b) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=\sqrt{x}\ +4 \quad \rightarrow \quad h(x)=-\left(\sqrt{x}\ +4\right)$

Up 4 — Reflect in the *x*-axis

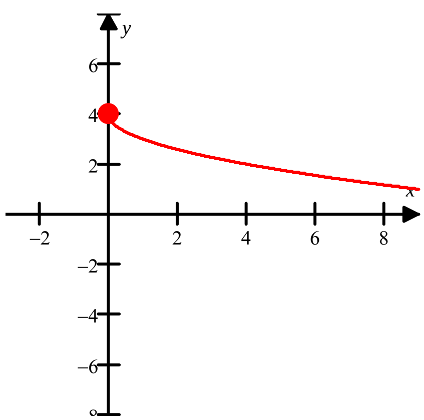
Note: In part (b), $h(x)$ can also be written as $h(x)=-\sqrt{x}-4$.

(c) No, parts (a) and (b) do not yield the same function, since $-\sqrt{x}+4\neq-\sqrt{x}-4$.

Both graphs are shown below to emphasize the difference in the final results (but we can see that the above functions are different without graphing the functions).

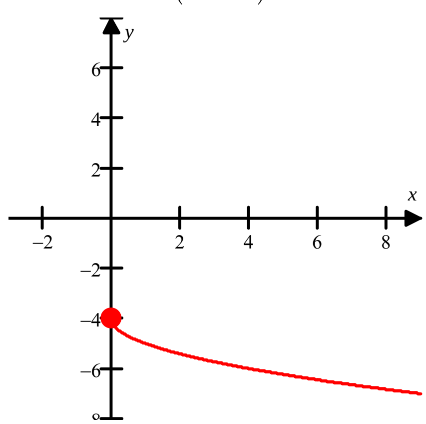
Part (a):

$h(x)=-\sqrt{x}+4$



Part (b):

$h(x)=-\left(\sqrt{x}+4\right)=-\sqrt{x}-4$



Example Problem 6: Start with the function $f(x)=\sqrt{x}$, and write the function which results from the given transformations. Then decide if the results from parts (a) and (b) are equivalent.

(a) Shrink horizontally by a factor of $\frac{1}{3}$, then shift right 6 units.

(b) Shift right 6 units, then shrink horizontally by a factor of $\frac{1}{3}$.

(c) Do parts (a) and (b) yield the same function? (You should be able to tell without graphing.)

SOLUTION

(a) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=\sqrt{3x} \quad \rightarrow \quad h(x)=\sqrt{3(x-6)}$

Shrink horizontally by a factor of $\frac{1}{3}$ Right 6

Note: In part (a), $h(x)$ can also be written as $h(x)=\sqrt{3x-18}$.

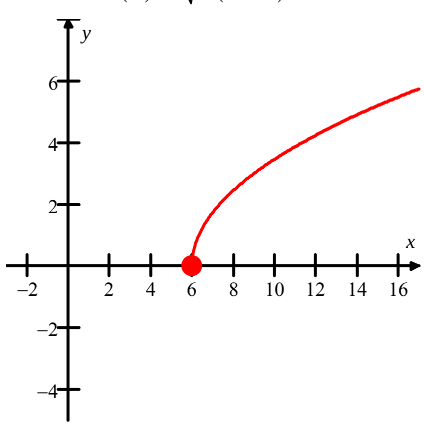
(b) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=\sqrt{x-6} \quad \rightarrow \quad h(x)=\sqrt{3x-6}$

Right 6   Shrink horizontally by a factor of $\frac{1}{3}$

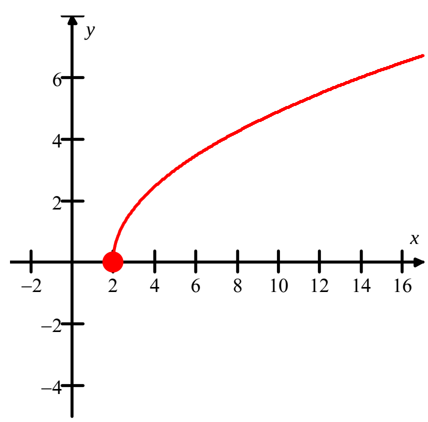
(c) No, parts (a) and (b) do not yield the same function, since $\sqrt{3x-18} \neq \sqrt{3x-6}$.

Both graphs are shown below to emphasize the difference in the final results (but we can see that the above functions are different without graphing the functions).

Part (a): $h(x)=\sqrt{3(x-6)}=\sqrt{3x-18}$



Part (b): $h(x)=\sqrt{3x-6}$



Looking for a Pattern – When Does the Order of Transformations Matter?

