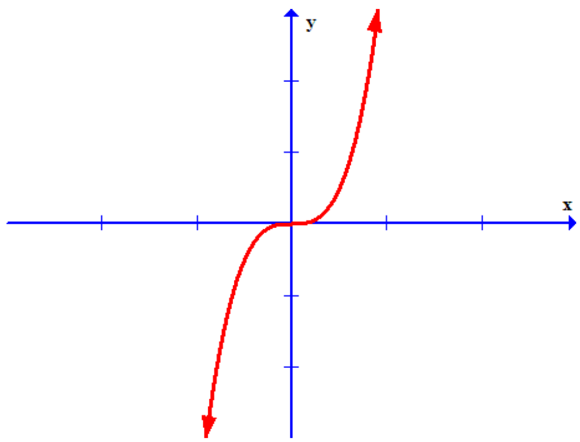
Section 1.5 – Analyzing of Functions

In this section, we'll look briefly at four types of functions: polynomial functions, rational functions, exponential functions and logarithmic functions.

Example 1: What do you know about the graph of the function $f(x) = x^3 - 9x$?

Solution: We know that it is a polynomial function with odd degree and a positive leading coefficient. Its basic shape is like this:



This shows that as x gets bigger, the y values will get bigger and the graph will be in the first quadrant. As x moves towards $-\infty$, (really big, but negative), the y values will do the same, and the graph will be in the third quadrant. Recall that cubic functions often have "humps" in the graph.

You should know the basic shapes of positive quadratic functions, negative quadratic functions, positive cubic functions, negative cubic functions, positive quartic (4th degree) functions and negative quartic (4th degree) functions.

Next, we can find the zeros (x intercepts) of the function. To do this, we set the function equal to 0 and solve for x.

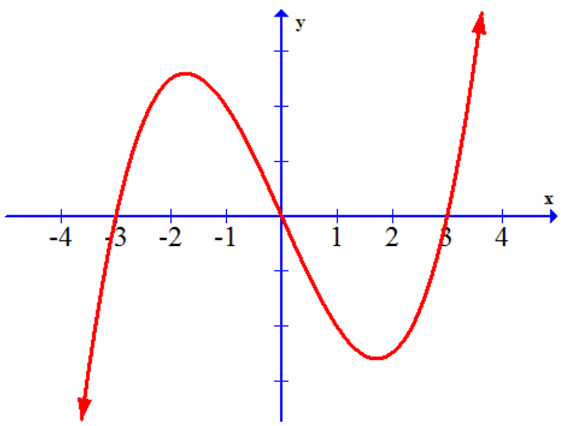
$$0 = x^3 - 9x$$
$$0 = x(x-3)(x+3)$$
$$x = 0, \quad x = 3, \quad x = -3$$

We have three x intercepts: $(0,0),(3,0)$ and $(-3,0)$.

See Section 1.2 for help with factoring.

We can also find the y intercept of the function. $f(0)=0^2-9(0)=0$, so the graph passes through the origin.

With no more information than this, we can graph the function. Plot the x and y intercepts. Use the shape information to fill in the rest.



You will not know how high or low the "humps" need to be, but you can draw a fairly accurate graph just from this information. Here, we generated a graph, but with the information we found you should be able to pick out the correct graph from a selection of graphs.

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Example 2: Suppose $f(x)=\dfrac{x^2-3x+2}{x^2-9}$. Find the x intercept(s), y intercept(s), vertical asymptote(s) and horizontal asymptote(s).

Solution: This is a rational function, as it is a function of the form $f(x)=\dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are both polynomials. To start, factor everything that can be factored:

$$f(x)=\frac{(x-2)(x-1)}{(x-3)(x+3)}$$

Recall that if you have a common factor in the numerator and the denominator, you can reduce by that factor. This will generate a hole in the graph. We don't have that here, so the graph of the function has no holes.

To find the x intercepts, set $f(x)$ equal to 0 and solve for *x*.

$$0 = \frac{(x-2)(x-1)}{(x-3)(x+3)}$$

Multiply both sides of the equation by the denominator to clear the problem of fractions. We have:

$$\begin{aligned}(x-2)(x-1) &= 0\\ x-2=0, &\quad x-1=0\\ x=2, &\quad x=1\end{aligned}$$

We have two *x* intercepts, $(2,0)$ and $(1,0)$.

