Section 2.1 – Limits

You've already spent some time studying algebra. When you solve an equation using algebra, you solve for a specific value of the variable. You can use algebra to answer questions such as these:

- What are a company's profits at a specific production and sales level?
- What is the population of a country at a given point in time?
- What is the area of a common geometric shape?

Calculus, however, deals with dynamic situations – situations that change. Typical problems that you will solve using calculus include:

- At what rate are the profits of a company changing?
- When does the rate of population growth start to slow down?
- What is the area of an arbitrary region?

In this course, you will learn to solve problems dealing with *differential calculus* and *integral calculus.*

Differential calculus focuses on the tangent line question: What is the slope of a line tangent to a curve at a specific point? You can use this branch of calculus to solve many problems dealing with rates of change.

Integral calculus focuses on the area question: What is the area between a curve and the x axis on an interval, or what is the area between two curves? In a typical problem, you'll be given an expression which gives the rate at which a quantity is changing and asked to find the total change over a period of time.

These two branches of calculus are connected by the *fundamental theorem of calculus*.

Both differential calculus and integral calculus rely on the concept of a limit.

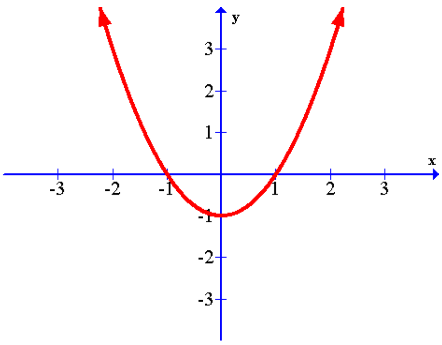
What is a Limit?

When you are asked to find a limit, you are asked to describe the behavior of a function, $f(x)$, as x gets really close to a specific number, c. If you can find a number that $f(x)$ is approaching as x gets close to the target number c, then you can state the limit, as x approaches c.

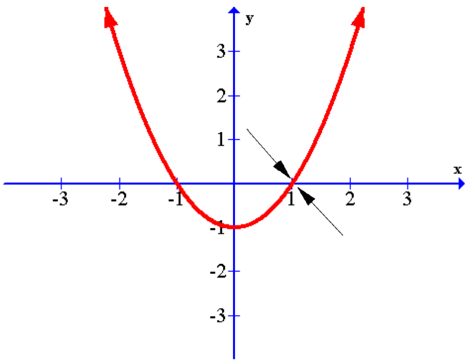
Definition: A function f has limit L as x approaches the number c if the value $f(x)$ gets close to the number L as we let x get sufficiently close to (but not equal to) c.

The notation, $\lim_{x \to c} f(x) = L$, summarizes this idea and is read "limit as x approaches c of $f(x)$ is L." The notation $x \to c$ is read "as x approaches c," meaning "as the value of x gets closer to the number c."

Example 1: Use the graph to find $\lim_{x \to 1} f(x)$.



Solution: Using this graph, as $x \to 1$, $f(x) \to 0$.



So, $\lim_{x \to 1} f(x) = 0$.

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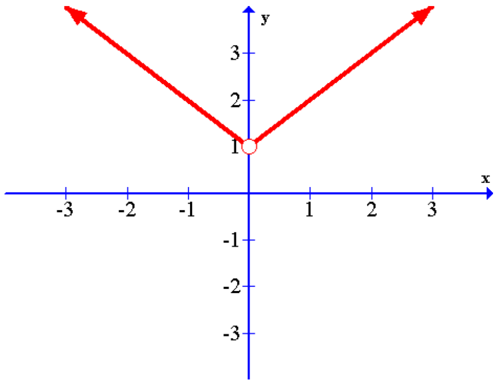
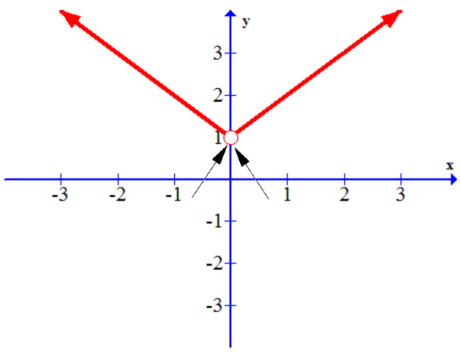
In the first example, the point (1, 0) was a point on the graph of the function. According to the definition, x does not have to ever equal the target number c.

