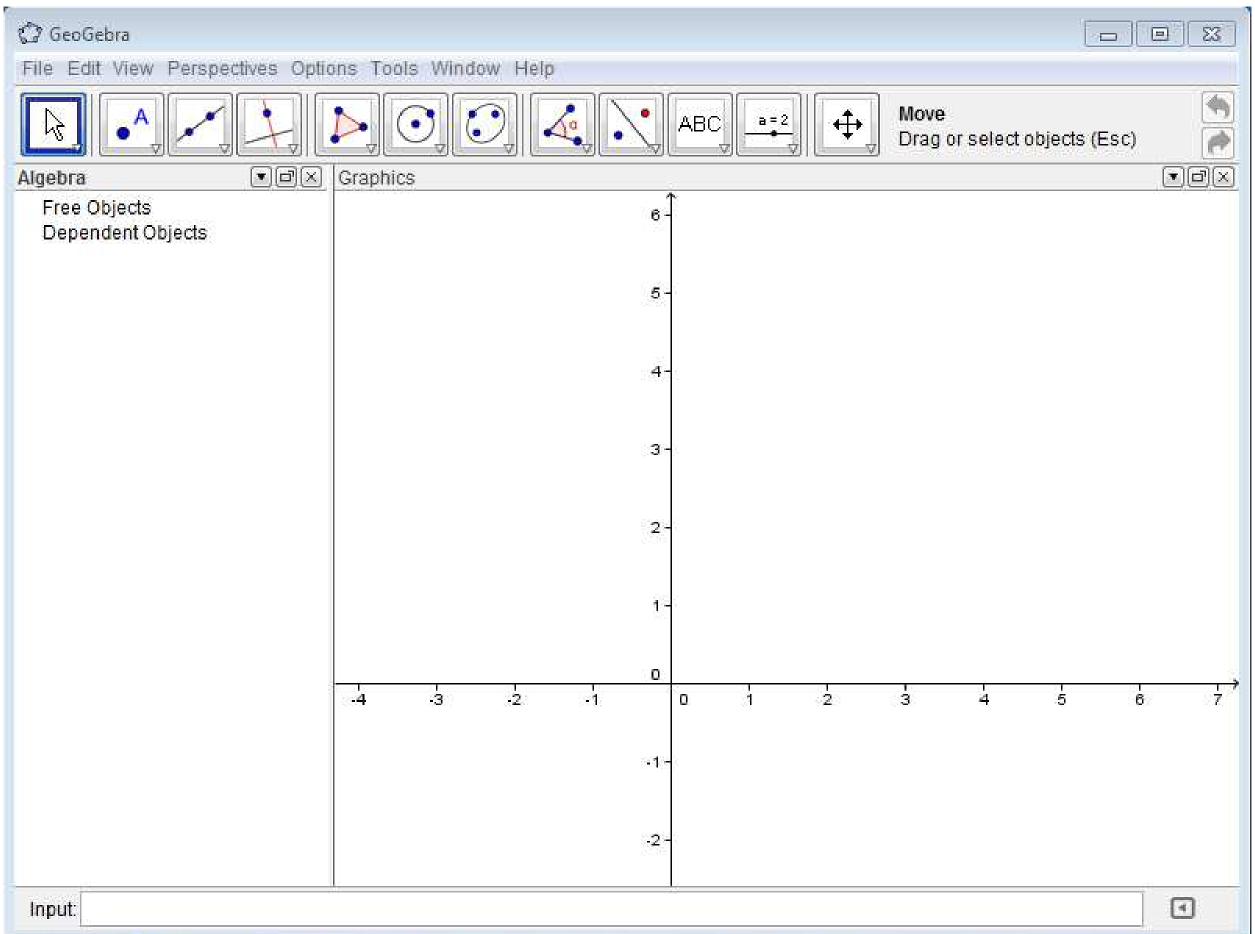
Appendix B – Using Other Technologies

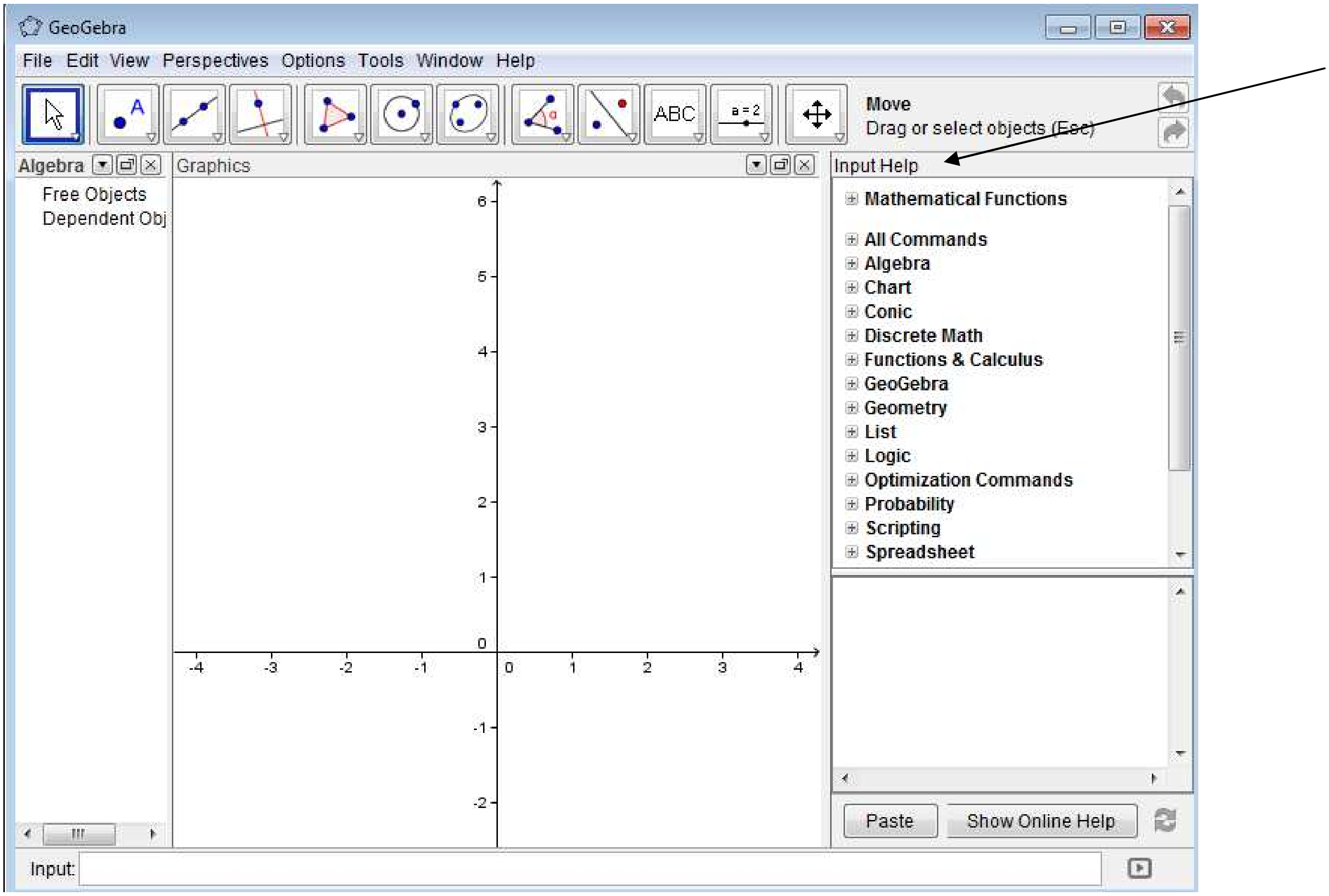
Section 3: GeoGebra

So far, we have looked at the features of a graphing calculator and at Excel as aids in working problems in this course. There are many other software packages available online to assist students who are taking calculus. In this section, the focus will be on a free software called GeoGebra. GeoGebra is a graphing utility and more. The URL for the software is www.geogebra.org. There you will find a link to download the software and links to some help materials. This section shows version 4.0.18.0 of GeoGebra as displayed on a PC. GeoGebra can be downloaded on a Mac and working with it on a Mac is the same as on a PC. GeoGebra is not yet available as a smartphone or iPad application.

When you open GeoGebra, this is what you will see.



The display has an algebra window on the left and a graphing area on the right. At the bottom is the input line. On the bottom at the right, you'll find an arrow in a box. Click on it to display **Input Help**.

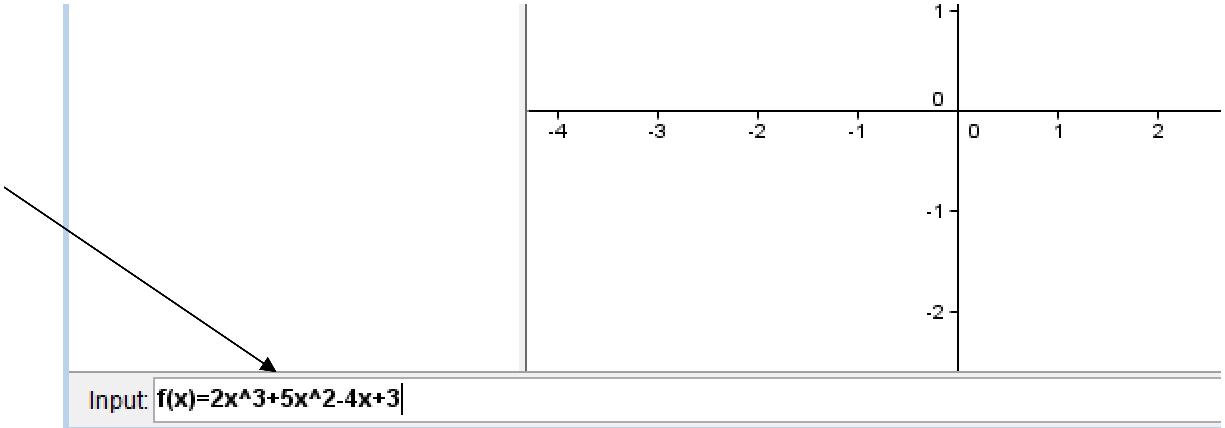


You work with GeoGebra using commands. **Input Help** organizes commands by topic and that can help you find the command you need if you don't remember its name. If you aren't sure where your command might be located, select **All Commands** and scroll down until you find it.

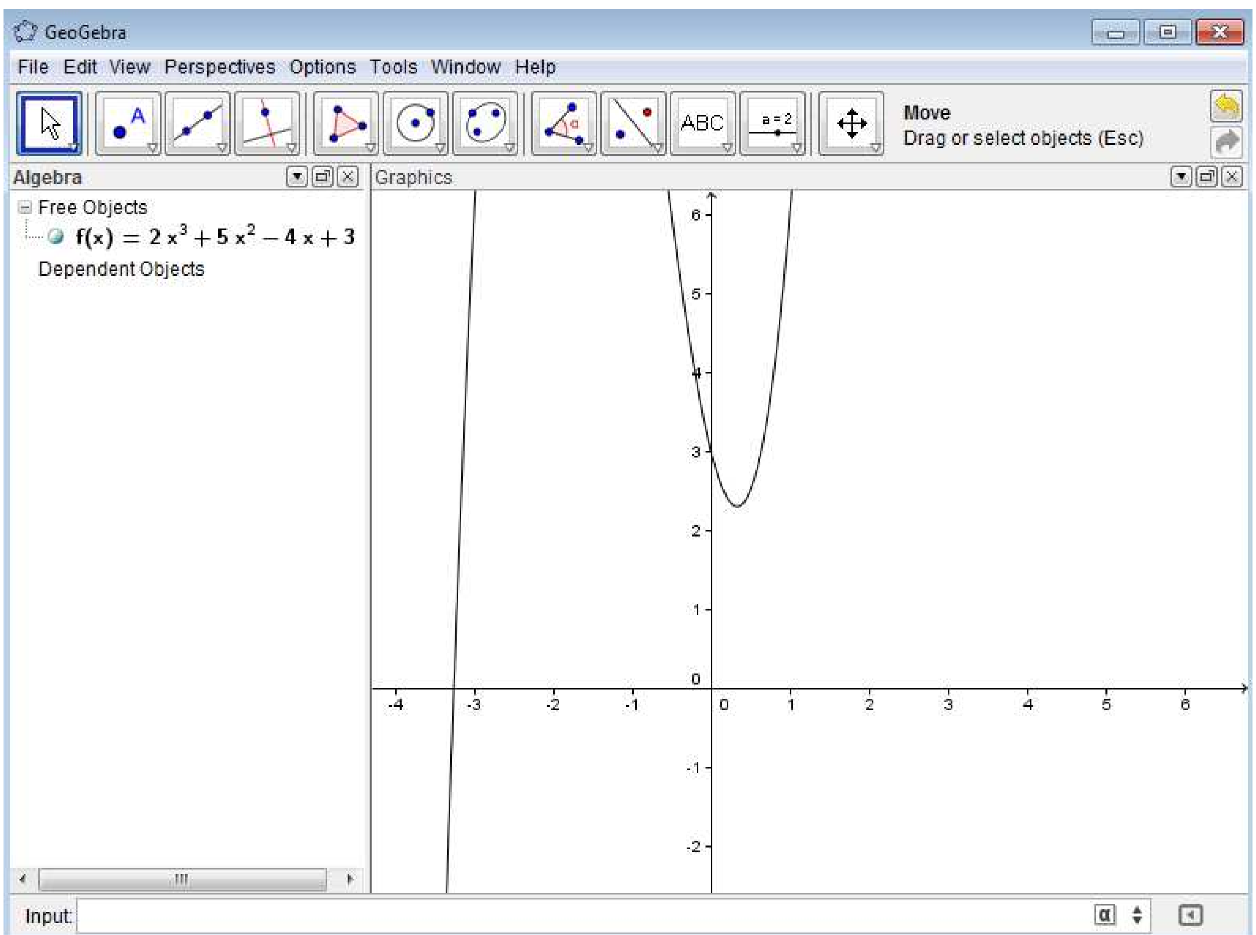
One of the advantages of GeoGebra is that graphing is easier. Instead of hunting for an appropriate window, you can just grab the axes and pull until you get a good view.

Example 1: Enter the function $f(x) = 2x^3 + 5x^2 - 4x + 3$ using GeoGebra, and format the graph so that you have can see all zeros and extrema of the function.

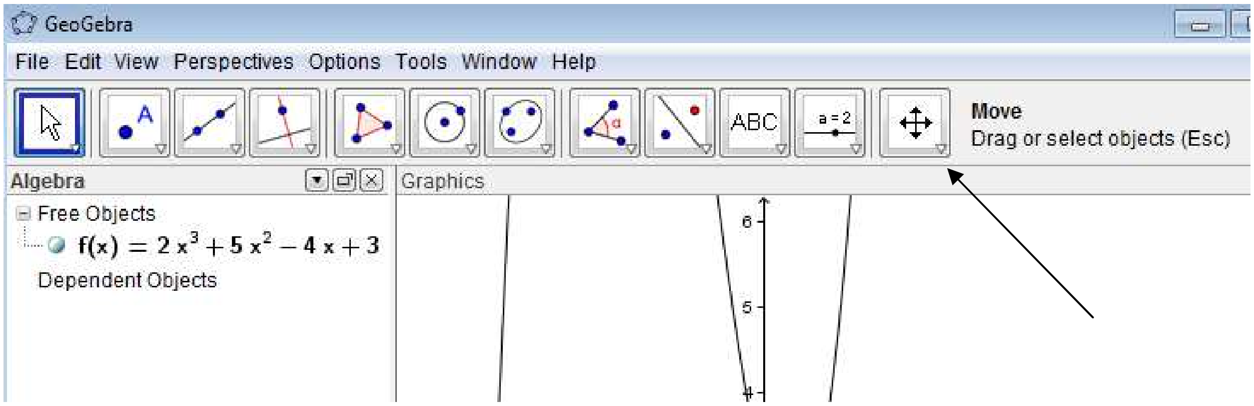
Solution: Type the equation into the input line.