When deciding whether the order of the transformations matters, it helps to think about whether a transformation affects the graph vertically (i.e. changes the *y*-values) or horizontally (i.e. changes the *x*-values).

Transformation	Vertical or Horizontal Effect?
Shifting up or down	Vertical
Shifting left or right	Horizontal
Reflecting in the *y*-axis	Horizontal
Reflecting in the *x*-axis	Vertical
Vertical stretching/shrinking	Vertical
Horizontal stretching/shrinking	Horizontal

A summary of the results from Examples 1 through 6 are below, along with whether or not each transformation had a vertical or horizontal effect on the graph.

Summary of Results from Examples 1 – 6
with notations about the vertical or horizontal effect on the graph, where
V = Vertical effect on graph H = Horizontal effect on graph

	First Set of Transformations **(with notations about horizontal/vertical effect)**	**Second Set of Transformations** **(with notations about horizontal/vertical effect)**	**Did (a) and (b) yield the same function?**
Ex 1	(a) Up 7 (V) Right 2 (H)	(b) Right 2 (H) Up 7 (V)	The functions were the same.
Ex 2	(a) Vertical stretch, factor of 2 (V) Down 5 (V)	(b) Down 5 (V) Vertical stretch, factor of 2 (V)	The functions were NOT the same.
Ex 3	(a) Reflect in *y*-axis (H) Up 6 (V)	(b) Up 6 (V) Reflect in *y*-axis (H)	The functions were the same.
Ex 4	(a) Reflect in *y*-axis (H) Left 2 (H)	(b) Left 2 (H) Reflect in *y*-axis (H)	The functions were NOT the same.
Ex 5	(a) Reflect in *x*-axis (V) Up 4 (V)	(b) Up 4 (V) Reflect in *x*-axis (V)	The functions were NOT the same.
Ex 6	(a) Horizontal shrink, factor of $\frac{1}{3}$ (H) Right 6 (H)	(b) Right 6 (H) Horizontal shrink, factor of $\frac{1}{3}$ (H)	The functions were NOT the same

Notice that in examples 1 and 3, the order of the transformations did not matter. In both of those examples, one of the transformations had a vertical effect on the graph, and the other transformation had a horizontal effect on the graph.

In examples 2, 4, 5 and 6, the order of the transformations did matter. Notice that example 2 had two vertically-oriented transformations, example 4 had two horizontally-oriented transformations, example 5 had two vertically-oriented transformations, and example 6 had two horizontally-oriented transformations.

When you perform two or more transformations that have a vertical effect on the graph, the order of those transformations may affect the final results. Similarly, when you perform two or more transformations that have a horizontal effect on the graph, the order of those transformations may affect the final results. The vertically-oriented transformations do not affect the horizontally-oriented transformations, and vice versa.

Let us now return to the function used at the start of this discussion:

Example Problem 7: Suppose that you want to graph $f(x) = 3\sqrt{x+2} - 7$. In what order can you perform the transformations to obtain the correct graph?

SOLUTION:

First, decide on the transformations that need to be performed on $f(x) = 3\sqrt{x+2} - 7$ (without consideration of correct order). Make a note of whether each transformation has a horizontal or vertical effect on the graph.

$f(x) = 3\sqrt{x+2} - 7$ (↑ 3)	$f(x) = 3\sqrt{x+2} - 7$ (↑ $x+2$)	$f(x) = 3\sqrt{x+2} - 7$ (↑ -7)
Vertical Stretch, factor of 3 (Vertical Effect)	Shift left 2 (Horizontal Effect)	Shift down 7 (Vertical Effect)

Notice that the shift to the left is the only transformation that has a horizontal effect on the graph. This transformation can be performed at any point in the graphing process.

We need to be more careful about the order in which we perform the vertical stretch and the downward shift, since they both have a vertical effect on the graph. Perform the following transformations algebraically on $g(x) = \sqrt{x}$ to see which one gives the desired function, $f(x) = 3\sqrt{x+2} - 7$. (The shift left is written first, but we could put that transformation at any point in the process and get the same result.)

Choice 1:
Shift left 2 units, then stretch vertically by a factor of 3, then shift downward 7 units:

$$g(x)=\sqrt{x} \quad \rightarrow \quad h(x)=\sqrt{x+2} \quad \rightarrow \quad k(x)=3\sqrt{x+2} \quad \rightarrow \quad f(x)=3\sqrt{x+2}-7$$

Left 2 — Vertical stretch, factor of 3 — Down 7

Choice 2:
Shift left 2 units, then shift downward 7 units, then stretch vertically by a factor of 3:

$$g(x)=\sqrt{x} \rightarrow h(x)=\sqrt{x+2} \rightarrow k(x)=\sqrt{x+2}-7 \rightarrow f(x)=3\left(\sqrt{x+2}-7\right)$$

Left 2 — Down 7 — Vertical stretch, factor of 3

Notice that our final result for Choice 2 can be written as $f(x)=3\sqrt{x+2}-21$, which is not the desired function.