Example 2: Use the graph to find $\lim_{x \to 0} f(x)$.



Solution: Note that the graph does not include the point at (0, 1). The definition of a limit requires that, as x approaches a specific number, $f(x)$ must *approach* a specific number. In this problem, as $x \to 0$, $f(x) \to 1$.



So $\lim_{x \to 0} f(x) = 1$.

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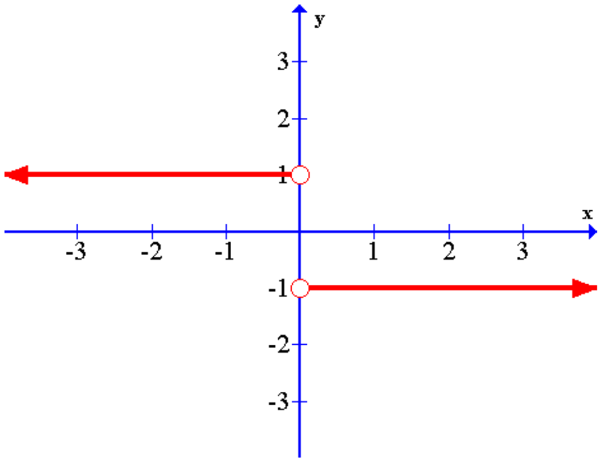
When Does a Limit Fail to Exist?

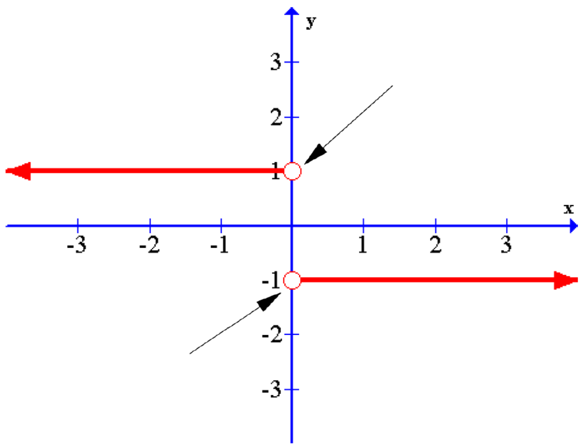
We will look at two situations where a limit does not exist:

Case 1: The value for $f(x)$ approaches one value for *x* values smaller than the target number *c* and approaches a different value for *x* values larger than the target number *c*.

Look at the graph below. As $x \to 0$, $f(x)$ approaches one value if you consider numbers smaller than 0 and $f(x)$ approaches a different value if you consider numbers bigger than 0.



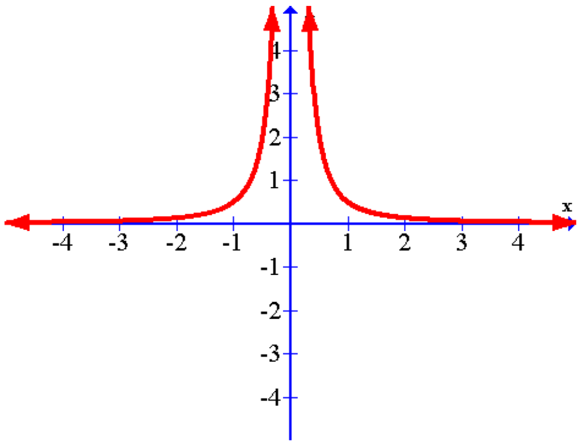
To the left of 0, $f(x) \to 1$ and to the right of 0, $f(x) \to -1$.



In this case, you'll state that $\lim_{x \to 0} f(x)$ does not exist. You may see this abbreviated as "dne."