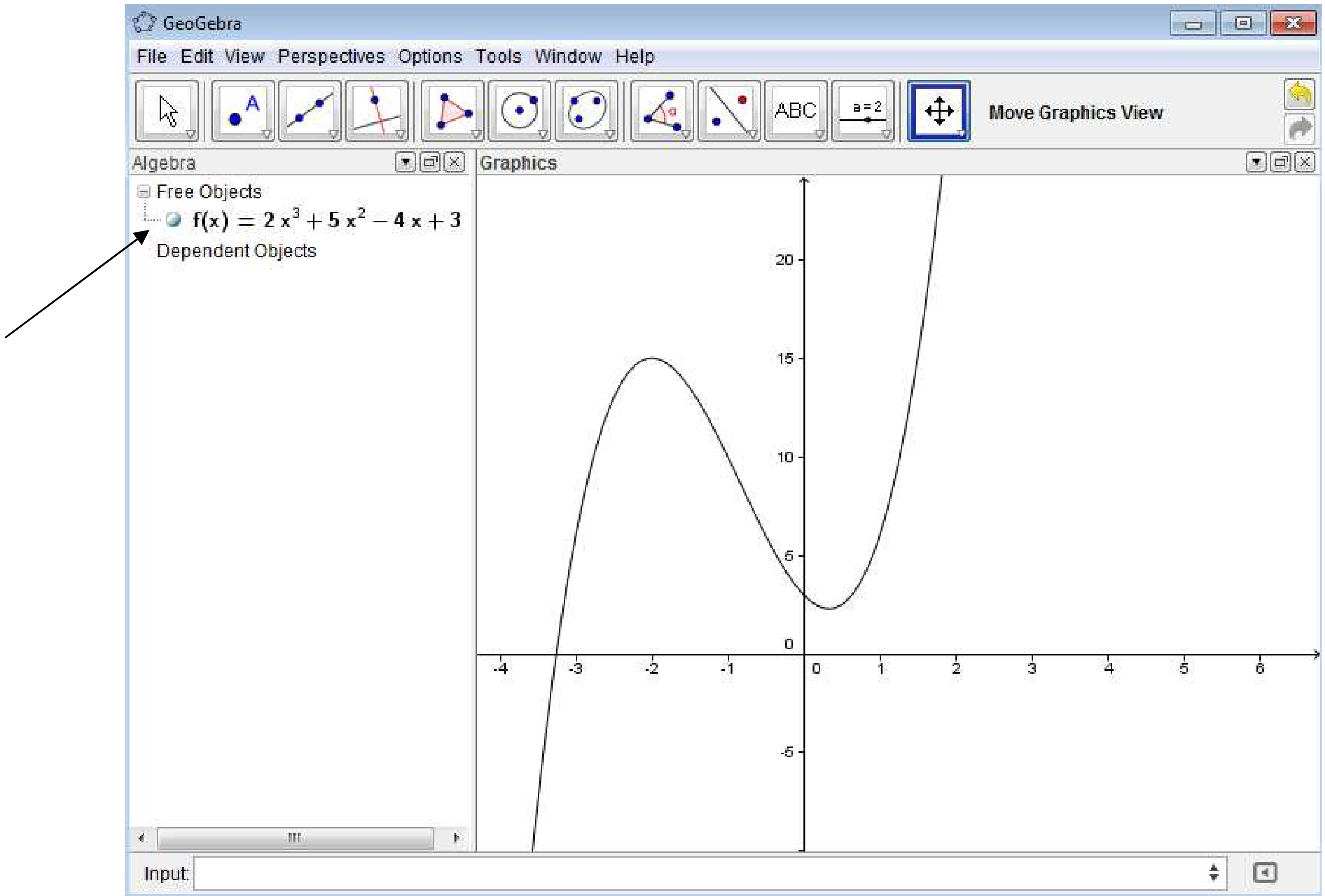
Press **Enter** on your keyboard to view the graph.



This does not give a good view of the function. There is a relative maximum in Quadrant 2, and we can't see it. To get a good view of the graph, we need more positive *y* values. Click on the last icon in the list of icons near the top of the screen.



Then put the cursor near the *y* axis somewhere between Quadrants 1 and 2 and pull down towards the origin. Keep pulling down until you have a good view of the graph of the function. The *y* axis labels you see may vary depending on how far down you pull the graph.



Notice that the function appears in the algebra list to the left of the graph as a "Free Object."

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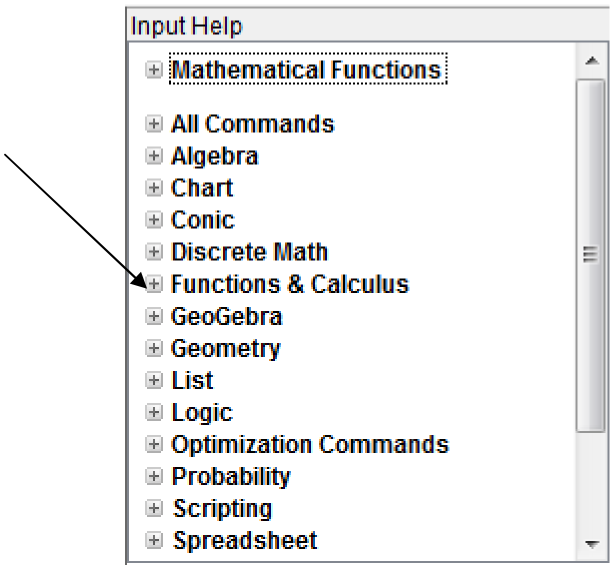
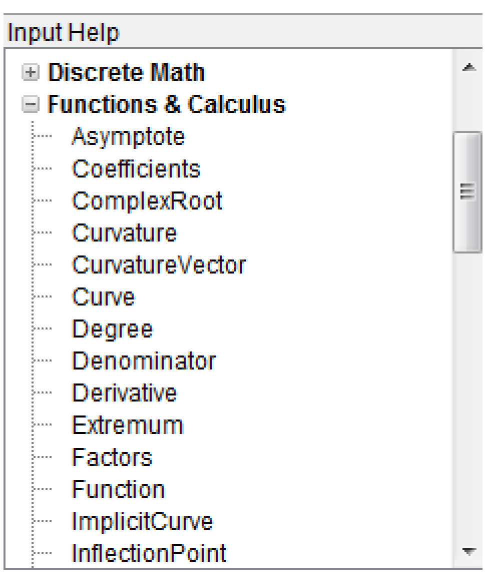
Now that you have a graph of a function to work with, you can find relevant points on the graph of the function, such as the zero(s), relative extrema and inflection points. When using GeoGebra, you will need to know the names that the program has assigned to the value you wish to find. Often these will be different that the names that are given in class or in the text. For example, if you want to find the zero of a function, you won't find "Zero" given in the list of commands. Geogebra calls zeros of a function "Roots."

Example 2: Suppose $f(x) = 2x^3 + 5x^2 - 4x + 3$. Find the zero of the function. Then find the relative maximum, relative minimum and inflection point of the graph of the function.

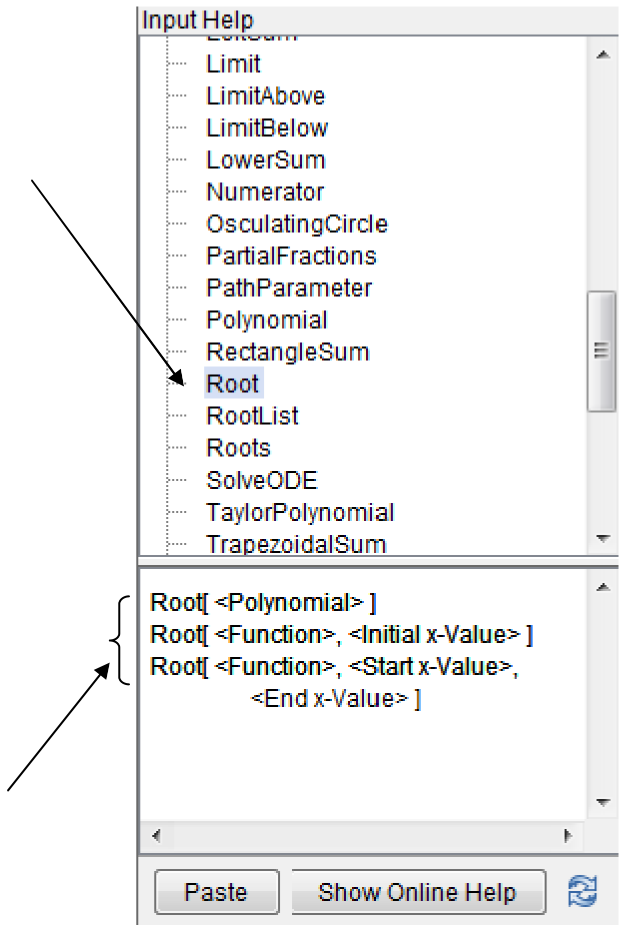
Solution: We can find these values using the list of commands in the **Input Help** box. We'll find all of these in the list under **Functions & Calculus**. Click on the plus sign to the left of the title to display the list.



Here is the result. You can scroll up and down the list using the scroll bar on the right side of the box. You can click on the little minus sign that now appears to the left of **Functions & Calculus** if you want to collapse the list.

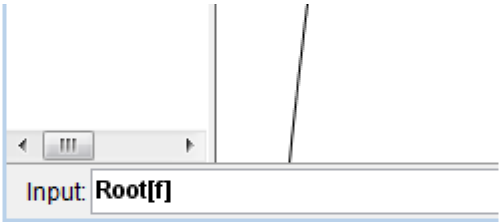


To find the zero of the function, scroll down the list of commands until you see **Root**, click on **Root**, and all of the possibilities for the **Root** command will appear in the box below the list.

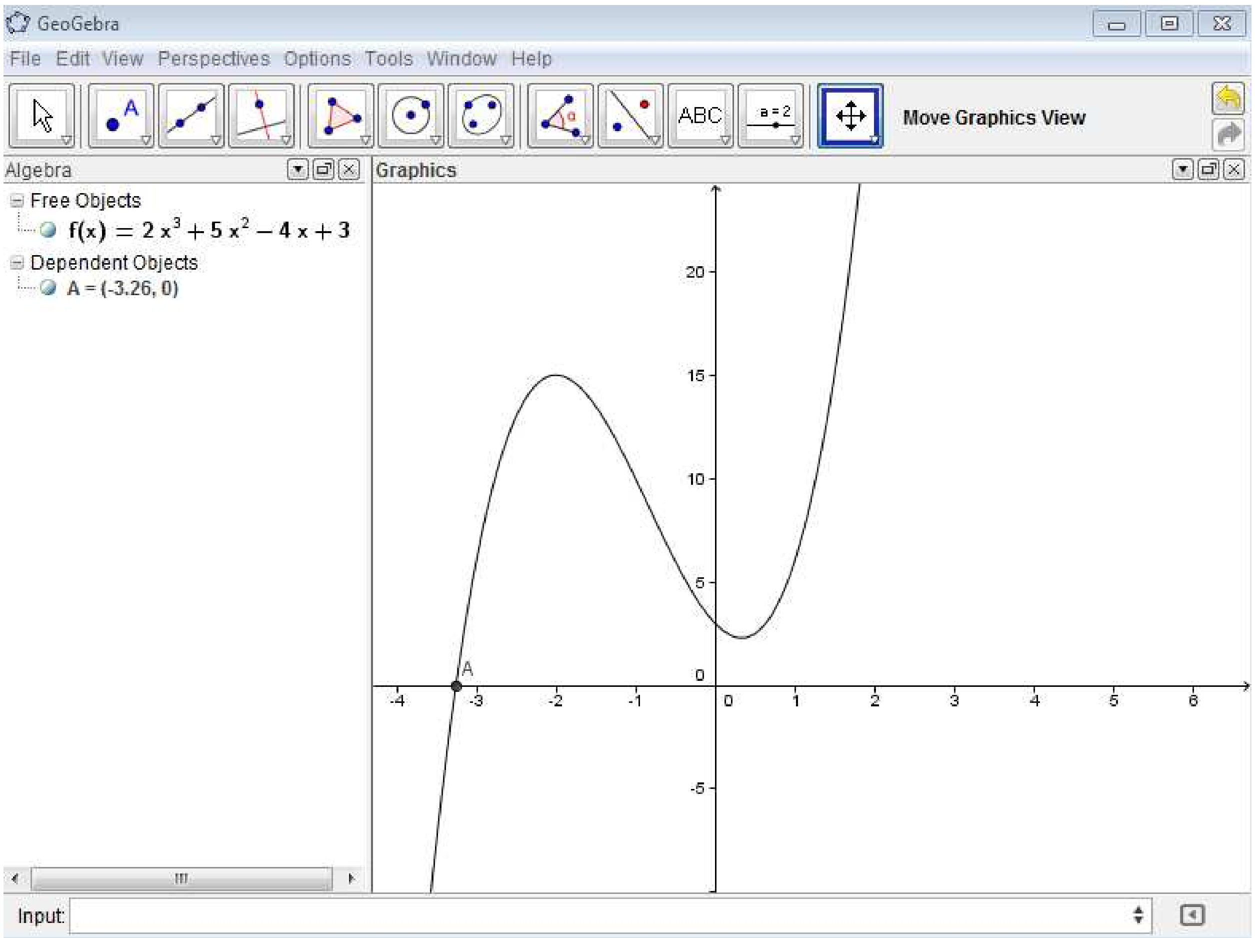


Click on **Paste** to display the **Root** command in the Input line.

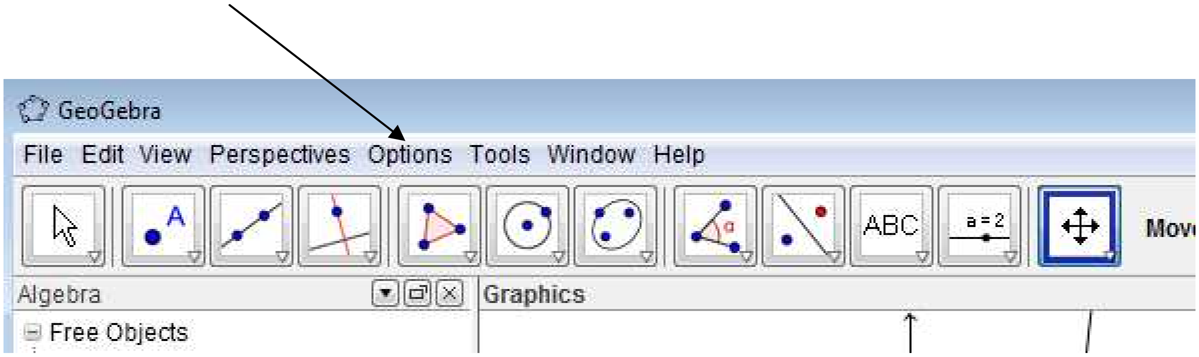
The red brackets indicate that you need to enter more information, called the arguments. You can find the roots of either a polynomial function or any general function. In this example, you want the roots of a polynomial function, so you only need to type in the letter name for the function, in this case, *f*, for $f(x)$. You'll see what to do for other types of functions in later examples.



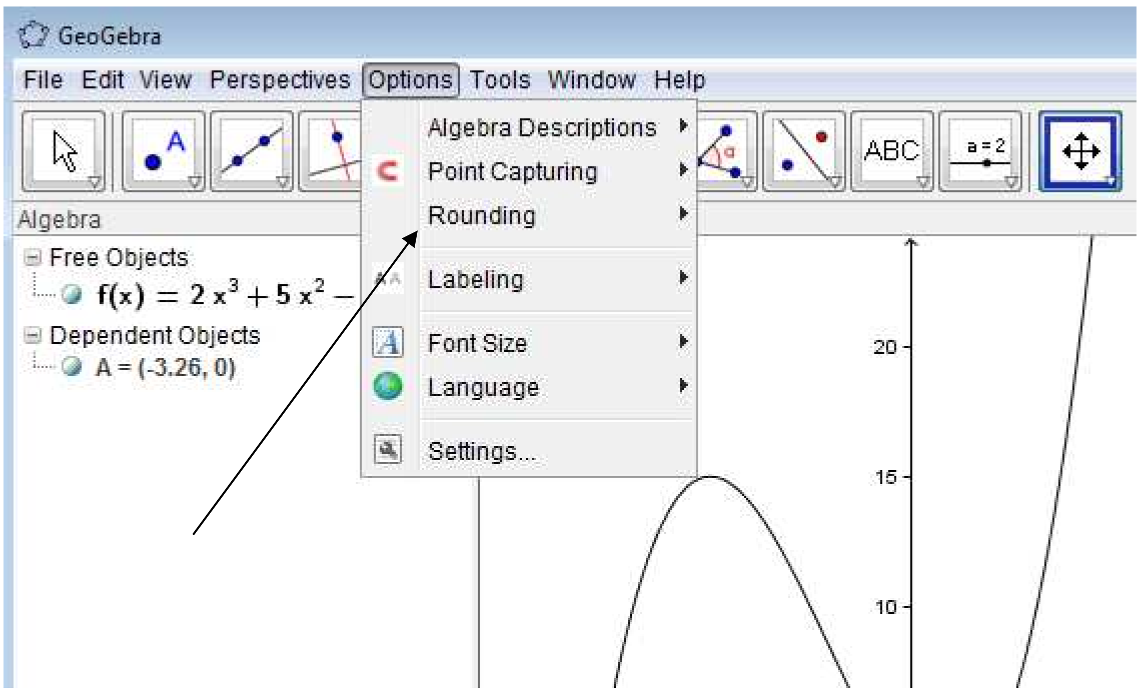
Now press **Enter**, and the program will display the values for the roots, or zeros, of the function. You can collapse the **Input Help** menu by clicking on the arrow, so that you have a larger display for the graph.