We can see from the analysis above that Choice 1 yields the desired function for Example Problem 7, which means that in this particular example, the vertical stretch needs to be performed before the downward shift. Since the left shift can be performed at any point in the process, any of the following order of transformations would yield the correct graph:

Shift left 2 units, then stretch vertically by a factor of 3, then shift downward 7 units.
Stretch vertically by a factor of 3, then shift left 2 units, then shift downward 7 units.
Stretch vertically by a factor of 3, then shift downward 7 units, then shift left 2 units.

In this course, you would only need to give one of the answers from the above list (not all three). This explanation is given to help you understand that there can be multiple solutions for a given problem – and how to determine an acceptable order of transformations for the given problem.

Example Problem 8: Suppose that you want to graph $f(x)=\sqrt{-x+2}-7$. In what order can you perform the transformations to obtain the correct graph?

SOLUTION:
First, decide on the transformations that need to be performed on $f(x)=\sqrt{-x+2}-7$ (without consideration of correct order). Make a note of whether each transformation has a horizontal or vertical effect on the graph.

$f(x)=\sqrt{-x+2}-7$ — Reflect in the y-axis (Horizontal Effect)

$f(x)=\sqrt{-x+2}-7$ — Shift left 2 (Horizontal Effect)

$f(x)=\sqrt{-x+2}-7$ — Shift down 7 (Vertical Effect)

Notice that the downward shift is the only transformation that has a vertical effect on the graph. This transformation can be performed at any point in the graphing process.

We need to be more careful about the order in which we perform the reflection in the *y*-axis and the shift to the left, since they both have a horizontal effect on the graph. Perform the following transformations algebraically on $g(x)=\sqrt{x}$ to see which one gives the desired function, $f(x)=\sqrt{-x+2}-7$. (The downward shift is written first, but we could put that transformation at any point in the process and get the same result.)

<u>Choice 1:</u>
Downward shift of 7 units, then reflect in the *y*-axis, then shift left 2 units.

$$g(x)=\sqrt{x} \rightarrow h(x)=\sqrt{x}-7 \rightarrow k(x)=\sqrt{-x}-7 \rightarrow f(x)=\sqrt{-(x+2)}-7$$

Down 7 — Reflect in the *y*-axis — Shift left 2 units

Notice that our final result for Choice 1 can be written as $f(x)=\sqrt{-x-2}-7$, which is not the desired function.

<u>Choice 2:</u>
Downward shift of 7 units, then shift left 2 units, then reflect in the *y*-axis.

$$g(x)=\sqrt{x} \rightarrow h(x)=\sqrt{x}-7 \rightarrow k(x)=\sqrt{x+2}-7 \rightarrow f(x)=\sqrt{-x+2}-7$$

Down 7 — Shift left 2 units — Reflect in the *y*-axis

We can see from the analysis above that Choice 2 yields the desired function for Example Problem 8, which means that in this particular example, the shift to the left needs to be performed before the reflection in the *y*-axis. Since the downward shift can be performed at any point in the process, any of the following order of transformations would yield the correct graph:

Shift downward 7 units, then shift left 2 units, then reflect in the *y*-axis.
Shift left 2 units, then shift downward 7 units, then reflect in the *y*-axis.
Shift left 2 units, then reflect in the *y*-axis, then shift downward 7 units.

As mentioned in Example Problem 7, you would only need to give one of the answers from the above list (not all three). This explanation is given to help you understand that there can be multiple solutions for a given problem – and how to determine an acceptable order of transformations for the given problem.

Alternate Analysis of Example Problem 8:
It is possible to use a different set of transformations to yield the same result in Example Problem 8. Remember that the desired function is $f(x)=\sqrt{-x+2}-7$. You may notice that $f(x)=\sqrt{-x+2}-7=\sqrt{-(x-2)}-7$.

Looking at $f(x)=\sqrt{-(x-2)}-7$, notice that the $x+2$ from the original function is now $(x-2)$, which indicates a shift right of 2 units (rather than a shift left of 2 units).

The order of the downward shift again does not matter, but we need to decide if we should reflect in the *y*-axis and then shift right, or shift right and then reflect in the *y*-axis.