Case 2: The value for $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to c$. This means that the graph of the function has a vertical asymptote at $x = c$.

In the graph below, the function values approach infinity as $x \to 0$. So $\lim_{x \to 0} f(x)$ does not exist.



Finding Limits Using Substitution

Most often, you will be asked to evaluate a limit using algebraic methods. You can use these properties to evaluate limits.

Properties of limits:

Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then,

1. $\lim_{x \to a} [f(x)]^r = [\lim_{x \to a} f(x)]^r = L^r$ for any real number *r*.
2. $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x) = cL$ for any real number *c*.
3. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M$.
4. $\lim_{x \to a} [f(x)g(x)] = [\lim_{x \to a} f(x)][\lim_{x \to a} g(x)] = LM$.
5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L}{M}$, provided $M \neq 0$.

Example 3: Evaluate $\lim_{x\to 2}\left(4x^2 - x + 7\right)$.

Solution: Using Property 3, you can think of the problem as

$$\lim_{x\to 2}\left(4x^2\right) - \lim_{x\to 2}\left(x\right) + \lim_{x\to 2}\left(7\right).$$

Then evaluate each limit by substituting 2 for *x* and evaluating.

$$\lim_{x\to 2}\left(4x^2\right) - \lim_{x\to 2}\left(x\right) + \lim_{x\to 2}\left(7\right) = 4\left(2\right)^2 - 2 + 7 = 16 - 2 + 7 = 21.$$

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Example 4: Evaluate $\lim_{x\to -1}\left(\dfrac{2x^2\sqrt{x+5}}{x^2+2}\right)$.

Solution: Using the properties, you can think of this problem as

$$\frac{\lim_{x\to -1}\left(2x^2\right)\cdot \lim_{x\to -1}\sqrt{x+5}}{\lim_{x\to -1}\left(x^2+2\right)}.$$

Now substitute -1 for *x* and evaluate:

$$\frac{2(-1)^2\cdot\sqrt{-1+5}}{(-1)^2+2} = \frac{2(1)\cdot\sqrt{4}}{1+2} = \frac{2\cdot 2}{3} = \frac{4}{3}$$

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Example 5: Evaluate $\lim_{x\to 4}\left(\dfrac{2x+5}{x-4}\right)$.

Solution: Using the properties, you can think of this problem as

$$\frac{\lim_{x\to 4}\left(2x+5\right)}{\lim_{x\to 4}\left(x-4\right)}.$$

In this problem, substituting 4 for *x* and evaluating would give a value of 0 in the denominator. Since division by zero is not defined, this limit does not exist.

So $\lim_{x \to 4} \left(\frac{2x+5}{x-4} \right)$ does not exist.

This will be true any time substitution gives a non-zero number in the numerator and zero in the denominator.

The graph of $f(x) = \frac{2x+5}{x-4}$ has a vertical asymptote at $x = 4$. This is one of the cases we saw earlier where a limit will fail to exist.

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Indeterminate Forms

If substitution results in $\frac{0}{0}$, then you cannot immediately determine the limit of the function. This is called an *indeterminate form* – the limit cannot be determined using the function as given.

You may be able to change the form of the function into one that agrees with the given function, except at the target number, and then apply the limit properties. Examples of two such methods are shown.

Example 6: Evaluate $\lim_{x \to -1} \left(\frac{x^2 + 3x + 2}{x+1} \right)$.

Solution: If you substitute -1 for *x* and evaluate, the result is $\frac{0}{0}$, so this is an indeterminate form.

Factor both the numerator and the denominator and see if you can simplify.

$$\lim_{x \to -1} \left(\frac{x^2 + 3x + 2}{x+1} \right) = \lim_{x \to -1} \left(\frac{(x+2)(x+1)}{x+1} \right)$$

Note the factor $x+1$ in both the numerator and the denominator. You can simplify as follows:

$$\lim_{x\to -1}\left(\frac{(x+2)(x+1)}{x+1}\right)=\lim_{x\to -1}(x+2)$$

Now you can substitute -1 for *x* to find the limit:

$$\lim_{x\to -1}(x+2)=-1+2=1$$

So $\lim_{x\to -1}\left(\frac{x^2+3x+2}{x+1}\right)=1$.