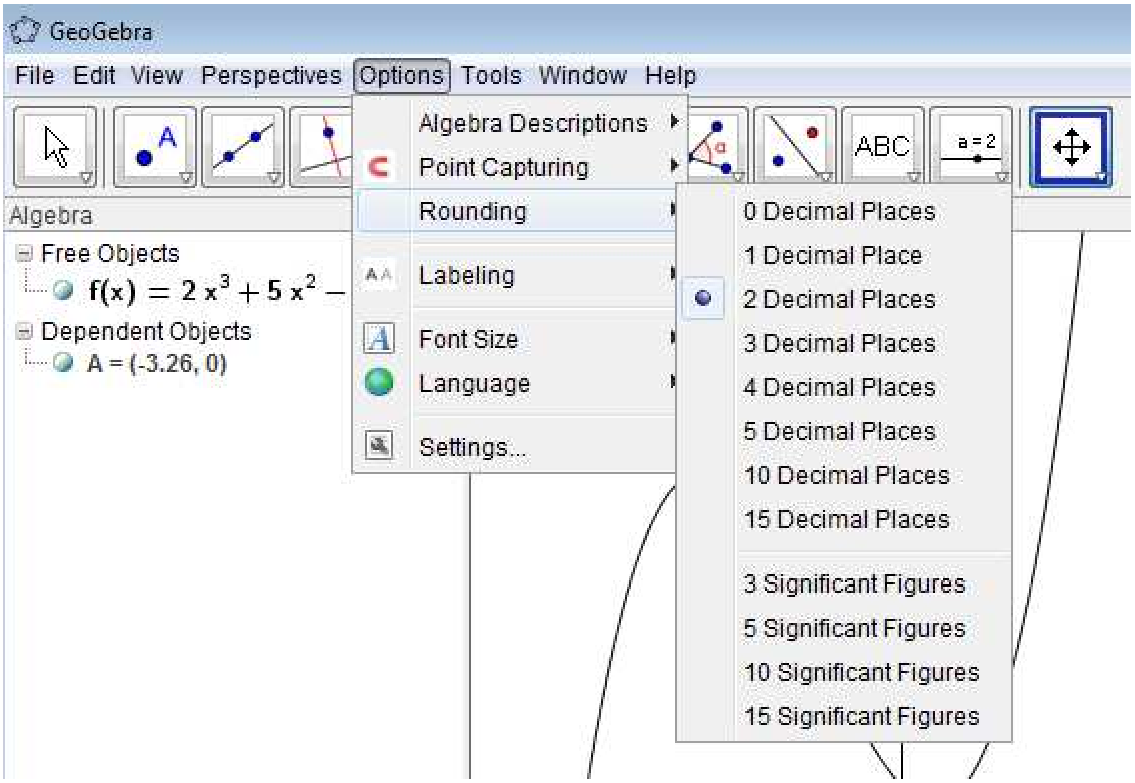
Note that the zero is listed as a dependent object in the algebra window and is plotted on the graph of the function. If you want or need to display more decimal places, you can change from the default of two decimal places to 3 or 4 or more. Click on **Options** in the menu at the top.



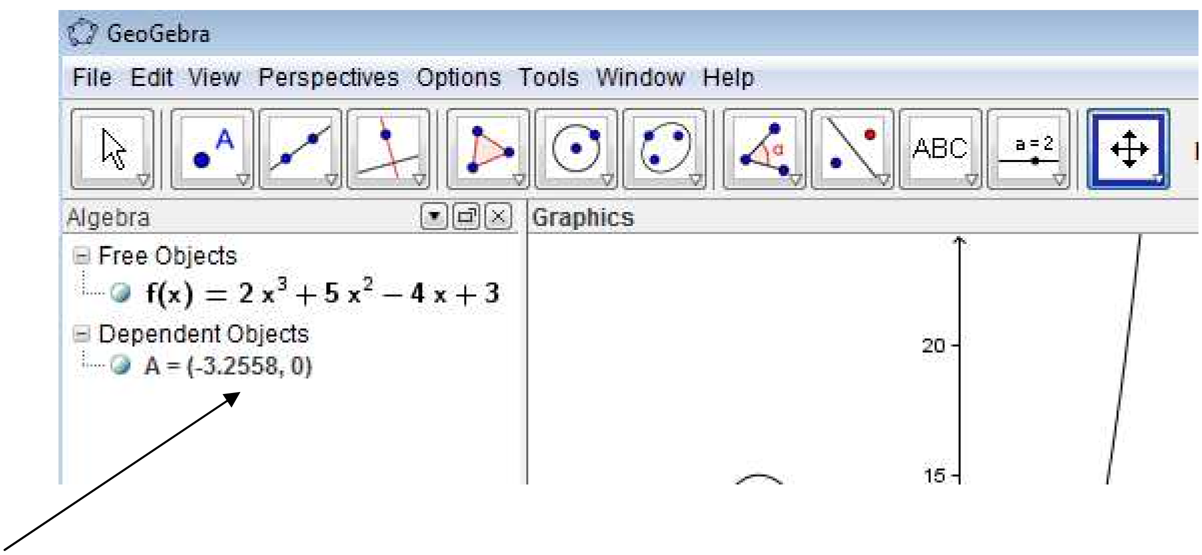
One of the **Options** is **Rounding**.



Move the cursor over the word **Rounding** until another menu pops up. You can then choose the number of decimal places or significant digits that you wish to display.



Select **4 Decimal Places**, and the display in the algebra window will adjust accordingly.



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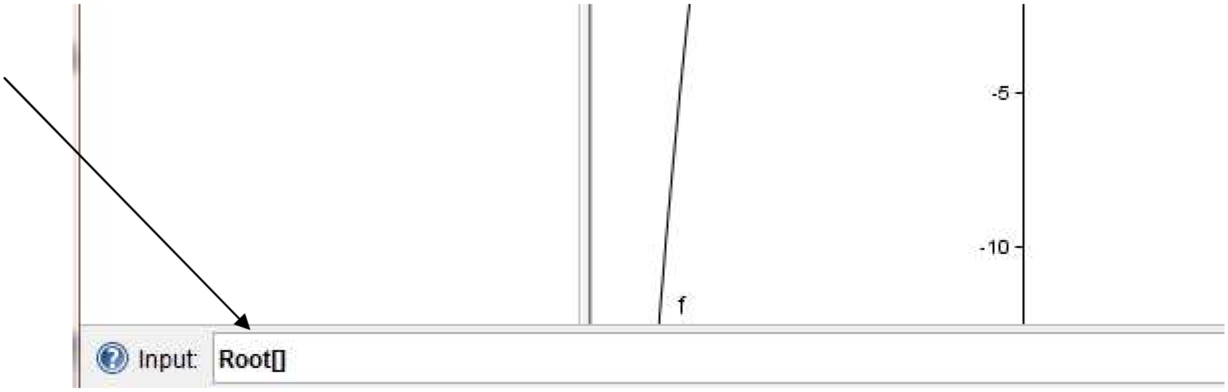
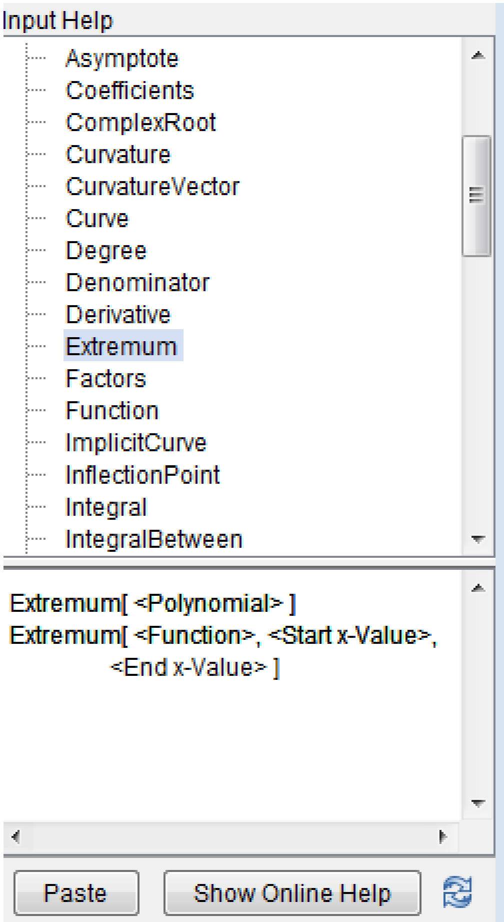
Once you are more familiar with GeoGebra and know the commands that you want to use, you can just start typing the word "Root" until **Root[]** appears in the **Input** line. The same options that you saw in the box on the left will pop up as you type. Select the command that you want to use and press **Enter**.



Put the cursor inside the brackets, and type *f* and then press **Enter** on your keyboard. GeoGebra will return the same display that you saw in the last example.

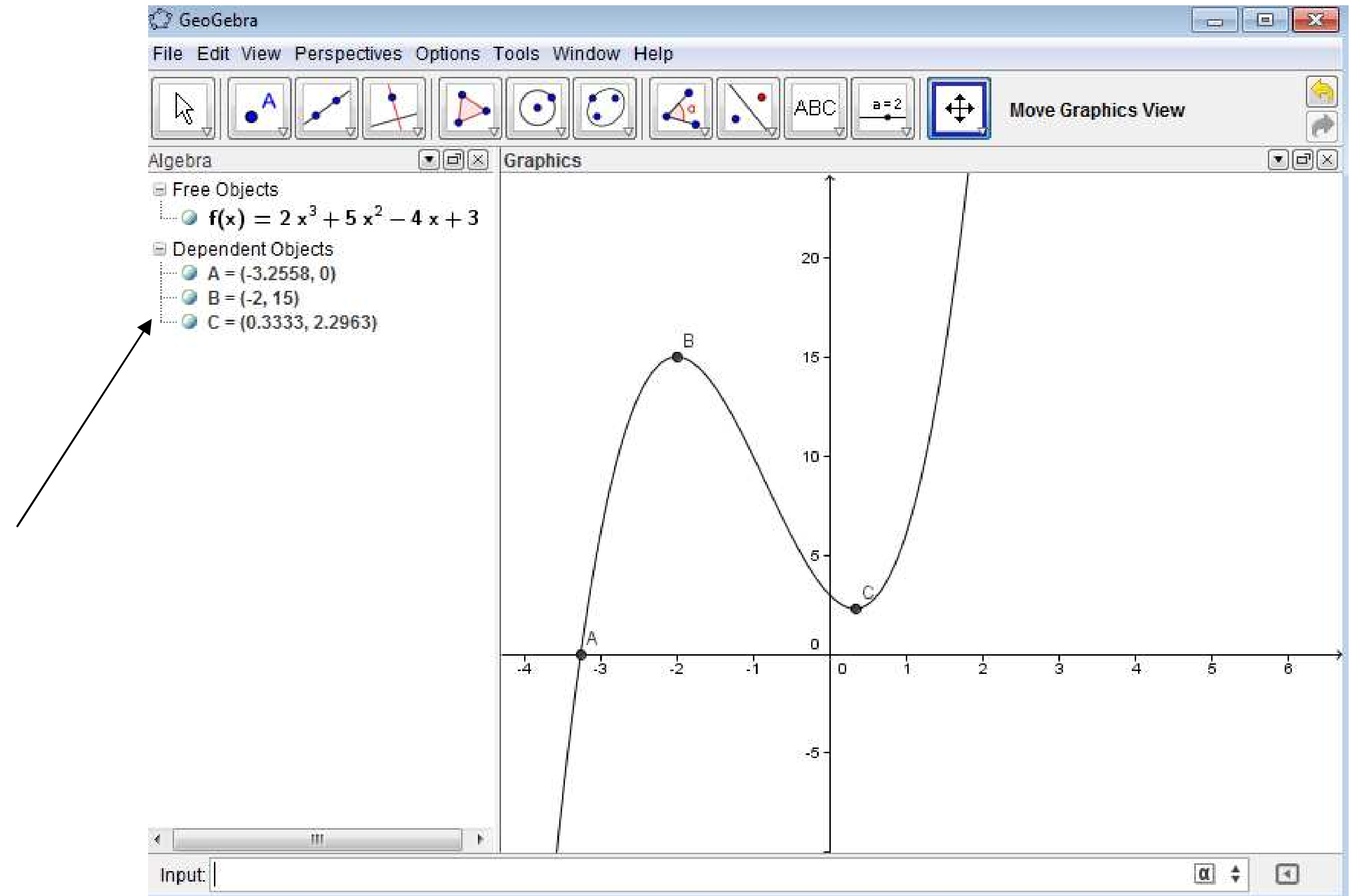
In this example, the function had only one zero. If your polynomial function has more than one zero, GeoGebra will return all of them at once. For other types of functions, you will need to find one zero at a time and include a start value of *x* and an end value of *x* inside the brackets in the input line to tell GeoGebra which zero you wish to find.

Next you can find the relative maximum and the relative minimum of the graph of the polynomial function. These are called the extrema of a function. You'll find the word "Extremum" in the **Input Help** list under **Functions and Calculus**.

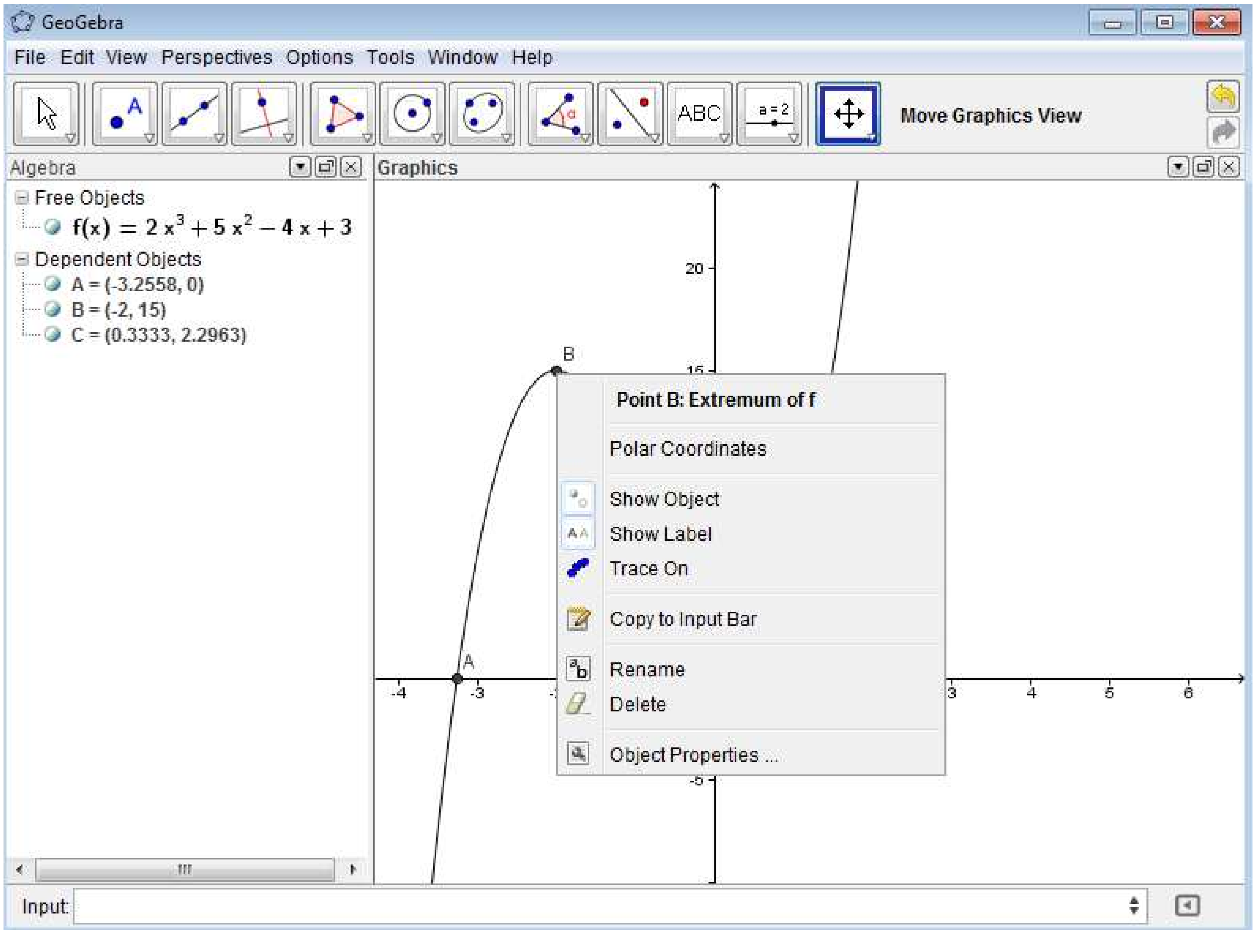


Click on **Paste**, and the **Extremum** command will appear in the input line.