Choice 1:
Shift downward 7 units (can be done at any time), reflect in the *y*-axis, then shift right 2 units:

$$g(x)=\sqrt{x} \rightarrow h(x)=\sqrt{x}-7 \rightarrow k(x)=\sqrt{-x}-7 \rightarrow f(x)=\sqrt{-(x-2)}-7$$

Down 7 — Reflect in the *y*-axis — Shift right 2 units

Notice that our final result for Choice 1 can be written as $f(x)=\sqrt{-x+2}-7$, which is the desired function.

Choice 2:
Shift downward 7 units (can be done at any time), shift right 2 units, then reflect in the *y*-axis.

$$g(x)=\sqrt{x} \rightarrow h(x)=\sqrt{x}-7 \rightarrow k(x)=\sqrt{x-2}-7 \rightarrow f(x)=\sqrt{-x-2}-7$$

Down 7 — Shift right 2 units — Reflect in the *y*-axis

Notice that Choice 2 does not yield the desired function.

We can see from the alternate analysis above that Choice 1 yields the desired function for Example Problem 8, which means that in this particular example, the reflection in the y-axis needs to be performed before the shift to the right. Since the downward shift can be performed at any point in the process, any of the following order of transformations would yield the correct graph:

Shift downward 7 units, then reflect in the *y*-axis, then shift right 2 units.
Reflect in the *y*-axis, then shift downward 7 units, then shift right 2 units.
Reflect in the *y*-axis, then shift right 2 units, then shift downward 7 units.

Remember that when we used a shift to the left instead, we obtained the following solutions as well:

Shift downward 7 units, then shift left 2 units, then reflect in the *y*-axis.
Shift left 2 units, then shift downward 7 units, then reflect in the *y*-axis.
Shift left 2 units, then reflect in the *y*-axis, then shift downward 7 units.

In all of the examples above, we have discussed problems where there were two or three transformations. **What if there are four or more transformations?** In that case, look at the vertically-oriented transformations and decide the order in which they need to be done. Look at the horizontally-oriented transformations and decide the order in which they need to be done. Then remember that the vertically-oriented transformations have no effect on the horizontally-oriented transformations, and vice versa.

Example Problem 9: Suppose that you want to graph $f(x) = 3\sqrt{-x+2} - 7$.

(a) In what order can you perform the transformations to obtain the correct graph?
(b) Graph the function.

SOLUTION:

(a) In order to save time, we have chosen an example that is similar to Example Problems 7 and 8.

We have the following transformations, not necessarily to be performed in this order:

Stretch vertically, factor of 3	Vertically-oriented transformation
Reflection in the *y*-axis	Horizontally-oriented transformation
Shift left 2 units*	Horizontally-oriented transformation
Shift downward 7 units	Vertically-oriented transformation

*For simplicity, we are choosing to focus on a left shift for this problem, instead of factoring out a negative under the radical, where $f(x) = 3\sqrt{-(x-2)} - 7$, and focusing on a right shift instead. As seen in Example Problem 8, we would obtain additional solutions if we considered a right shift as an alternate means of solving this problem. (The point of these detailed explanations is not for the student to be able to list all possible orders of transformations, but to be able to determine one order of transformations for any given problem which would yield a correct graph.)

Let us look first at the vertically-oriented transformations. (Temporarily put aside the horizontally-oriented transformations, and just look at what is happening ‘outside’ the radical sign.) As in Example Problem 7, the vertical stretch needs to be performed before the downward shift.

Next look at the horizontally-oriented transformations. (Temporarily put aside the vertically-oriented transformations, and just look at what is happening under the radical sign.) As in Example Problem 8, the shift to the left needs to be performed before the reflection in the *y*-axis.

Remember that the vertically-oriented transformations do not affect the horizontally-oriented transformations, and vice versa. There are many correct solutions to this problem. Just be sure that in your answer, the vertical stretch is performed before the downward shift, and the shift to the left is performed before the reflection in the *y*-axis.

All the solutions involving a left shift are shown below. **You do not need to find all of the answers below; any one of the solutions below would be acceptable.** (Note: If we looked at all the possibilities involving a right shift, we would add six more solutions which would yield the same graph.)

Stretch vertically by a factor of 3, then shift downward 7 units, then shift left 2 units, then reflect in the *y*-axis.

Shift left 2 units, then reflect in the *y*-axis, then stretch vertically by a factor of 3, then shift downward 7 units.

Stretch vertically by a factor of 3, then shift left 2 units, then shift downward 7 units, then reflect in the *y*-axis.

Stretch vertically by a factor of 3, then shift left 2 units, then reflect in the *y*-axis, then shift downward 7 units,.