To find the y intercept, compute $f(0)$. In this case,

$$f(0) = \frac{(0-2)(0-1)}{(0-3)(0+3)} = \frac{(-2)(-1)}{(-3)(3)} = \frac{2}{-9} = -\frac{2}{9}$$

The y intercept is $\left(0, \frac{-2}{9}\right)$.

We could also have easily found the *y* intercept by looking at the original problem. Sub in 0 for *x* in that function, and you can probably find the *y* intercept in your head.

By the way, you may be asked to find the *y* intercept(s), implying that there can be more than one. Don't be fooled. If there are two or more *y* intercepts for your graph, it's not a function. We'll only deal with functions in this class.

To find the vertical asymptote(s), set the denominator equal to 0 and solve for *x*. Note, if you have a common factor in the numerator and the denominator (which we *don't* have here), you will reduce the fraction before finding any vertical asymptote(s).

$$(x-3)(x+3)=0$$
$$x-3=0, \quad x+3=0$$
$$x=3, \quad x=-3$$

These are the two vertical asymptotes.

To find the horizontal asymptote, you will need to compare the highest power of *x* in the numerator with the highest power of *x* in the denominator. This number is called the degree of the function. There are three possibilities:

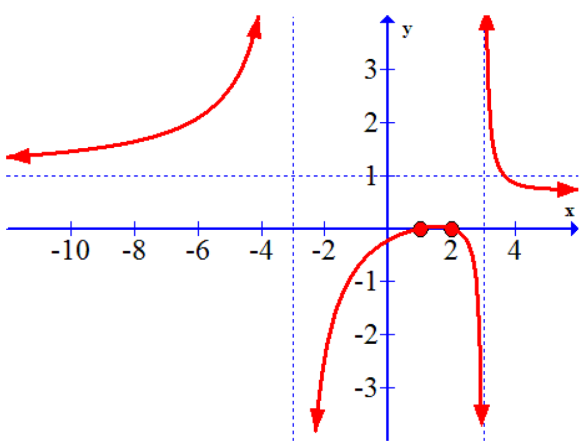
Degree of numerator > Degree of denominator. In this case, there is no horizontal asymptote.

Degree of numerator < Degree of denominator. In this case, the horizontal asymptote is 0. We write this as $y = 0$.

Degree of numerator = Degree of denominator. In this case, we make a fraction out of the coefficients of the terms with the highest power and reduce it to lowest terms. This is the horizontal asymptote.

Our problem falls into the third category, since we have x^2 in both the numerator and the denominator. Both coefficients are 1, so our horizontal asymptote is $y = 1$.

We can use all of this information to graph the function.



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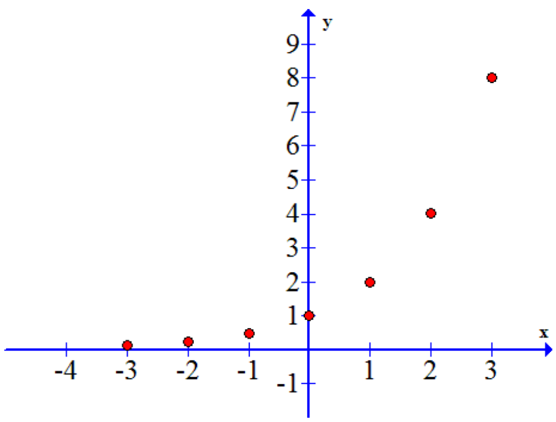
Exponential Functions

An exponential function is a function of the form $f(x) = a \cdot b^x$. Note that the variable is in the exponent. You should be able to recognize and draw the graph of an exponential function.

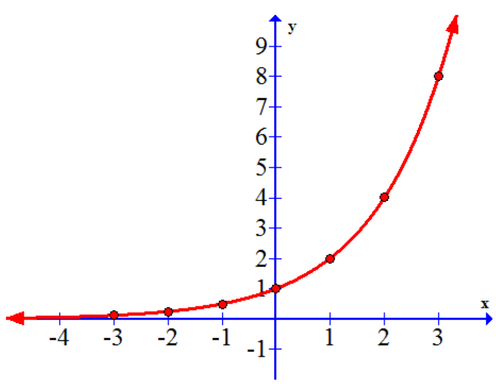
Example 3: Sketch $f(x) = 2^x$.