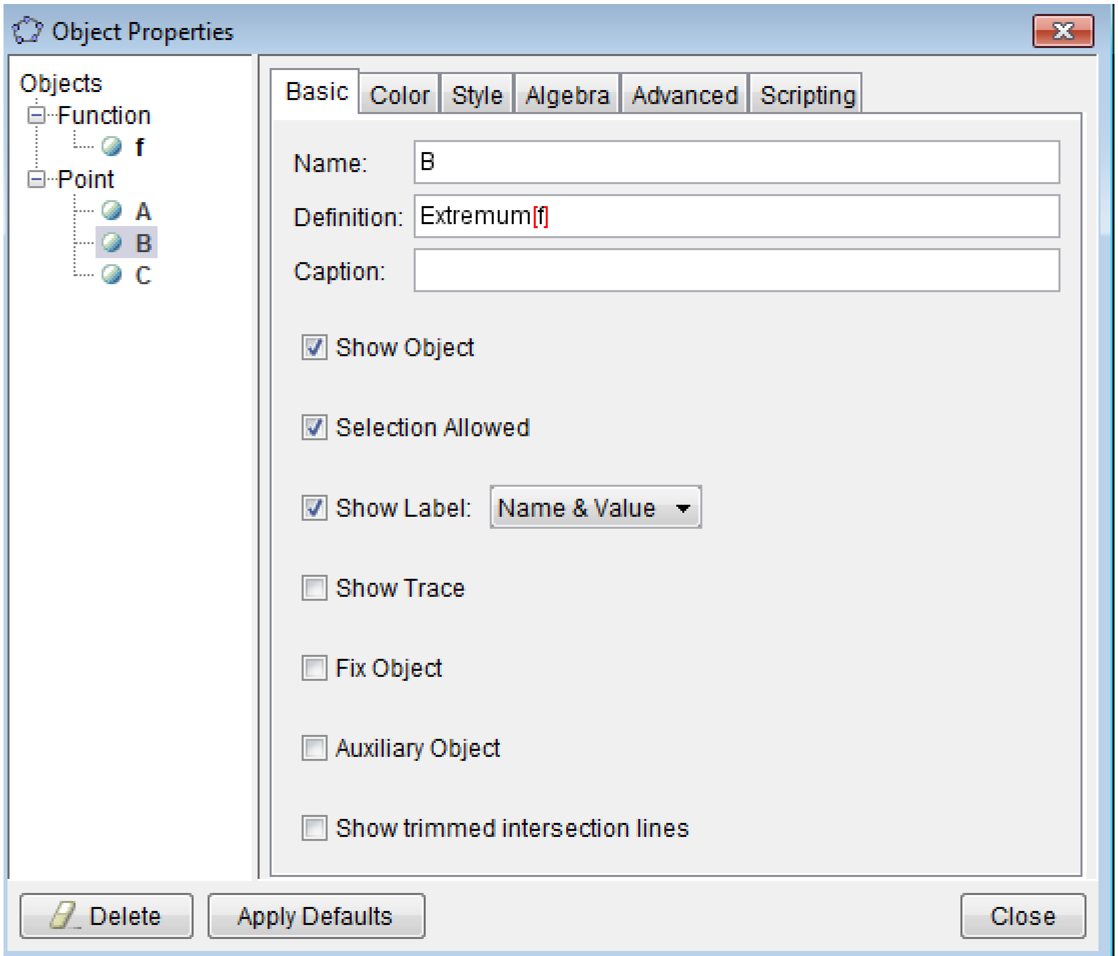
To find the extrema of a polynomial function, as we have in this example, put the cursor inside the brackets and type the name of the function you are working with, in this case, *f*. Then press enter. GeoGebra will return values for all relative extrema of the polynomial function at once. Note that points appear on the graph at the relative maximum and the relative minimum, and the coordinates for these points are listed under **Dependent Objects** in the algebra window.



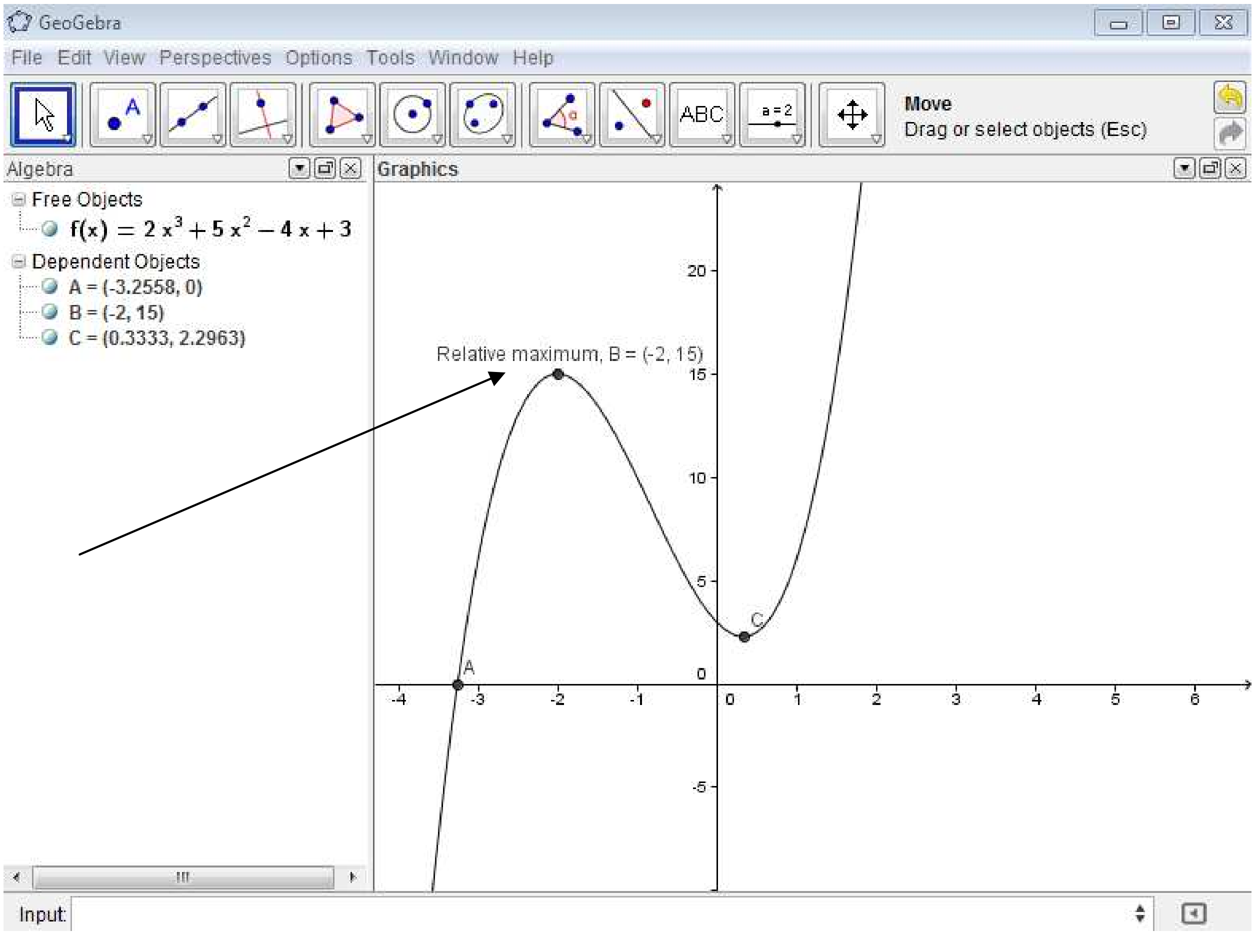
Note that the items in the list of Dependent Objects are not labeled. Just by looking at the list, you don't know what A, B and C represent (although you can probably figure them out by looking at the graph). Put your cursor on point B on the graph and right click. You'll see a menu that gives the details of the point.



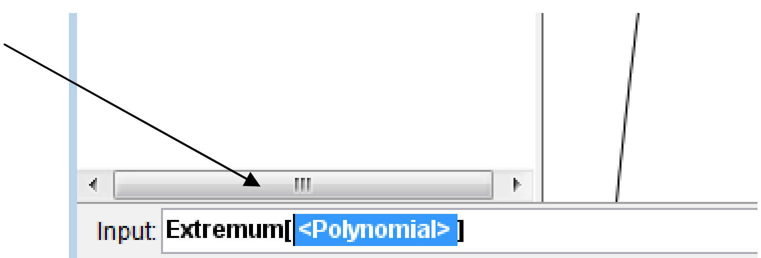
The top line identifies point B as an extremum. You can tell by looking at the graph if it is a relative maximum or a relative minimum. The last option on this menu is **Object Properties**. Click on it to see what you can do from this menu.



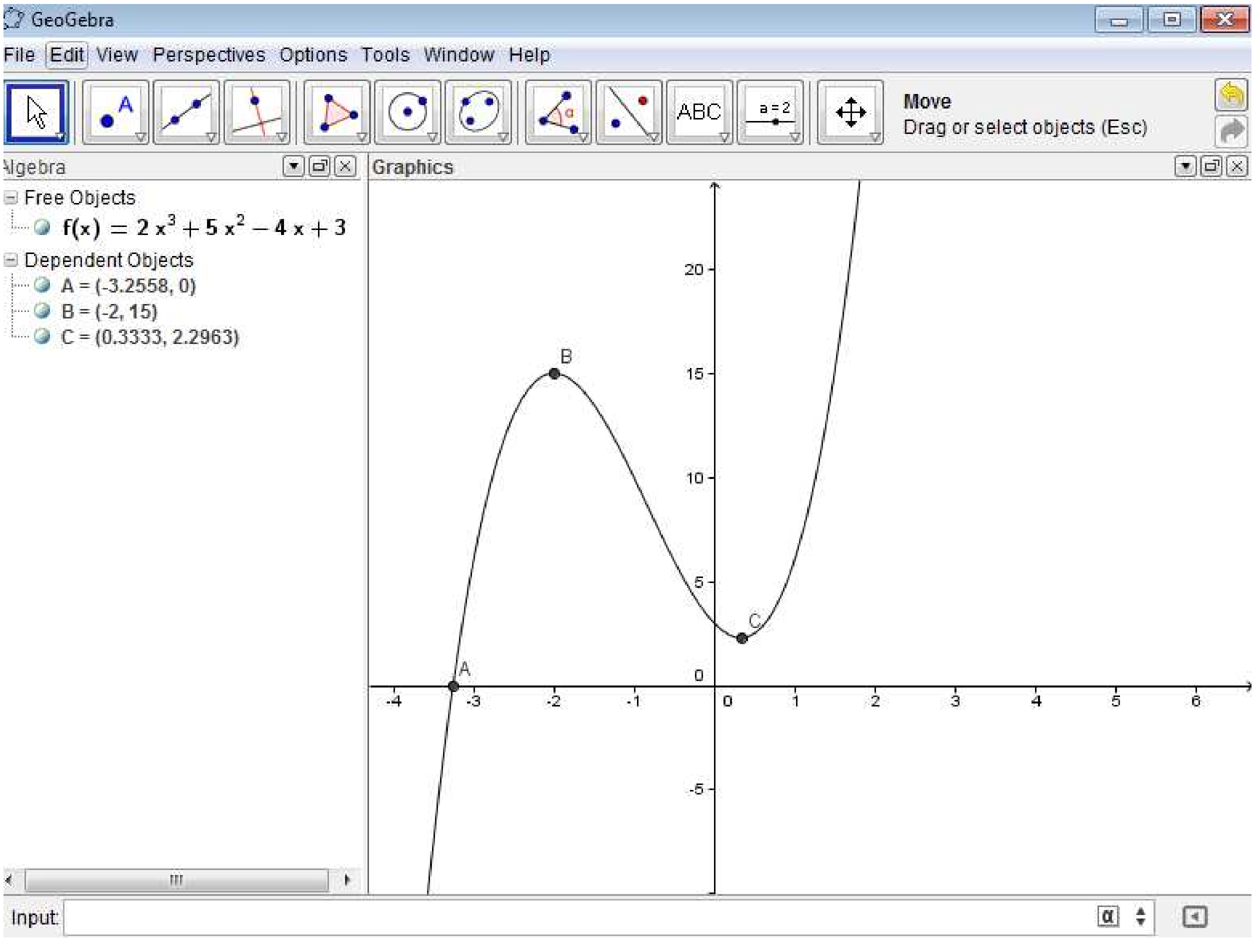
You can change many things, such as the color of the point or the size of the point, using the **Color** and **Style** options at the top. You can also change the label. Click on the drop down menu next to **Show Label**, and you'll see four options, **Name** (the default, which just gives the letter from the list), **Name & Value** (which gives both the name and the coordinates), **Value** (which just gives the coordinates) and **Caption** (which allows you to manually enter a caption in the **Caption** line above the menu, and that is what will appear on the graph). In the graph on the next page, you'll see a customized caption which gives all information about the point. Captions are given in text boxes, so you can move them by left-clicking on the caption and dragging to the desired location.



Once you are more familiar with GeoGebra, you can by-pass the **Input Help** box and just enter your command in the input line. For the relative extrema, put the cursor in the input line and start typing "Extremum."

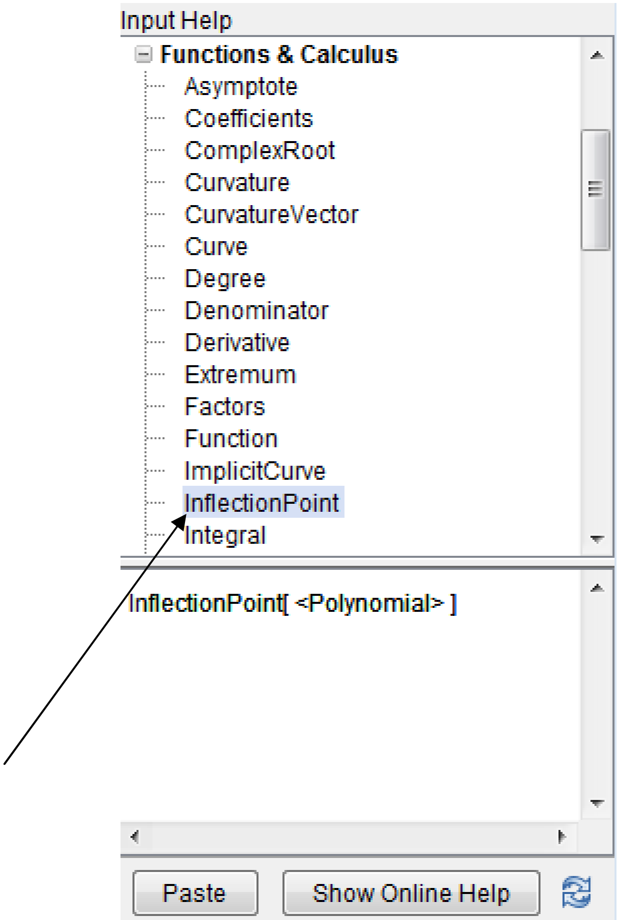


GeoGebra wants you to specify the function name, so put the cursor inside the brackets and type f(x). Then press **Enter** on your keyboard. You will get the same result that you saw when you used the **Input Help** box.