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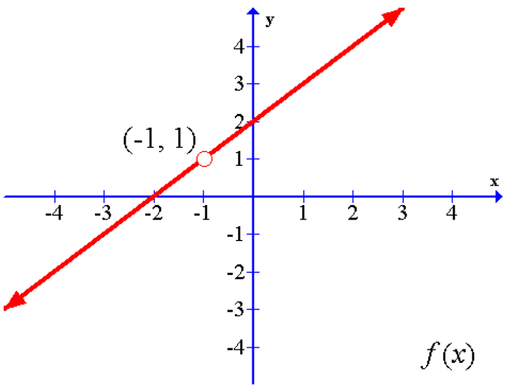
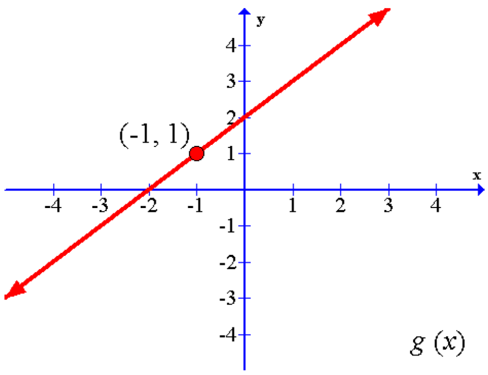
Why does this method work? Let's examine the functions $f(x)=\frac{x^2+3x+2}{x+1}$ and $g(x)=x+2$. The first function is what was given in the statement of the problem in Example 6, and the second function is the function you used to find the limit in that problem – the simplified version of $f(x)$. The graphs of the two functions are given here.

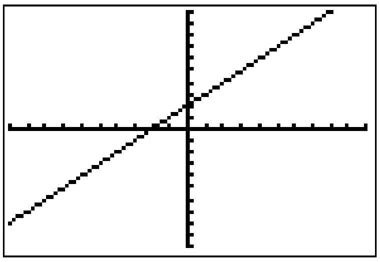




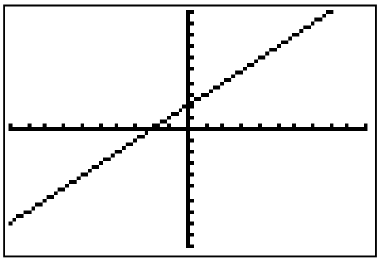
The two graphs are identical, except where $x=-1$. The graph of $g(x)$ contains the point (-1, 1); however, since $f(x)$ is undefined when $x=-1$, the graph of $f(x)$ does not contain a point with $x=-1$. From the graph of $f(x)$, you can see that the limit exists. To find the limit, you can use $g(x)$ to find the *y* value of the point, which is the limit.

You may not notice this difference between the two graphs if you use a graphing calculator to graph them both.

$f(x)$



$g(x)$



However, if you look at the table with values near $x = -1$ for each function, you'll see that the graph of $g(x)$ contains the point $(-1,1)$ while the graph of $f(x)$ does not. Note the ERROR message in the table for $f(x)$.

$f(x)$

X	Y1
-1.003	.997
-1.002	.998
-1.001	.999
-1	ERROR
-.999	1.001
-.998	1.002
-.997	1.003

X=-1.003

$g(x)$

X	Y2
-1.003	.997
-1.002	.998
-1.001	.999
-1	1
-.999	1.001
-.998	1.002
-.997	1.003

X=-1.003

Example 7: Evaluate $\lim_{x \to 3} \left(\frac{x^2 - 4x + 3}{x^2 - 9} \right)$.

Solution: If you substitute 3 for *x* and evaluate, the result is $\frac{0}{0}$, so this is an indeterminate form.