Shift left 2 units, then stretch vertically by a factor of 3, then reflect in the *y*-axis, then shift downward 7 units.

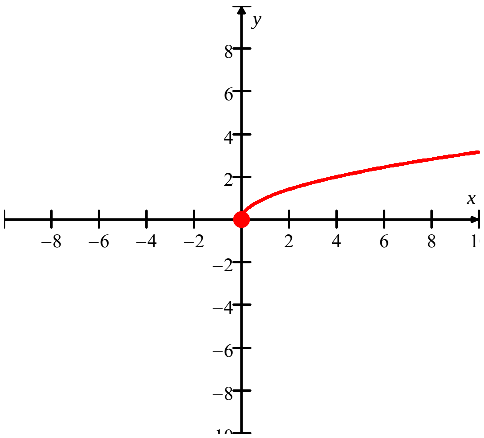
Shift left 2 units, then stretch vertically by a factor of 3, then shift downward 7 units, then reflect in the *y*-axis.

(b) To graph the function, we can choose any one of the solutions shown above (and obtain the same result). We will perform the transformations in the order listed for the first solution listed above:

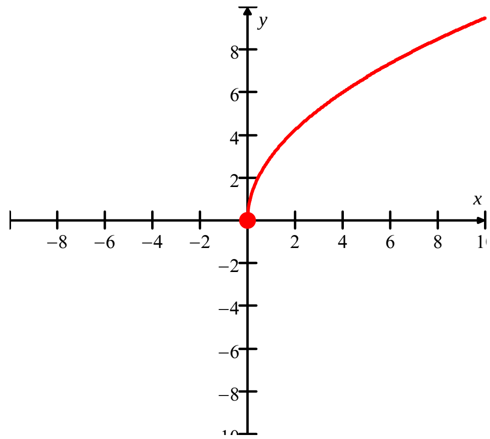
Starting with the base graph $y = \sqrt{x}$:

Stretch vertically by a factor of 3, then shift downward 7 units, then shift left 2 units, then reflect in the *y*-axis. All steps are shown consecutively below.

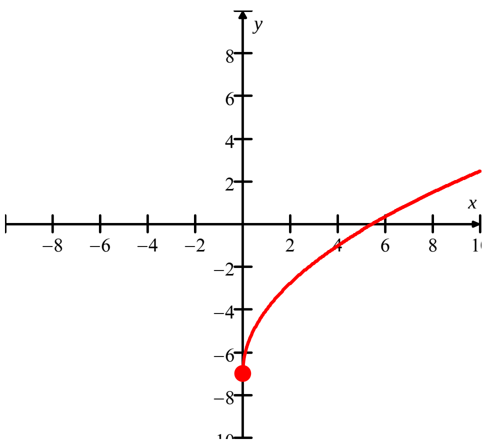
Step 1: $y=\sqrt{x}$



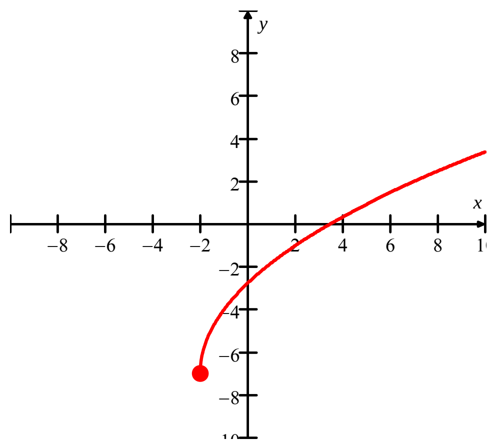
Step 2: $y=3\sqrt{x}$
(Stretch vertically by a factor of 3)



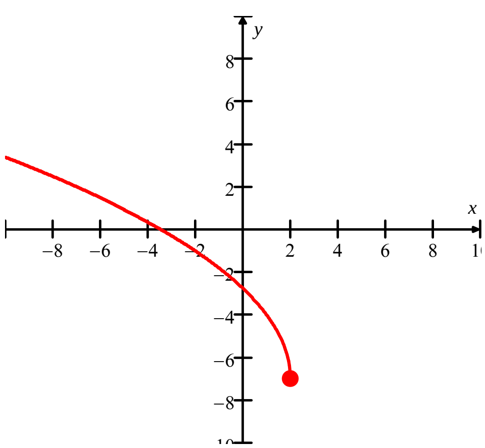
Step 3: $y=3\sqrt{x}-7$
(Shift downward 7 units)



Step 4: $y=3\sqrt{x+2}-7$
(Shift left 2 units)



Step 5: $y=f(x)=3\sqrt{-x+2}-7$
(Reflect in the *y*-axis)



Step 5 represents the final result.