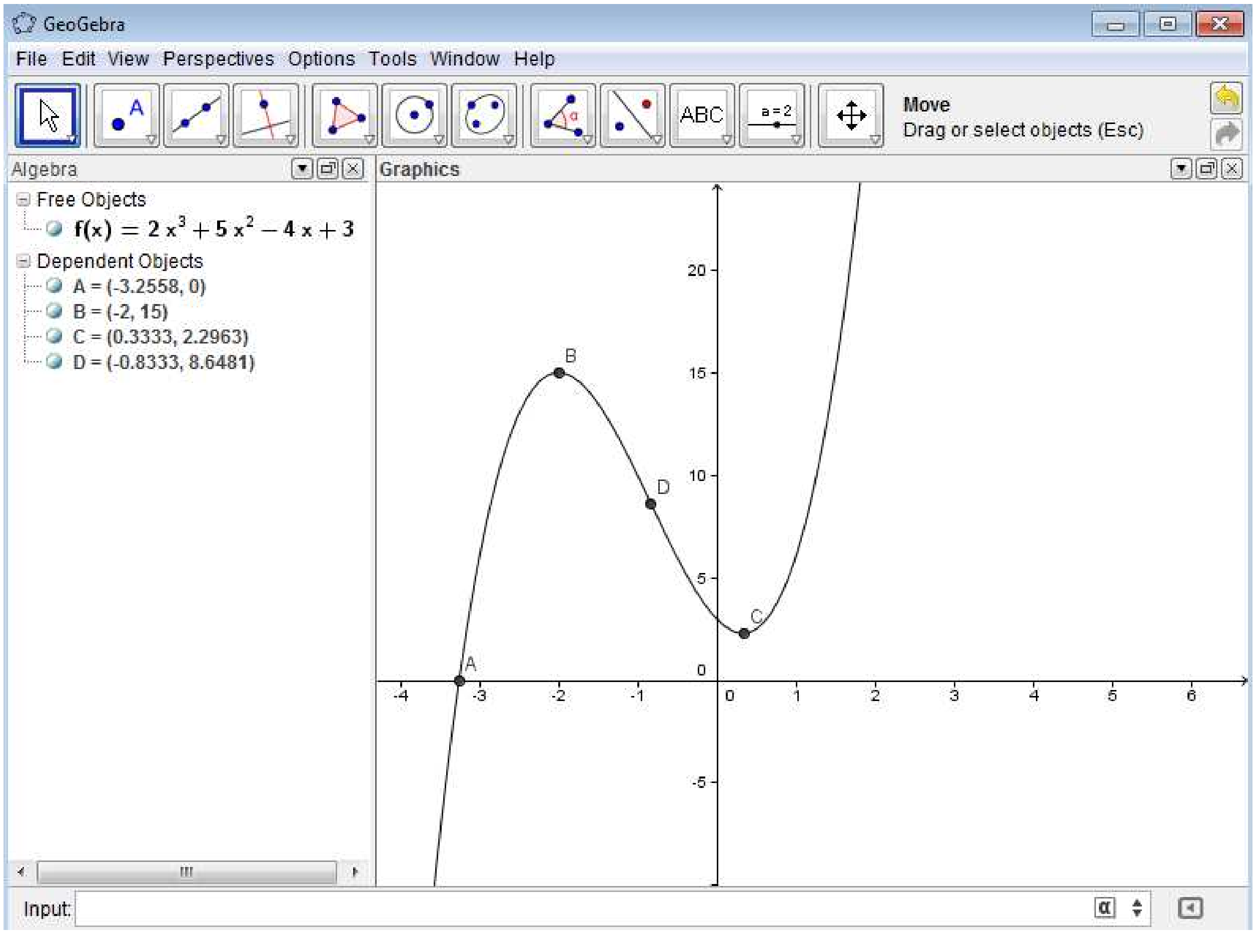


Finally, you want to find the location and coordinates of any inflection points for the graph of this function. An inflection point is a point on the graph of a function where the graph changes concavity, that is, where it changes from looking like an upward-opening parabola to looking like a downward-opening parabola, or vice versa. The change is sometimes very subtle, and you can use calculus methods to locate these. You can also use GeoGebra.

Using the **Input Help** box, expand **Functions & Calculus** and scroll down until you see **Inflection Point**.



Click on **Paste**, and the command will appear in the input line. Insert the function name inside the brackets and press **Enter**.

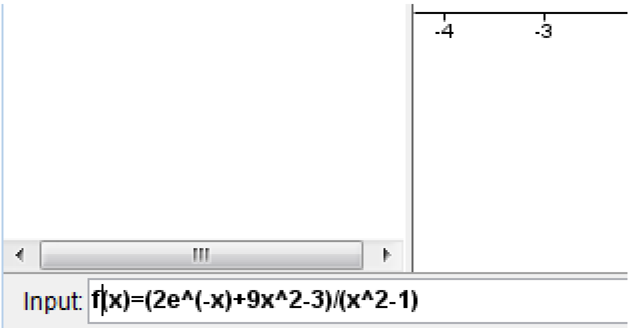


Point D appears on the graph and in the list of **Dependent Objects**. This is the inflection point. Note that you can only use this method to find inflection points of polynomial functions using GeoGebra (no other options appeared in the **Input Help** box). You will see how to use GeoGebra to find inflections points of other types of functions in the next example.

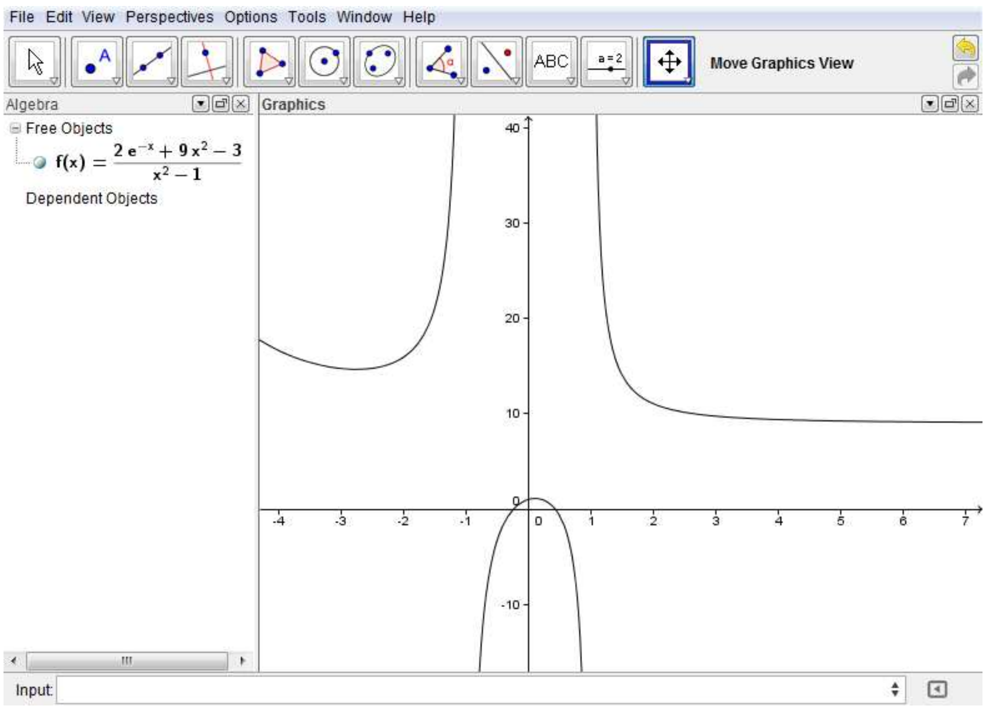
You can find the same values with a general function, but the selections that you make and the required inputs will be different.

Example 3: Suppose $f(x) = \dfrac{2e^{-x} + 9x^2 - 3}{x^2 - 1}$. Find the zeros of the function. Then find the relative maximum, relative minimum and inflection point of the graph of the function.

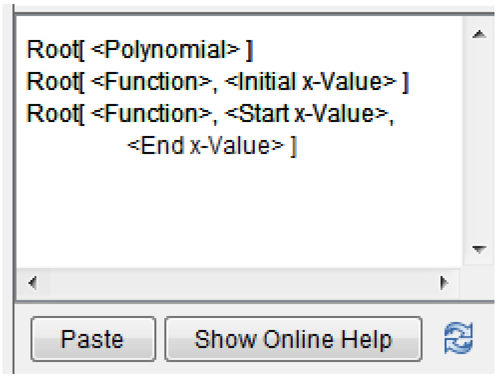
Solution: This is a rational function, and the numerator contains an exponential term. This is not a polynomial function, so using GeoGebra has a few more requirements. You will enter the function in the input line exactly as for Example 2, but you will need to put parentheses around both the numerator and the denominator.



Then graph the function and use the **Move Graphics View** icon to grab the axes and resize the graph, so that you have a good view of the graph of the function.

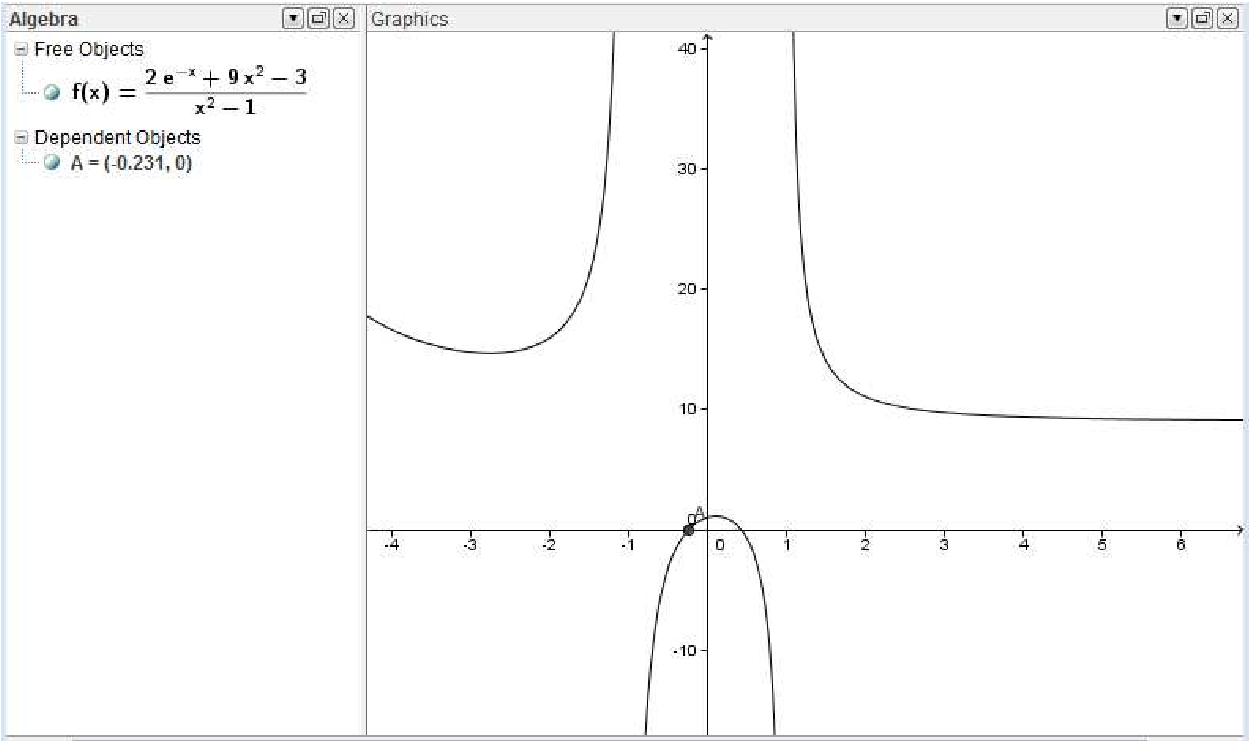


Next, we need to find the zeros of the function. Open the **Input Help** menu and select **Functions and Calculus**. Then scroll down until you see “Root.” Alternatively, you can just type “Root” in the input line below the graph. You’ll see that you have several options for this command, and since you no longer have a polynomial function to work with, you’ll need a different selection.

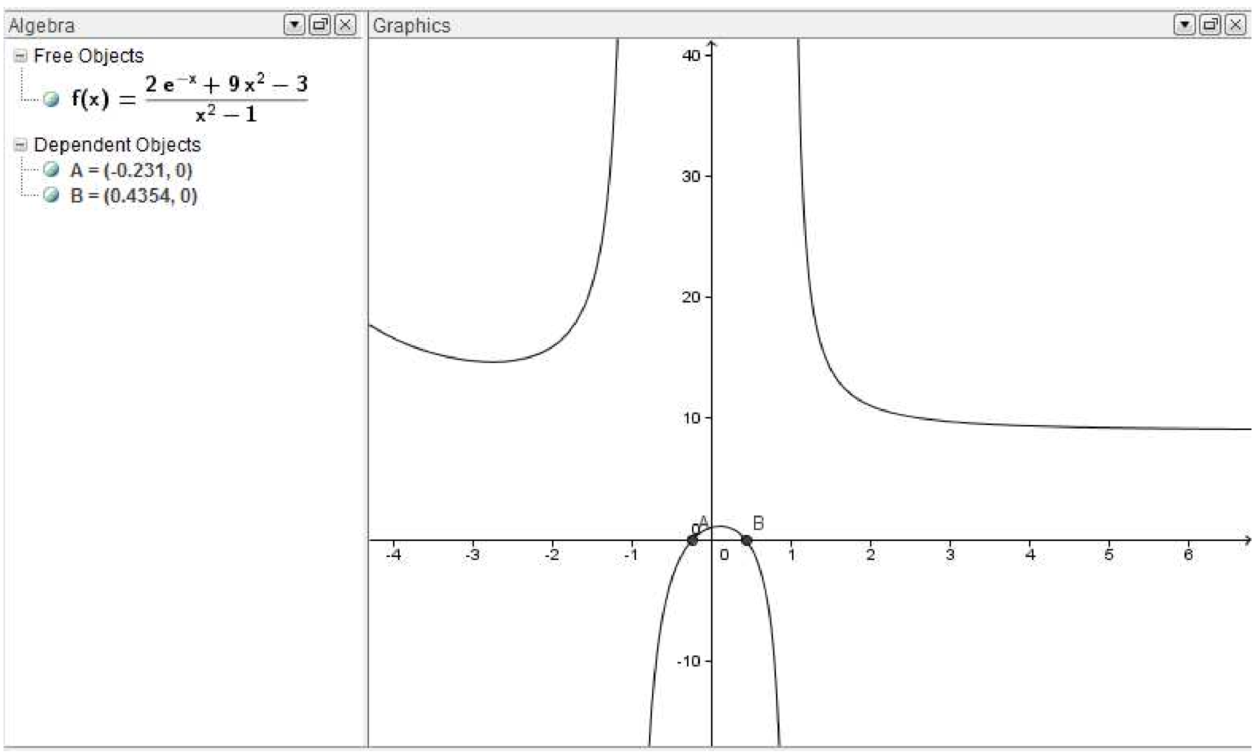


To find the zeros of a general function, you'll need to choose the third option, that is, the option where you will enter the function, the starting *x* value and the ending *x* value. This is very similar to what you do when using a graphing calculator – you need to mark the interval within which you want the software to look for a zero. Note that you have to do these one at a time, which is different from the method used for polynomials. You will use the same method for finding extrema. Click on **Paste**, and then insert the inputs inside the brackets. In this case, the function name is *f*, and the left-most zero occurs between -1 and 0.

Now press **Enter**, and the zero will appear as a point in the algebra window and on the graph of the function. You can select more decimal places using the **Rounding** feature that is on the **Options** menu. You can also change the labeling, as show in Example 2, if you'd like to.



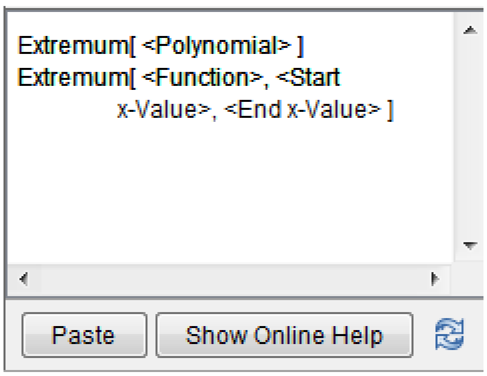
Now find the second zero, using the same method. This zero occurs between 0 and 1.



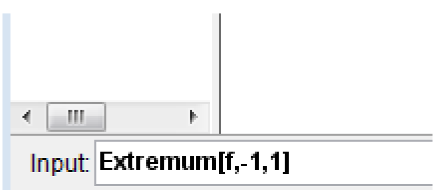
So the zeros of the function are $(-0.231, 0)$ and $(0.4353, 0)$.

Next, you want to find the relative maximum and the relative minimum. The relative maximum occurs between the two zeros, and the relative minimum occurs in Quadrant 2. You'll use the **Extremum** command, and you will need to specify the function and the interval on which you want GeoGebra to look for an extremum. Geogebra will only give you one answer at a time.