To sketch this, you can use a table of values, then plot the points. However, you should be able to draw a rough sketch from memory.

x	-3	-2	-1	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



Connect the points with a smooth curve.



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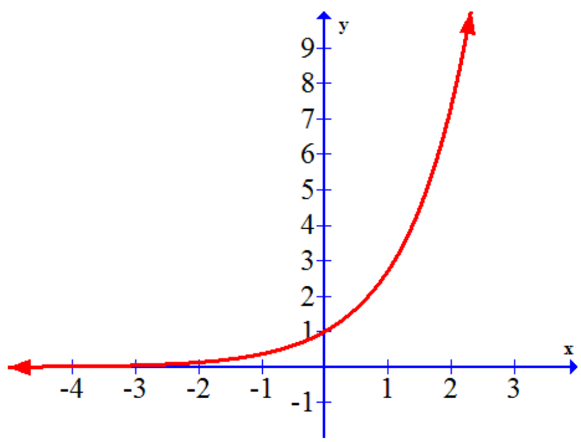
Recall that all basic exponential functions have this same shape. So any function of the form $f(x) = a^x$, $a > 1$, will basically look like the graph shown in Example 3.

Often the function we'll work with will involve the natural exponential function, $f(x) = e^x$. The base of this function is the number *e*, is the value that the expression $\left(1+\frac{1}{n}\right)^n$ approaches as you let *n* get bigger and bigger. This idea is called a limit and we will study them in this course. Note that $e \approx 2.718281828...$ You will find an e^x key on your calculator. (It is exp( ) on the online calculator.) It is sometimes convenient to remember that $e \approx 3$.

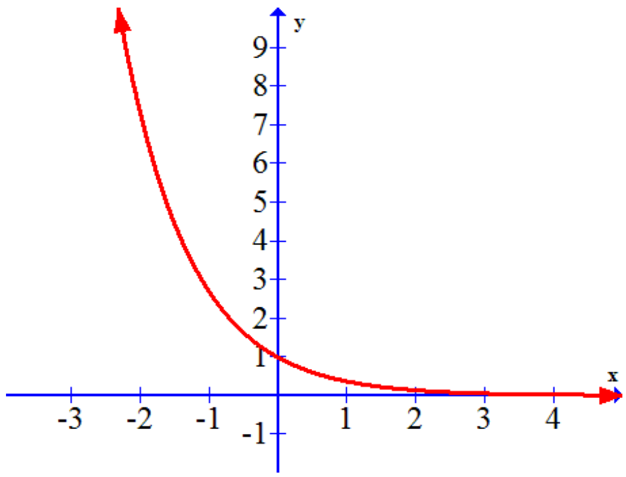
Since $e \approx 3$, $f(x) = e^x$ is an exponential function of the form $f(x) = a^x$, $a > 1$. Compare the graph of $f(x) = e^x$, shown below, with the graph of $f(x) = 2^x$ shown on the previous page.



There are some important features of this function:

Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Horizontal asymptote: as $x \to -\infty$, $f(x) \to 0$.
Passes through the point $(0,1)$

Any function of the form $f(x) = a^x$, $0 < a < 1$ will be a reflection of the above graph over the y axis:



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Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Horizontal asymptote: as $x \to \infty$, $f(x) \to 0$.
Passes through the point $(0,1)$

You should be able to solve exponential equations.

Example 4: Solve for *x*: $4^{x+4} = 32$

Solution: We can write both sides of the equation with the same base, 2. Since $4 = 2^2$ and $32 = 2^5$, we have $\left(2^2\right)^{x+4} = 2^5$. Solve this equation for x.

$$\left(2^2\right)^{x+4} = 2^5$$
$$2^{2x+8} = 2^5$$
$$2x + 8 = 5$$
$$2x = -3$$
$$x = \frac{-3}{2}$$

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Now we'll turn our attention to logarithms.

A logarithmic function is the inverse of an exponential function. So if we have $y = 2^x$, the inverse of this function is $x = 2^y$.