Factor both the numerator and the denominator and see if you can simplify.

$$\lim_{x \to 3} \left(\frac{x^2 - 4x + 3}{x^2 - 9} \right) = \lim_{x \to 3} \left(\frac{(x-3)(x-1)}{(x-3)(x+3)} \right)$$

Note the factor $x-3$ in both the numerator and the denominator. You can simplify as follows:

$$\lim_{x\to 3}\left(\frac{(x-3)(x-1)}{(x-3)(x+3)}\right)=\lim_{x\to 3}\left(\frac{x-1}{x+3}\right)$$

Now you can substitute 3 for *x* to find the limit:

$$\lim_{x\to 3}\left(\frac{x-1}{x+3}\right)=\frac{3-1}{3+3}=\frac{2}{6}=\frac{1}{3}$$

So $\lim_{x\to 3}\left(\frac{x^2-4x+3}{x^2-9}\right)=\frac{1}{3}$.

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Example 8: Evaluate $\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}$.

Solution: If you try to do this by substitution, the result will be the indeterminate form $\frac{0}{0}$. Factoring will not help to simplify this problem, so you'll need to use another method.

You'll use a technique called "rationalizing the numerator." In this problem, you'll multiply by $\frac{\sqrt{x+9}+3}{\sqrt{x+9}+3}$. Notice that $\sqrt{x+9}+3$ consists of the same two terms that are given in the numerator in the problem, but the two terms are added rather than subtracted.

$$\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}=\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}\cdot\frac{\sqrt{x+9}+3}{\sqrt{x+9}+3}$$
$$=\lim_{x\to 0}\frac{x+9+3\sqrt{x+9}-3\sqrt{x+9}-9}{x\left(\sqrt{x+9}+3\right)}$$
$$=\lim_{x\to 0}\frac{x}{x\left(\sqrt{x+9}+3\right)}$$
$$=\lim_{x\to 0}\frac{1}{\sqrt{x+9}+3}$$

Now substitute 0 for x and evaluate:

$$\lim_{x\to 0}\frac{1}{\sqrt{x+9}+3}=\frac{1}{\sqrt{0+9}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{3+3}=\frac{1}{6}$$

So $\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}=\frac{1}{6}$.

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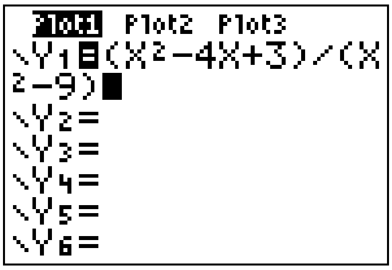
Finding Limits Using a Graphing Calculator

You can also use the table feature of a graphing calculator to help you determine a limit.

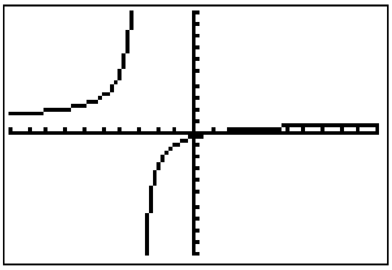
Example 9: Evaluate $\lim_{x\to 3}\left(\frac{x^2-4x+3}{x^2-9}\right)$.

Solution: First, use the graph of the function to determine if the limit exists.

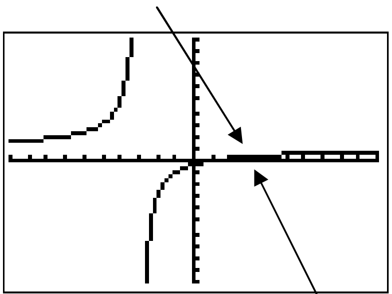
Enter the function as **y1** on the **Y=** screen of your calculator.



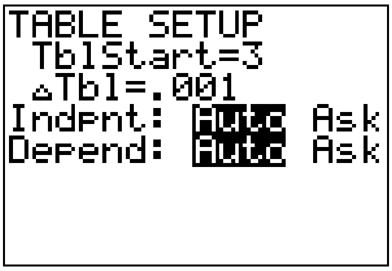
Now view the graph of the function.



Note that as $x \to 3$, $f(x)$ is approaching some number, although you can't tell right away what value $f(x)$ is approaching.