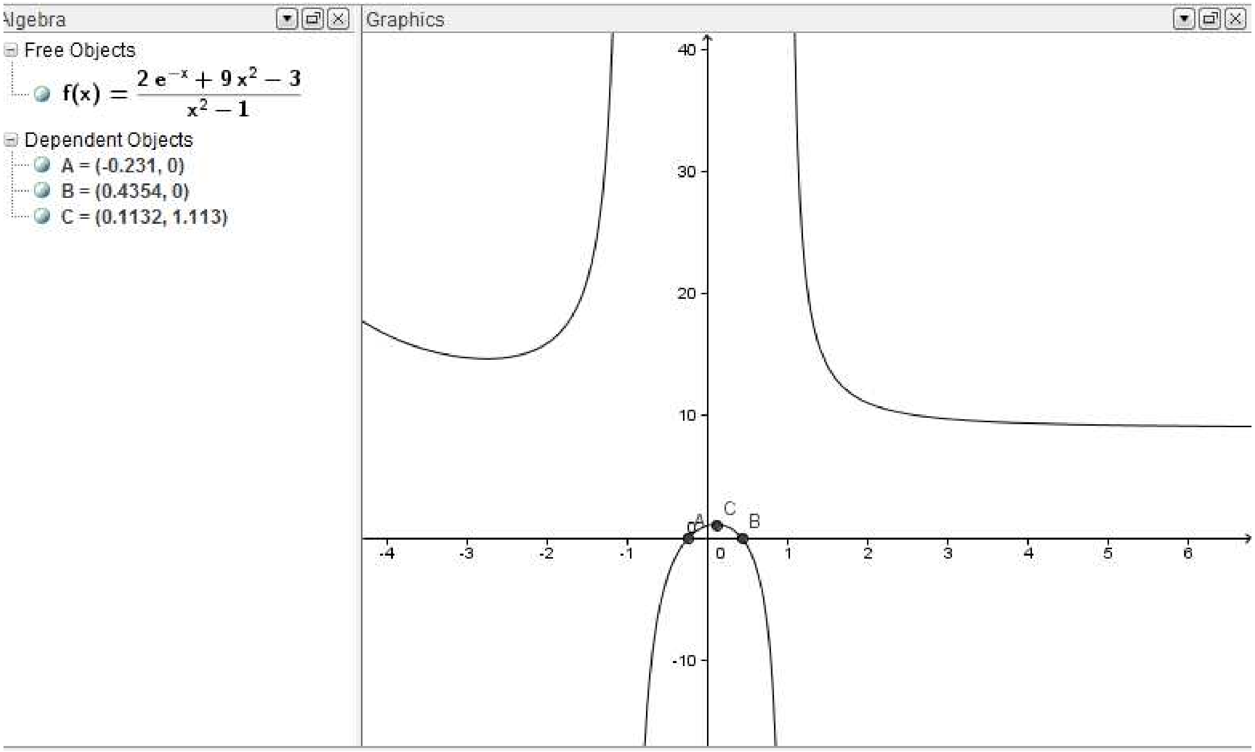
Find the relative maximum first. Using the **Input Help** menu, you can see the options. This is not a polynomial function, so you'll use the second option.



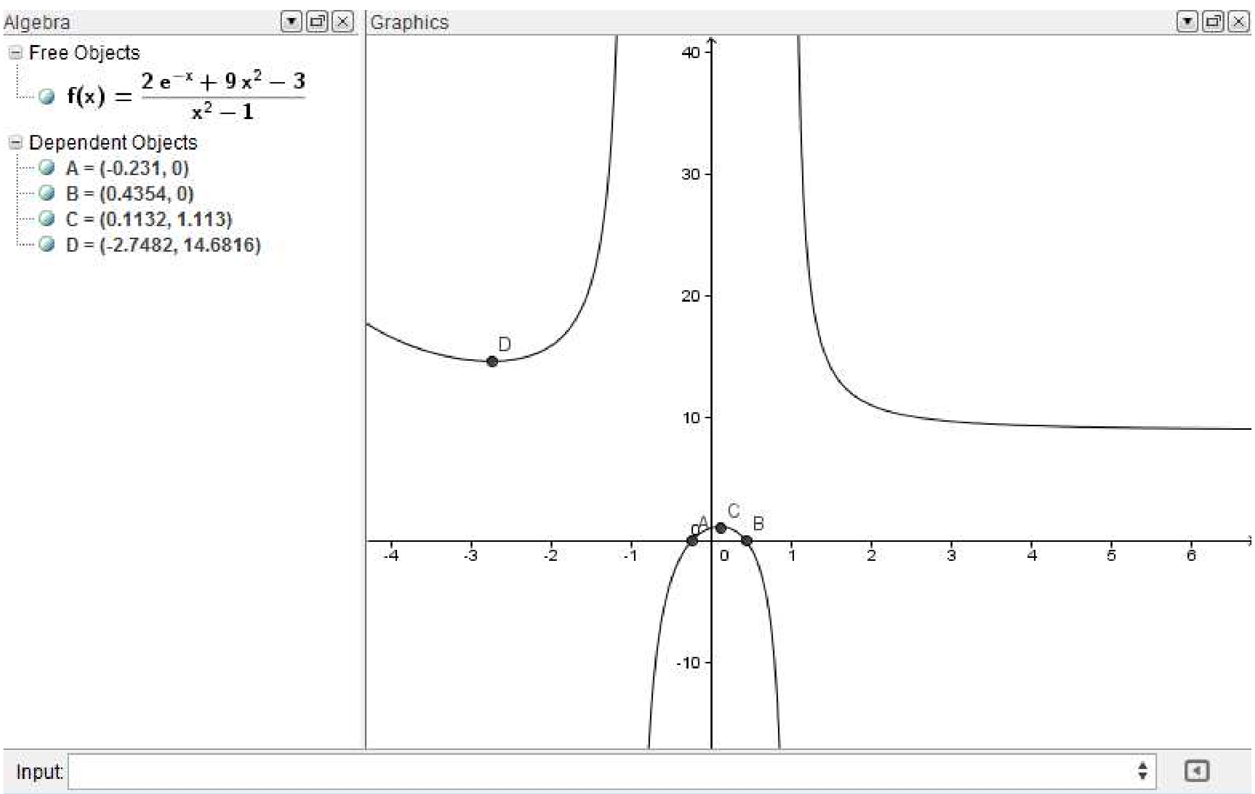
Paste **Extremum** into the input line and fill in the information. The zero is in the interval $[-1, 1]$.



Now press enter to display to relative maximum.



The relative maximum occurs at the point $(0.1132, 1.113)$. Now repeat the process to fine the relative minimun. It occurs between -4 and -1.



The relative minimum occurs at $(-2.7482, 14.6816)$. If you use the **Input Help** menu to find the **Inflection Point** command, you will see that it works only for polynomial functions. You will need to use a different method for finding inflection points of general functions. We will defer discussing inflection points until after seeing how to using GeoGebra to find derivatives.

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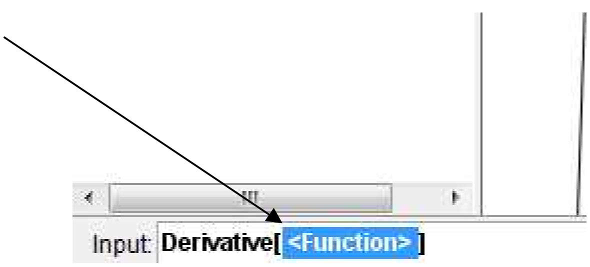
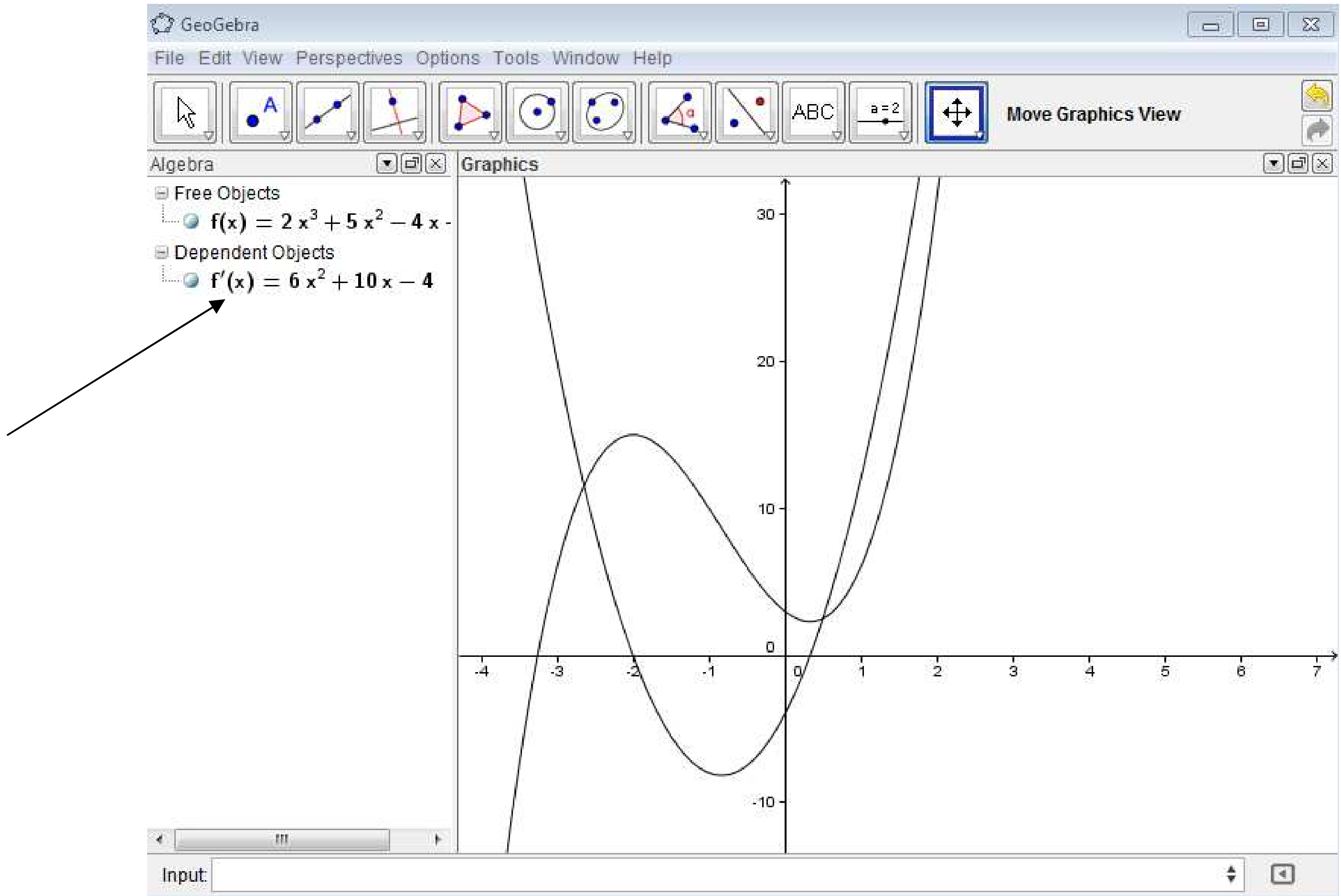
GeoGebra can also find the symbolic derivative (i.e., the derivative in terms of the variables).

Example 4: Find the derivative of $f(x) = 2x^3 + 5x^2 - 4x + 3$. Then find $f'(2.08)$ and $f'(10.47)$.

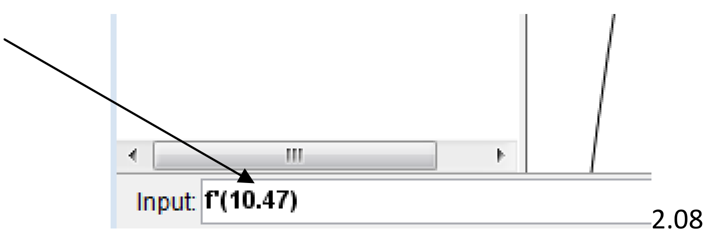
Solution: **Derivative** is one of the commands in the **Functions and Calculus** list. Start by entering the function. Then either find the **Derivative** command using the **Input Help** menu or just put the cursor in the **Input** line and start typing "Derivative".

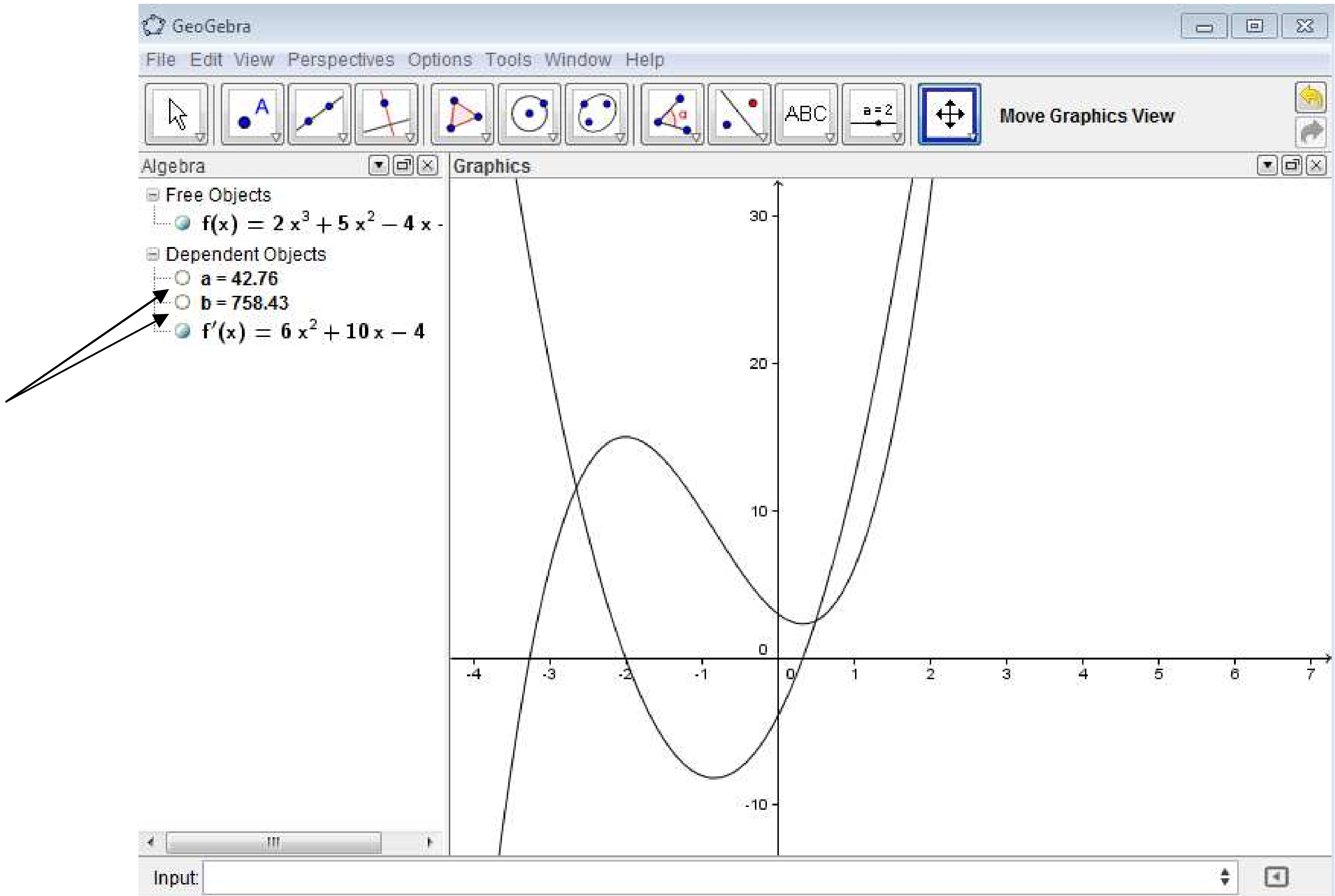


You want to find the derivative of $f(x)$ so type that in the brackets. F gNow press **Enter** onyour keyboard.



GeoGebra graphs $f'(x)$ and displays the symbolic derivative as $f'(x)$ in the algebra window. Now to find the value of the derivative at the two points that are given, you will need to evaluate $f'(x)$. Type $f'(2.08)$ in the **Input** line and press **Enter** on your keyboard. Then type $f'(10.47)$ (shown below) in the **Input** line and press **Enter** on your keyboard.





The two values are given as *a* and *b* in the algebra window. So $f'(2.08) = 42.76$ and $f'(10.47) = 758.43$.

