Math 1324
Section 3.5
The Inverse of a Matrix

The videos corresponding to this worksheet can be found at
https://online.math.uh.edu/Math1324/.
UH students can alternatively view the videos within the Math 1324 textbook.

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If A is a square matrix such that $AA^{-1} = A^{-1}A = I_n$, then A^{-1} is called the **inverse matrix of A**.
Note: Not every square matrix will have an inverse.

If a square matrix does not have an inverse, then it is said to be **singular**.

How to Find the Inverse of a Square Matrix

1. Adjoin the square matrix A with the identity matrix of the same size, $\left[A \mid I\right]$.
2. Use the Gauss-Jordan elimination method to reduce $\left[A \mid I\right]$ to the form $\left[I \mid B\right]$, if possible.

The matrix B is the inverse of the matrix A. It may be verified by checking $AB = BA = I_n$.

If in the process of reducing $\left[A \mid I\right]$ to $\left[I \mid B\right]$, there is a row of zeros to the left of the vertical line then the matrix does not have an inverse.

Example 1: Find the inverse of each matrix below, if possible.

a. $A = \begin{pmatrix} 2 & 4 \\ 1 & 6 \end{pmatrix}$

b. $B = \begin{pmatrix} -2 & 6 \\ -4 & 12 \end{pmatrix}$

Example 2: Find the inverse of $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{pmatrix}$ if possible.

If a matrix is of size 2X2 there is a formula for finding its inverse, if it exists. Before we state the formula, we need to define the determinant of a matrix.

A **determinant** is a real number associated with a square matrix. A square matrix will have an inverse if the determinant is not equal to 0.

Formula for Finding the Determinant of a 2X2 Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The determinant, *D*, is $D = ad - bc$. If $D \neq 0$, then the matrix A has an inverse.

Formula for Finding the Inverse of a 2X2 Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $D \neq 0$, then the inverse of the matrix A is given by

$$A^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 3: Find the inverse of $C = \begin{pmatrix} -7 & -8 \\ 10 & 1 \end{pmatrix}$, if possible.

Matrix Representation of a Linear System

Recall from Section 3.1 that we may write the coefficient and constant matrix of a system of linear equations:

Given $\begin{cases} -2x+8y=1 \\ 3x-y=10 \end{cases}$, the related matrices are:

Coefficient Matrix: $A=\begin{bmatrix} -2 & 8 \\ 3 & -1 \end{bmatrix}$

and

Constant Matrix: $B=\begin{bmatrix} 1 \\ 10 \end{bmatrix}$

We may also write a column matrix containing the variables in the system:

Variable Matrix: $X=\begin{bmatrix} x \\ y \end{bmatrix}$

By matrix multiplication we have that the following matrix equation is equivalent to the given system.

$$AX=B$$

$$\begin{bmatrix} -2 & 8 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

Example 4: Write the following system of linear equations in matrix form.

$$x+5y-3z=18$$
$$-6y+8z=-10$$
$$-4x+9y=8$$

Solving Systems of Linear Equations Using Inverses

Given a system of linear equations, we'll first represent the system using matrices:

$$AX = B$$

If the coefficient matrix A has an inverse, then we may solve the system using the inverse.

Assuming the coefficient matrix has an inverse, we'll first find its inverse, multiply it to both sides of the matrix equation $AX = B$, and solve for X:

$$\begin{aligned} A^{-1}(AX) &= A^{-1}B \\ (A^{-1}A)X &= A^{-1}B \\ I_nX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Once the product $A^{-1}B$ is found, then by equality of matrices we'll have the value of each variable in the system; hence, we'll have solved the system.

Using matrices to solve a system of linear equations is extremely useful when systems have the same coefficient matrix, but different constant matrices.

Example 5: Write a matrix equation that is equivalent to the given system of linear equations and then solve the system using the inverse of the coefficient matrix.

$$\begin{aligned} 2x + y &= b_1 \\ 5x + 3y &= b_2 \end{aligned}$$

where i. $b_1 = 5,\ b_2 = 13$

and ii. $b_1 = -9,\ b_2 = 1$

Example 6: A movie theatre had a special once a day showing of *Funny Girl*. The theatre in which the movie played has a capacity of 50. The price for an adult ticket was $12 and the price for a child ticket was $7. On Saturday, the movie was sold out and total receipts were $565. On Sunday, the combined number of adults and children attending the movie was 39 and total receipts were $428. How many adults and how many children attended the movie for each showing?