This function is inconvenient to deal with. Logarithms help us with this and allow us to rewrite the second function as $y = \log_2 x$. Now we can choose values for *x*, and evaluate the function to find *y*. If we choose carefully, we'll get integers.

The first thing you should be able to do is evaluate a logarithm. You'll use the definition of a logarithm to do this. The definition of a logarithm says that $y = \log_b x$ if and only if $b^y = x$. Note that $b > 0, b \neq 1$ and $x > 0$.

So, *y* is the power that you must raise *b* to if you want to obtain *x*.

Example 5: Evaluate $\log_3 81$.

Solution: We can rewrite this as $\log_3 81 = y$. We want to find *y*.

Use the definition of a logarithm: $3^y = 81$

Rewrite the right hand side as 3^4. We have $3^y = 3^4$, so $y = 4$.

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Example 6: Evaluate $\log(3000)$.

Solution: Note that no base is given in the problem. In this case, the base of the logarithm is 10. This is called the common logarithm. In this problem we want to find *x* in the problem $10^x = 3000$. We cannot write both sides of the equation using the same base. We can, however, use a calculator to find a decimal approximation. Locate the "log" key on your calculator. Type in $\log(3000)$ and press enter to find the answer: $\log(3000) \approx 3.4771$. You can check this by computing $10^{3.4771}$ using your calculator. Remember that 3.4771 is a decimal approximation, so when you check, you will not get exactly 3000.

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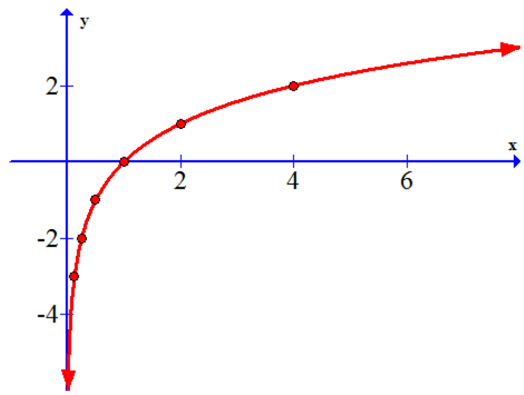
Example 7: Graph $f(x) = \log_2 x$

Solution: Start by filling in a table of values. In this case, the values for *x* have been strategically chosen to given rational answers (i.e., fractions).

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$f(x)$	-3	-2	-1	0	1	2	3



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All functions of the form $f(x) = \log_b x, b > 1$ will have the same basic shape.

All functions of the form $f(x) = \log_b x, 0 < b < 1$ will be the reflection of the above graph over the *x* axis.

Some features of basic logarithmic graphs:

Domain is $(0, \infty)$

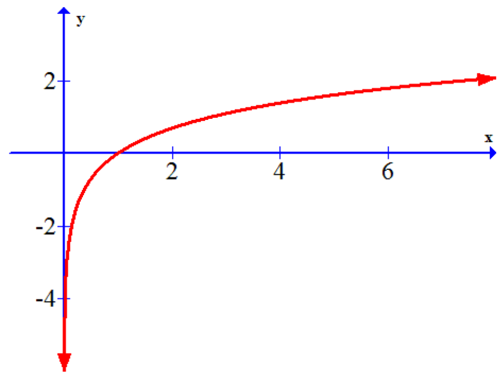
Range is $(-\infty, \infty)$

Graph passes through the point (1, 0)

The function is increasing on $(0, \infty)$ if $b > 1$, and it's decreasing on $(0, \infty)$ if $0 < b < 1$.

You will work with the natural logarithmic function most of the time in this course. This is the function $f(x) = \log_e x$, so it's a log function whose base is *e*. This function is so important in mathematics that it has its own notation: $f(x) = \ln x$.

The graph of $f(x) = \ln x$ is shown below. Note the similarities between the graph of this function and the graph of $f(x) = \log_2 x$.



There is an "ln" key on handheld calculators which will be useful.

Example 8: Evaluate $\ln 25$.

Solution: We'll work this problem in a similar manner as we worked Example 6, and for the same reason. The problem is asking us to find x that will solve the equation $e^x = 25$. Using the ln key on a calculator, we find that $\ln 25 \approx 3.219$.