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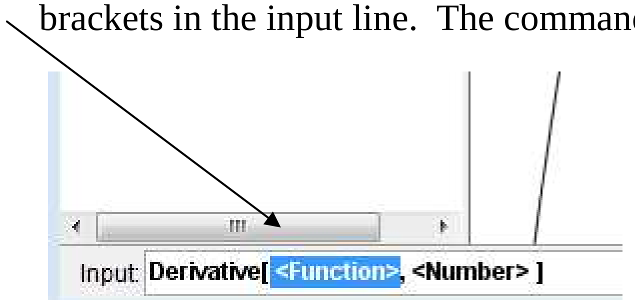
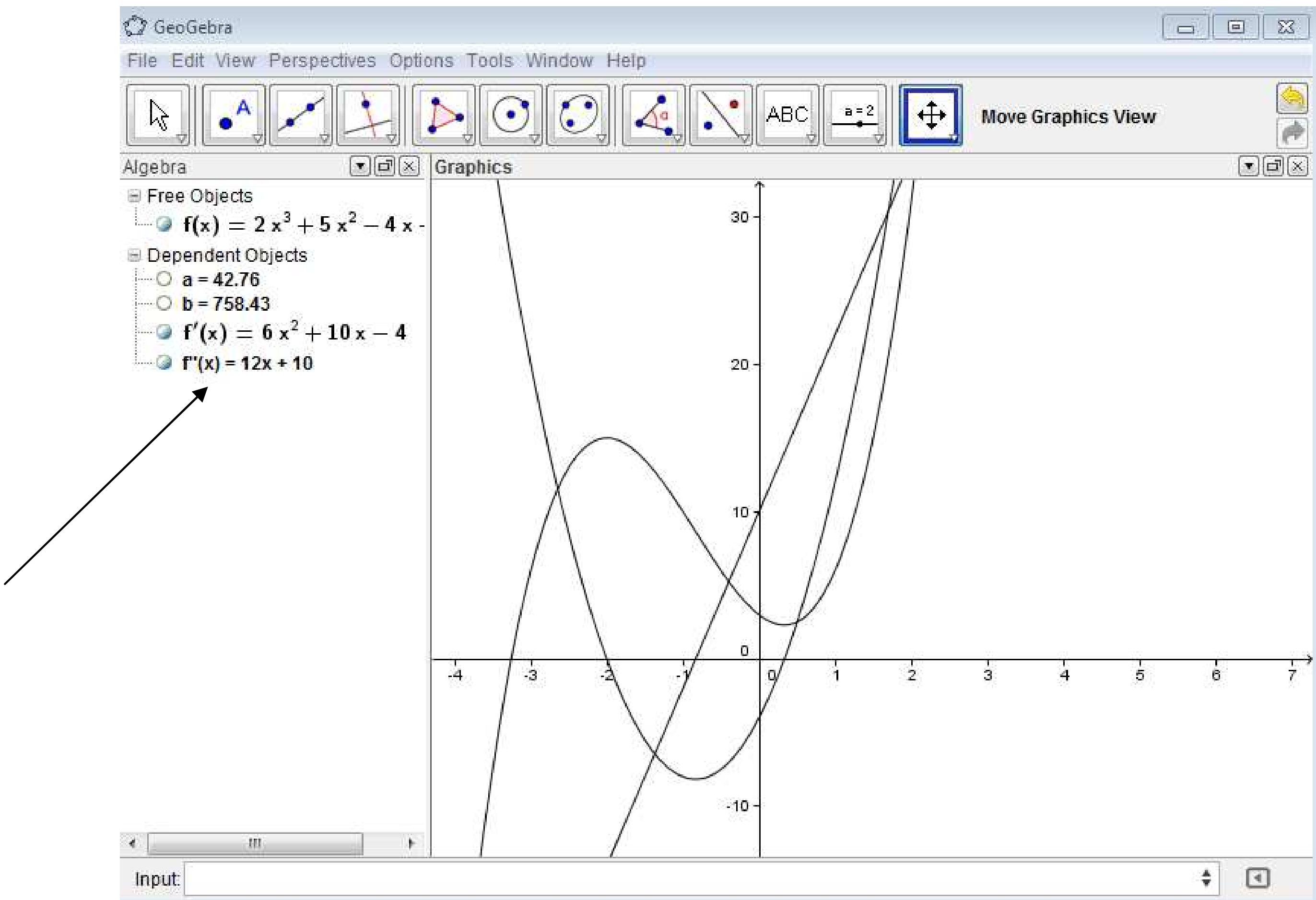
If you need to find the second derivative of a function, you will include this in the brackets in the input line. The command prompts you what to enter.



So in this example, you would type "f Tab 2" inside the brackets to find the second derivative.

Now press **Enter** on your keyboard to view the result.



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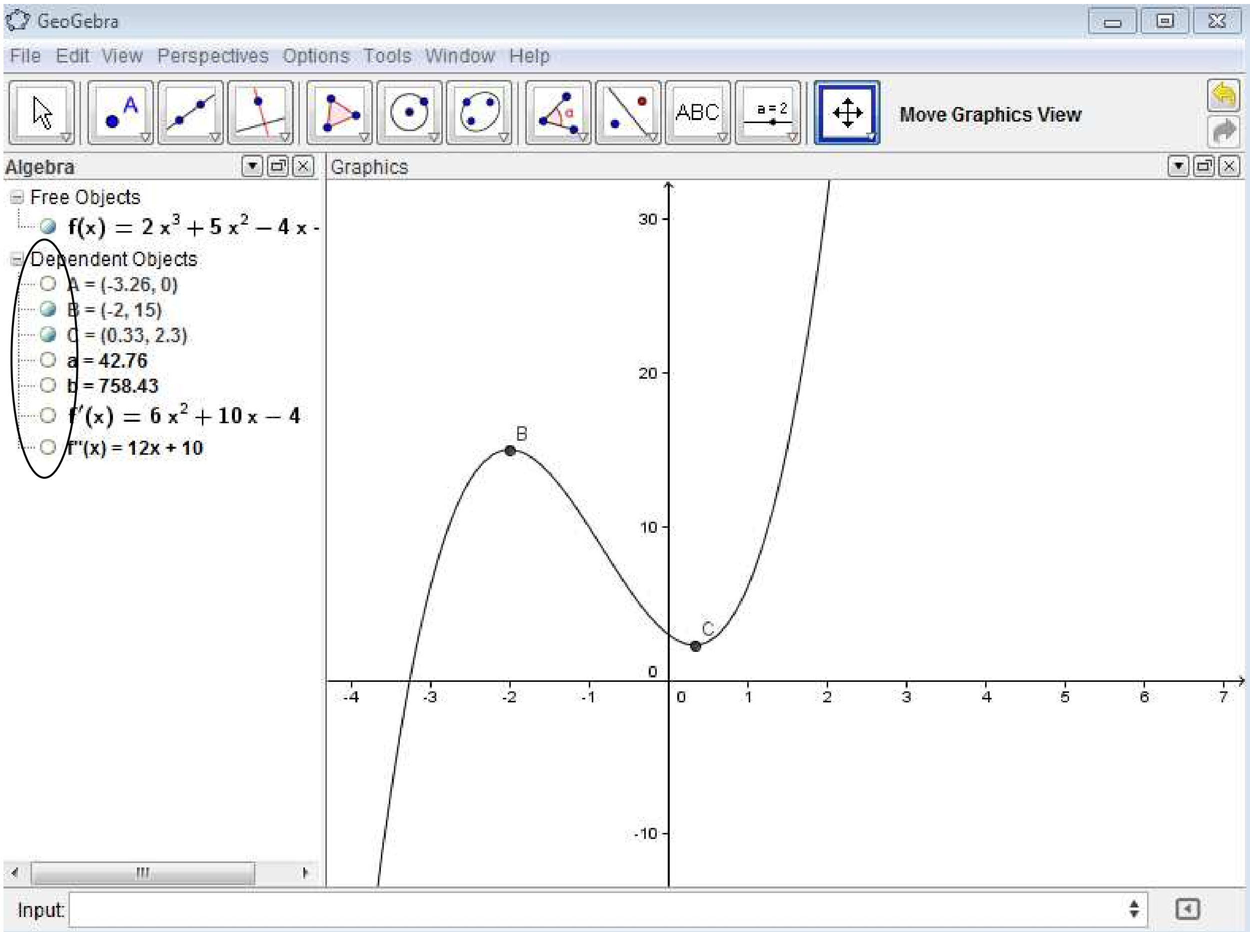
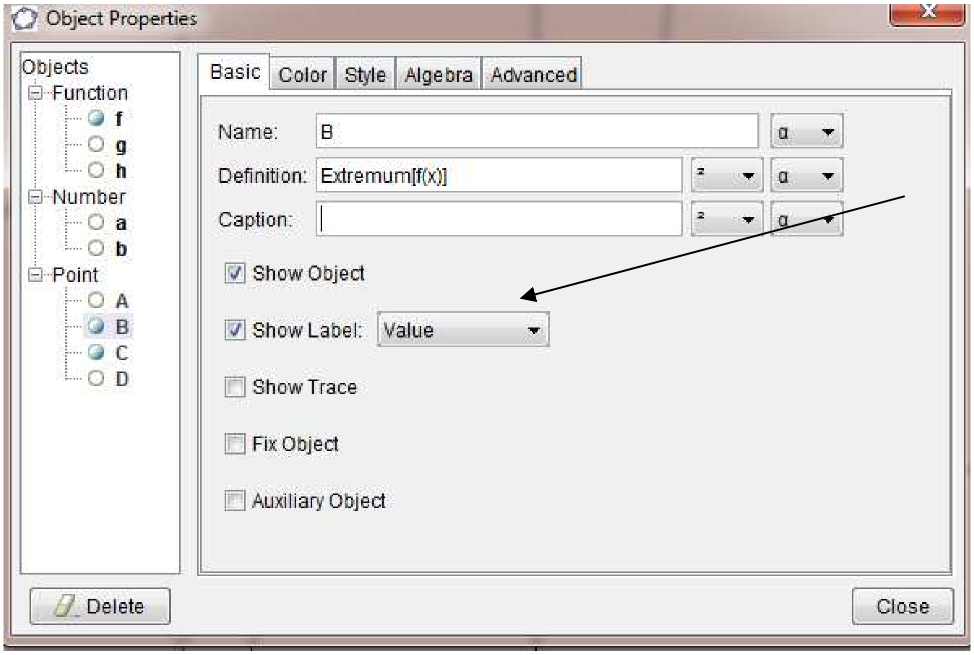
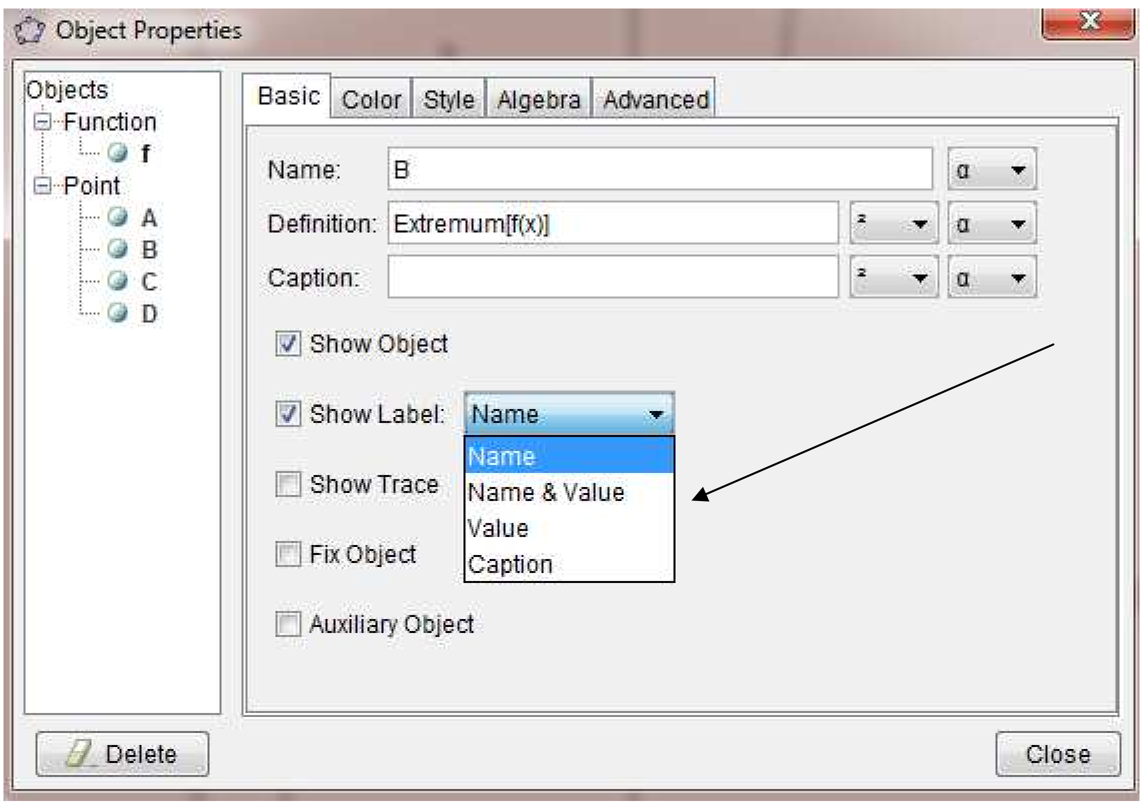
If you want to limit the items that are displayed on the graph, you can easily do so. Note that the bullets to the left of many of the items in the **Dependent Objects** list are shaded. That means that the items will appear on the graph. In the graph shown below, only the function and the relative extrema are displayed. Note that the other items have been disabled in the algebra window.



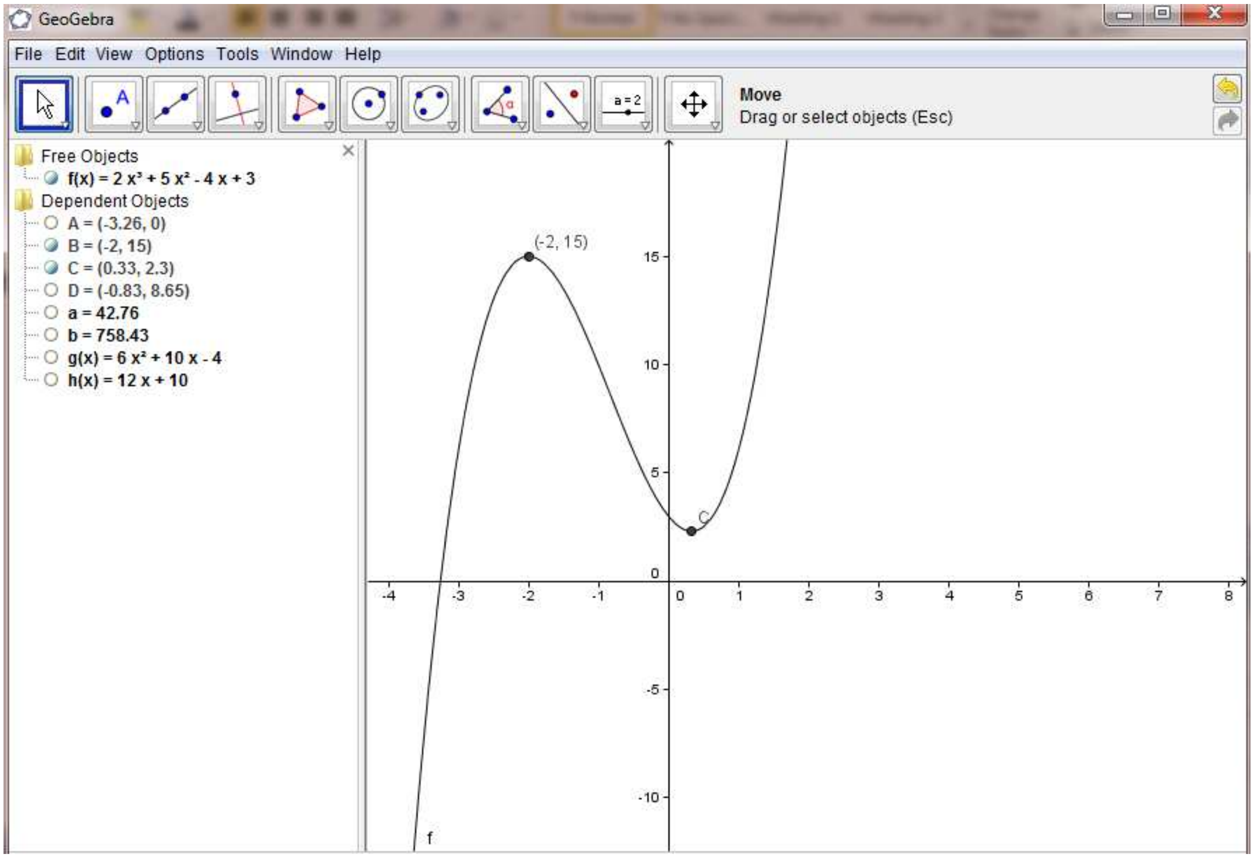
If you want to change the way GeoGebra labels points, right-click on the point in the algebra window, select **Object Properties**, click on the pull down menu titled **Show Label** and select the type of labeling you want.