You can use the table to determine the limit. Press **2nd WINDOW** to set up an automatic table. Use the target number as your **TblStart** value, and choose a small value such as 0.001 as your Δ **Tbl** value. For an automatic table, both the independent and dependent variables should be set to **Auto**.



Now view the table, by pressing **2nd GRAPH**.

X	Y1	
3	ERROR	
3.001	.33344	
3.002	.33356	
3.003	.33367	
3.004	.33378	
3.005	.33389	
3.006	.334	

X=3

Scroll up, so that the target number is in the middle of the list of values for *x*. Then look at the values for **y1**.

X	Y1	
2.997	.333	
2.998	.33311	
2.999	.33322	
3	ERROR	
3.001	.33344	
3.002	.33356	
3.003	.33367	

X=2.997

As the values for *x* approach the target number, the values for $f(x)$ are approaching the number 0.333… . Convert this to a fraction to find that $\lim_{x \to 3}\left(\frac{x^2-4x+3}{x^2-9}\right)=\frac{1}{3}$.

This is the same answer that you found algebraically in Example 7.

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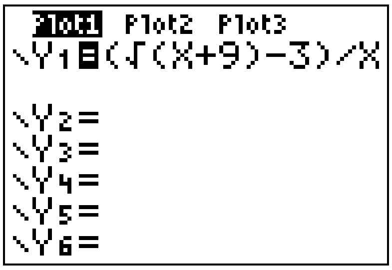
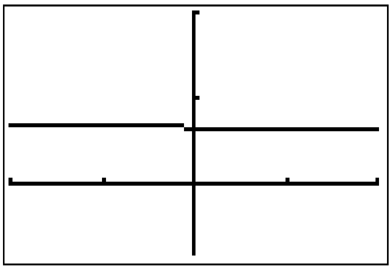
Example 10: Evaluate $\lim_{x \to 0}\frac{\sqrt{x+9}-3}{x}$.

Solution: Use the graph of the function to determine if the limit exists.

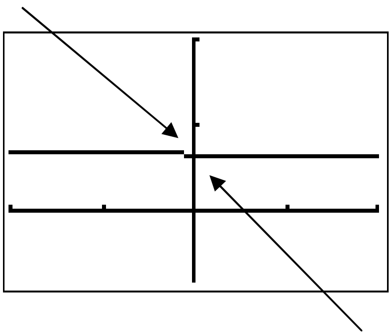
Enter the function as **y1** on the **Y=** screen of your calculator.



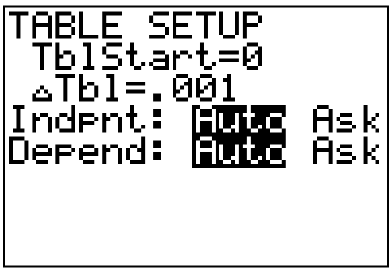
Now view the graph of the function. Use the window [-1, 1, -0.2, 0.5].



Note that as $x \to 0$, $f(x)$ is approaching some number, although you can't tell right away what value $f(x)$ is approaching.



Now press **2nd WINDOW** to set up an automatic table. Use the target number as your **TblStart** value, and choose a small value such as 0.001 as your Δ**Tbl** value. For an automatic table, both the independent and dependent variables should be set to **Auto**.



Now view the table, by pressing **2nd GRAPH**.

X	Y1	
0	ERROR	
.001	.16666	
.002	.16666	
.003	.16665	
.004	.16665	
.005	.16664	
.006	.16664	

X=0

Scroll up, so that the target number is in the middle of the list of values for *x*. Then look at the values for **y1**.

X	Y1	
-.003	.16668	
-.002	.16668	
-.001	.16667	
0	ERROR	
.001	.16666	
.002	.16666	
.003	.16665	

X=-.003

As the values for *x* approach the target number, the values for $f(x)$ are approaching the number 0.16666… . Convert this to a fraction to find that

$$\lim_{x \to 0} \frac{\sqrt{x+9}-3}{x} = \frac{1}{6}.$$

This is the same answer that you found algebraically in Example 8.

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