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Properties of logarithms are very useful in calculus.

Here are the log properties you need to know:

Suppose *M*, *N* and *b* are positive real numbers, $b \neq 1$, and *r* is any real number. Then:

1. $\log_b MN = \log_b M + \log_b N$
2. $\log_b \frac{M}{N} = \log_b M - \log_b N$
3. $\log_b M^r = r \log_b M$

Example 9: Expand using log properties: $\log[(x-3)(x+1)]$

Solution: Using property 1, $\log[(x-3)(x+1)] = \log(x-3) + \log(x+1)$. Note that in this case, b = 10.

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Example 10: Expand using log properties: $\ln\left(\frac{x^4}{y^2}\right)$

Solution: We'll use properties 2 and 3. $\ln\left(\frac{x^4}{y^2}\right) = \ln x^4 - \ln y^2 = 4\ln x - 2\ln y$.

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Example 11: Expand using log properties: $\log\left[\frac{x^3}{(x+1)^2(x-5)^4}\right]$.

Solution: We'll use all 3 properties.

$$\begin{aligned}
\log\left[\frac{x^3}{(x+1)^2(x-5)^4}\right] &= \log x^3 - \log[(x+1)^2(x-5)^4] \\
&= \log x^3 - \left[\log\left(x+1\right)^2 + \log\left(x-5\right)^4\right] \\
&= 3\log x - \left[2\log(x+1) + 4\log(x-5)\right] \\
&= 3\log x - 2\log(x+1) - 4\log(x-5)
\end{aligned}$$

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Example 12: Express as a single logarithm: $\log_2 7 - \log_2 14$

Solution: Reverse the process from the previous examples.

$$\begin{aligned}
\log_2 7 - \log_2 14 = \log_2 \frac{7}{14} &= \log_2 \frac{1}{2} \\
&= \log_2 2^{-1} \\
&= -1
\end{aligned}$$

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There are other properties of logarithms that you need to remember:

4. $\ln e^x = x$

5. $e^{\ln x} = x$

6. $\ln e = 1$

7. $\ln 1 = 0$

Example 13: Expand and simplify using log properties: $\ln\left(2x^3e^{4x}\right)$.

Solution: Using property 1, $\ln\left(2x^3e^{4x}\right) = \ln 2 + \ln x^3 + \ln e^{4x}$

Using property 3, $\ln 2 + \ln x^3 + \ln e^{4x} = \ln 2 + 3\ln x + 4x\ln e$

Using property 6, since $\ln e = 1$, we have $\ln 2 + 3\ln x + 4x\ln e = \ln 2 + 3\ln x + 4x$.

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Logarithms can help you solve equations.

Example 14: Solve for *x*: $2^x = 10$

Solution: We can't write both sides of this equation using the same base. However, we can rewrite it using the definition of a logarithm. Remember, $\log_b y = x$ if and only if $b^x = y$. We can rewrite this problem as $\log_2 10 = x$. This may be the correct answer. You can also use the change of bases formula to rewrite this as $\log_2 10 = \frac{\log 10}{\log 2}$. You can use a calculator to find a decimal approximation. $\log_2 10 = \frac{\log 10}{\log 2} \approx 3.322$.

You can also solve this equation by taking the common log (base 10) or the natural log of both sides of the equation. Then use log properties to simplify and solve.

$$
\begin{aligned}
\log 2^x &= \log 10 \\
x \log 2 &= \log 10 \\
x &= \frac{\log 10}{\log 2} \approx 3.322
\end{aligned}
$$

Using the natural log instead of the common log yields the same result.

$$
\begin{aligned}
\ln 2^x &= \ln 10 \\
x \ln 2 &= \ln 10 \\
x &= \frac{\ln 10}{\ln 2} \approx 3.322
\end{aligned}
$$

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Example 15: Solve for *x*: $2\ln(x-3)=8$

Solution: Start by dividing both sides of the equation by 2.

$$\ln(x-3)=4$$

Now use the definition of a logarithm to rewrite this in exponential form.

$$
\begin{aligned}
e^4 &= x-3 \\
e^4+3 &= x \\
57.598 &\approx x
\end{aligned}
$$

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