Section 2.5 – Average Rate of Change

Suppose that the revenue realized on the sale of a company's product can be modeled by the function $R(x) = 600x - 0.3x^2$, where x is the number of units sold and $R(x)$ is given in dollars. (Revenue is the income a company receives from selling its goods or services.)

You can use the revenue function to determine how much money the company takes in when it sells a given number of items. So when the company sells 100 items, its revenue is

$$\begin{aligned} R(100) &= 600(100) - 0.3(100)^2 \\ &= 60000 - 0.3(10000) \\ &= 60000 - 3000 \\ &= 57000 \end{aligned}$$

When 100 items are sold, the revenue is $57000.

Now suppose you are interested in the change in revenue. If you want to compute the change in revenue when the number of items sold increases from 100 items to 200 items, you can compute

$$\begin{aligned} R(200) - R(100) &= 600(200) - 0.3(200)^2 - 57000 \\ &= 120000 - 0.3(40000) - 57000 \\ &= 120000 - 12000 - 57000 \\ &= 51000 \end{aligned}$$

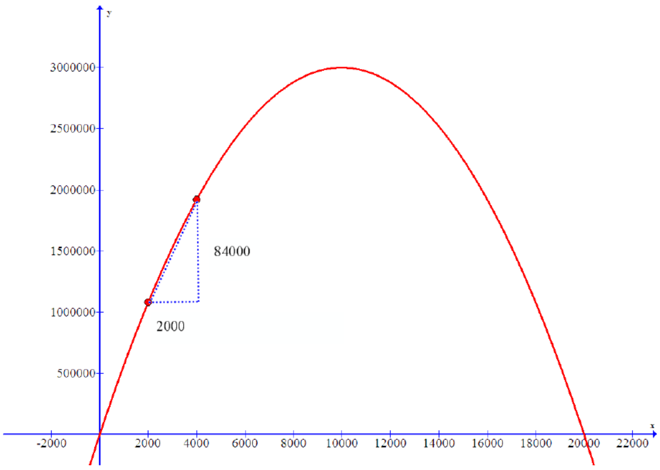
So a change in the number of items sold from 100 items to 200 items will generate an additional $51000 in revenue.

Now suppose you want to know the average change in revenue, *per additional item* that was sold. To do this, you will divide the change in revenue by the change in the number of items sold.

$$\frac{R(200)-R(100)}{200-100}=\frac{51000}{100}$$
$$=510$$

So the average increase in revenue, per additional item sold, is $510.

This is an example of an *average rate of change* problem. In this example, you are interested in finding the average change in the function value given a change in the number of items sold.



Definition: For $y=f(x)$, the average rate of change on an interval [a, b] is $\frac{f(b)-f(a)}{b-a}$, where $b-a\neq 0$.

In the example, we found the average rate of change of $R(x)$ on [100, 200].

You should already be familiar with one average rate of change: the slope of a line. When you find the slope of a line, you are really applying the definition stated above.

Example 1: Find the slope of the line that passes through the points (1, 5) and (3, 9).

Solution: You can find the slope using the familiar formula $m=\frac{y_2-y_1}{x_2-x_1}$.

Then $x_1=1, x_2=3, y_1=5$ and $y_2=9$.

$$\begin{aligned} m &= \frac{9-5}{3-1} \\ &= \frac{4}{2} \\ &= 2 \end{aligned}$$

So the slope of the line passing through the points is 2.

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You could also work this problem using the average rate of change definition. Consider the point (1, 5). The *y* coordinate can be written as $f(1)$, so the point is $(1,\ f(1))$, where $f(1) = 5$. Similarly the point (3, 9) is $(3,\ f(3))$, where $f(3) = 9$.

Now use the average rate of change formula:

$$\frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(1)}{3 - 1} = \frac{9 - 5}{3 - 1} = \frac{4}{2} = 2$$

You can determine the average rate of change over an interval from several different sources of information. You can be given two points, as in Example 1. You can also work with a table, or with a function.

Example 2: Use the values given in the table to find the average rate of change on these intervals:

A. [1, 4]
B. [2.8, 5.3]

x	1	2	2.8	3.1	4	5.3
y	8	14	17	21	25	29

Solution: A. Use the table to determine the ordered pairs that are needed to find the average rate of change. They are (1, 8) and (4, 25). Now use the average rate of change formula:

$$\frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(1)}{4 - 1} = \frac{25 - 8}{4 - 1} = \frac{17}{3} = 5\frac{2}{3}$$

The average rate of change on the interval [1, 4] is $\frac{17}{3}$ or $5\frac{2}{3}$.

B. Use the table to determine the ordered pairs that are needed to find the average rate of change. They are (2.8, 17) and (5.3, 29). Now use the average rate of change formula:

$$\frac{f(b)-f(a)}{b-a}=\frac{f(5.3)-f(2.8)}{5.3-2.8}=\frac{29-17}{5.3-2.8}=\frac{12}{2.5}=4.8$$

The average rate of change on the interval [2.8, 5.3] is 4.8.

Example 3: Find the average rate of change of $f(x)=0.001x^3+0.05x^2-0.25x+600$ on the interval [200, 300].

Solution: Use the average rate of change formula to find the answer.

$$\frac{f(b)-f(a)}{b-a}=\frac{f(300)-f(200)}{300-200}$$

Now use a calculator to compute $f(300)$ and $f(200)$.

$$\begin{aligned} f(300) &= 0.001(300)^3+0.05(300)^2-0.25(300)+600 \\ &= 27000+4500-75+600 \\ &= 32025 \\ f(200) &= 0.001(200)^3+0.05(200)^2-0.25(200)+600 \\ &= 10550 \end{aligned}$$

Then

$$\frac{f(b)-f(a)}{b-a}=\frac{f(300)-f(200)}{300-200}=\frac{32025-10550}{100}=214.75.$$

So the average rate of change of the function on the interval [200, 300] is 214.75.

One common rate of change problem involves velocity. The average velocity of an object tells you how fast it is moving per unit of time over an interval.

Example 4: Suppose an object is thrown upward with an initial velocity of 52 feet per second from a height of 125 feet. The height of the object t seconds after it is thrown is given by $h(t) = -16t^2 + 52t + 125$. Find the average velocity in the first three seconds after the object is thrown.

Solution: The problem is asking you to find the average rate of change of $h(t)$ on the interval [0, 3]. Use the average rate of change formula:

$$\frac{h(b)-h(a)}{b-a} = \frac{h(3)-f(0)}{3-0}.$$

Now use your calculator to compute $h(3)$ and $h(0)$:

$$\begin{aligned} h(3) &= -16(3)^2 + 52(3) + 125 \\ &= -144 + 156 + 125 \\ &= 137 \\ h(0) &= -16(0)^2 + 52(0) + 125 \\ &= 0 + 0 + 125 \\ &= 125 \end{aligned}$$

Then

$$\frac{h(b)-h(a)}{b-a} = \frac{h(3)-f(0)}{3-0} = \frac{137-125}{3} = \frac{12}{3} = 4.$$

So the average velocity on the interval [0, 3] is 4 feet per second.