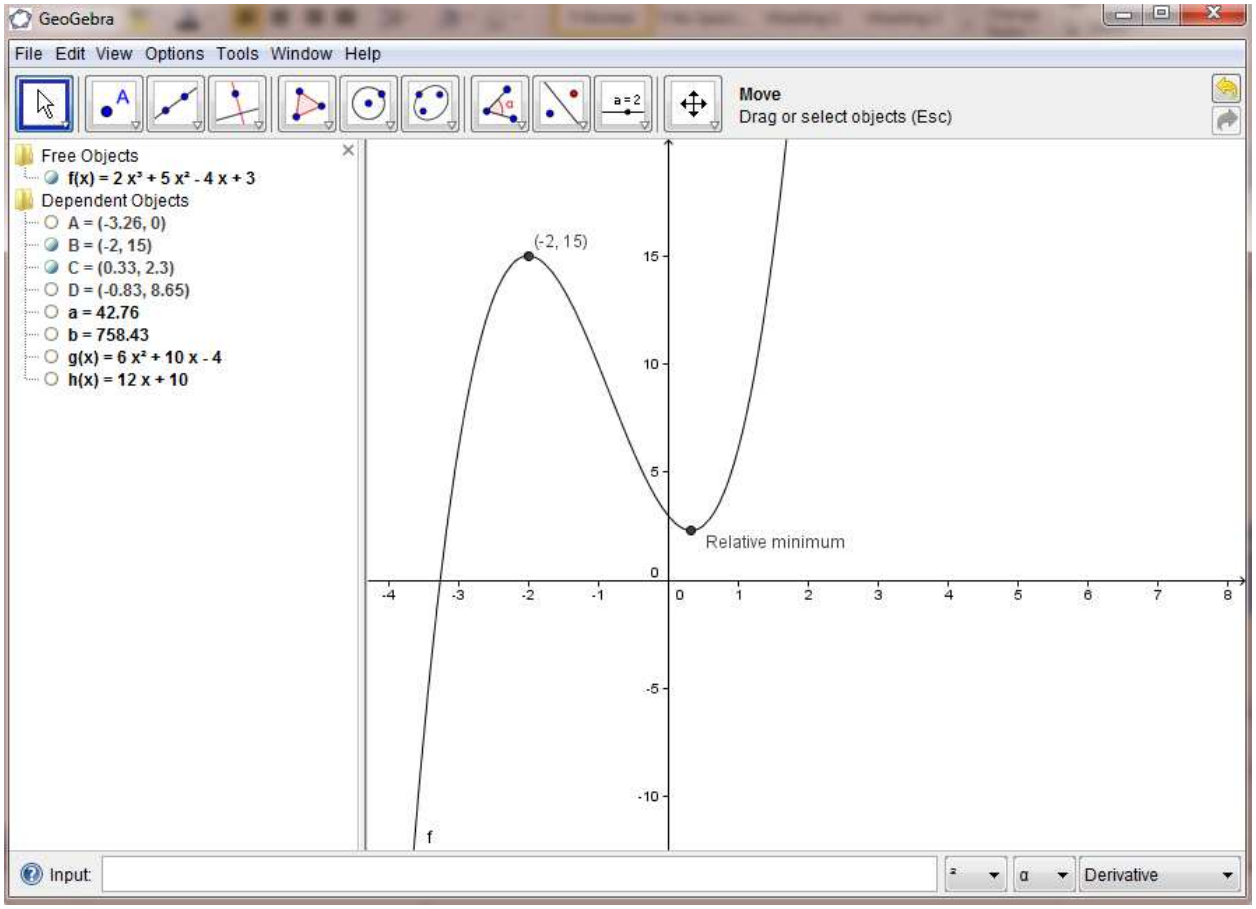
Here’s the result:



In the examples you’ve seen so far, points have been labeled with just the name, which is the letter given in the algebra window. In the graph shown below, GeoGebra displays Point B with just the coordinates. To do this, choose **Value** from the options.

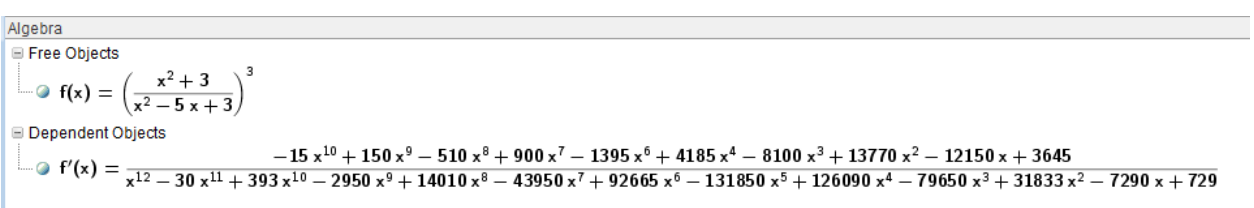


You can also give the point a customized caption. From the **Show Label** menu in **Object Properties**, select **Caption** and type in what you want to display. In the graph shown below, the relative minimum at Point C has been captioned "Relative minimum." The labels are text boxes and can be dragged to another location on the graph, if needed.

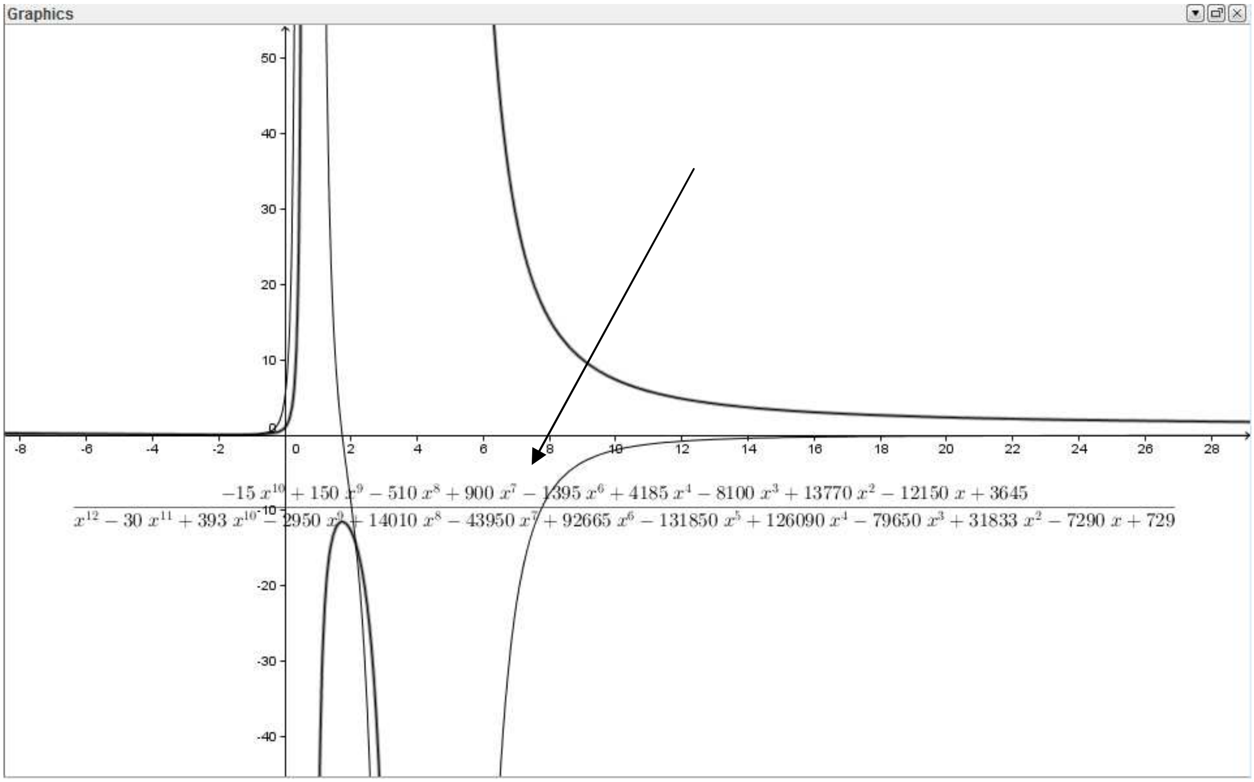


In the technology based version of this course, you will not be required to find complicated derivatives. In the tradition version, you will. If asked, GeoGebra will return symbolic derivatives of more complicated functions. However, they are usually not in simplified form and may be very difficult to work with, if you must do something with it using paper and pencil. Suppose the function is

$k(x) = \left(\dfrac{x^2+3}{x^2-5x+3} \right)^3$. See the display below for the symbolic derivative of this function.



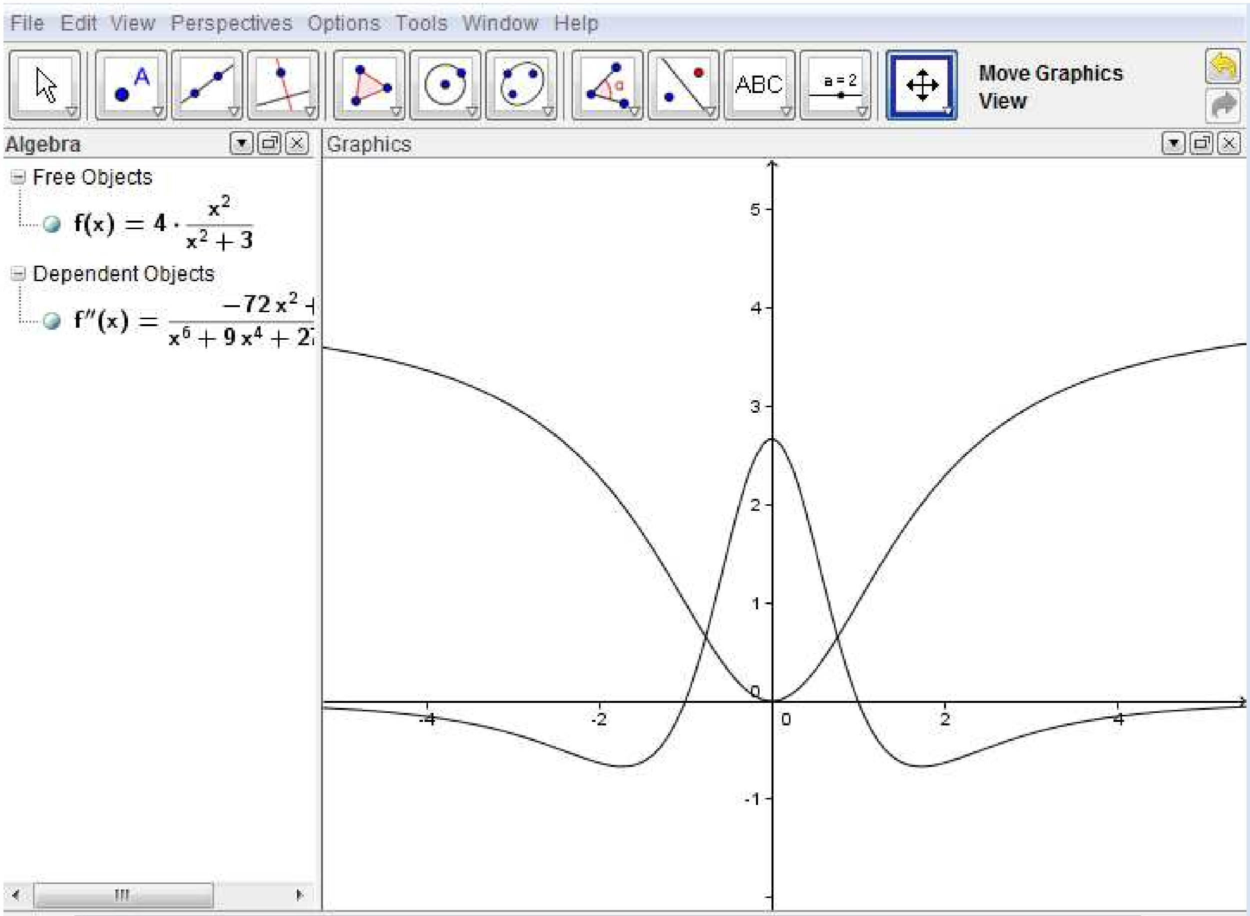
The graph given below shows the graphs of the function and the derivative, along with the symbolic derivative of the function.



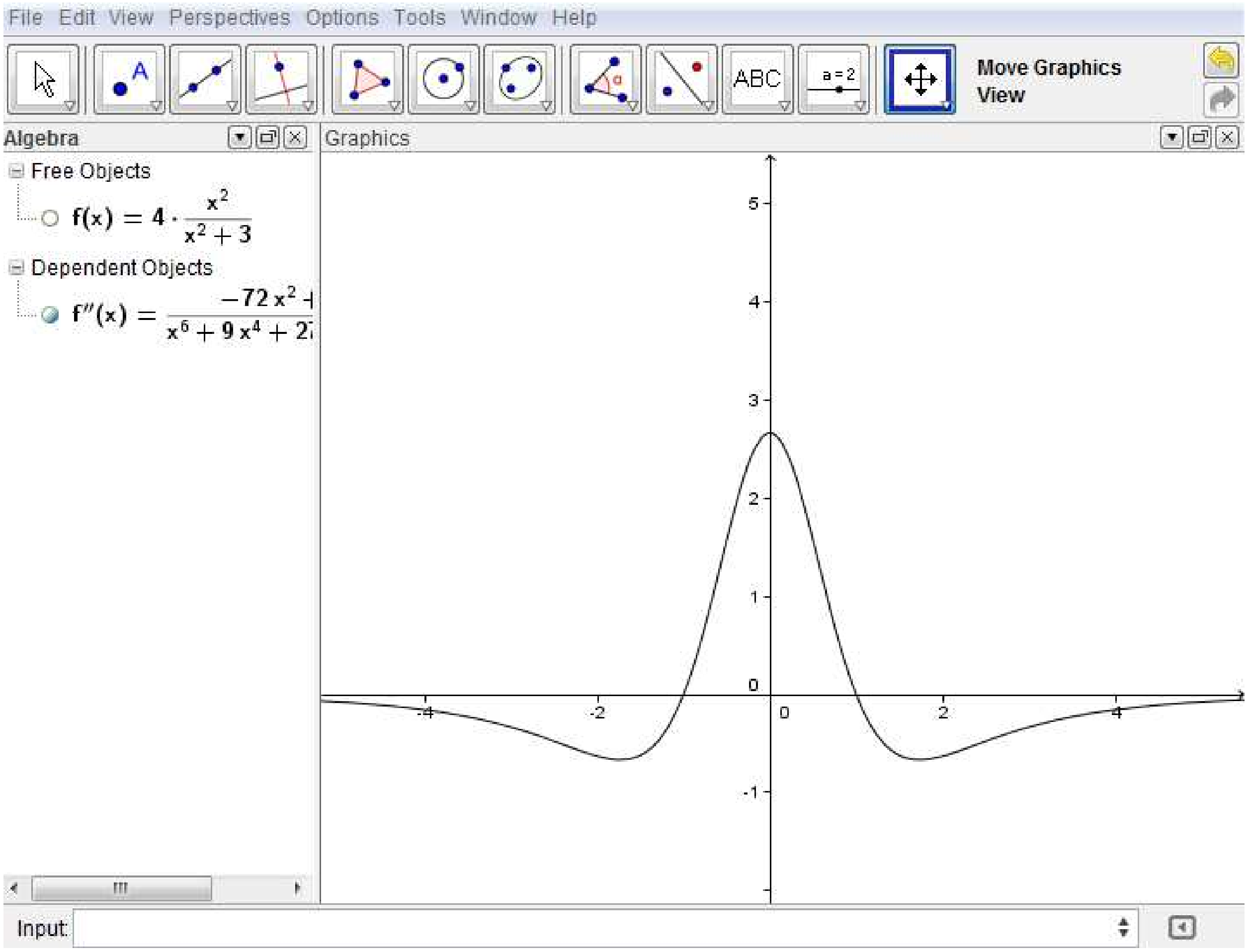
You will need to use the graph of the second derivative to help you find inflection points of the graph of a general function. (Recall that the Inflection Point command only returns inflection points for polynomial functions.) Start by graphing the function, along with its second derivative. Then disable the function so that only the second derivative is displayed. Now find the zeros of the second derivative. These will give you the *x* coordinates of the inflection points. Remember that an inflection point must be a point on the graph of the function and cannot occur at any point where the function is not defined.

Example 5: Find any inflection points: $f(x) = \dfrac{4x^2}{x^2+3}$.

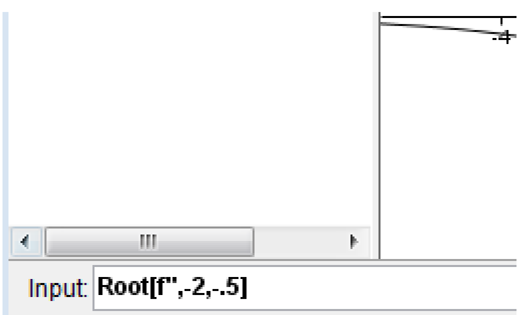
Solution: Start by graphing the function and the second derivative.



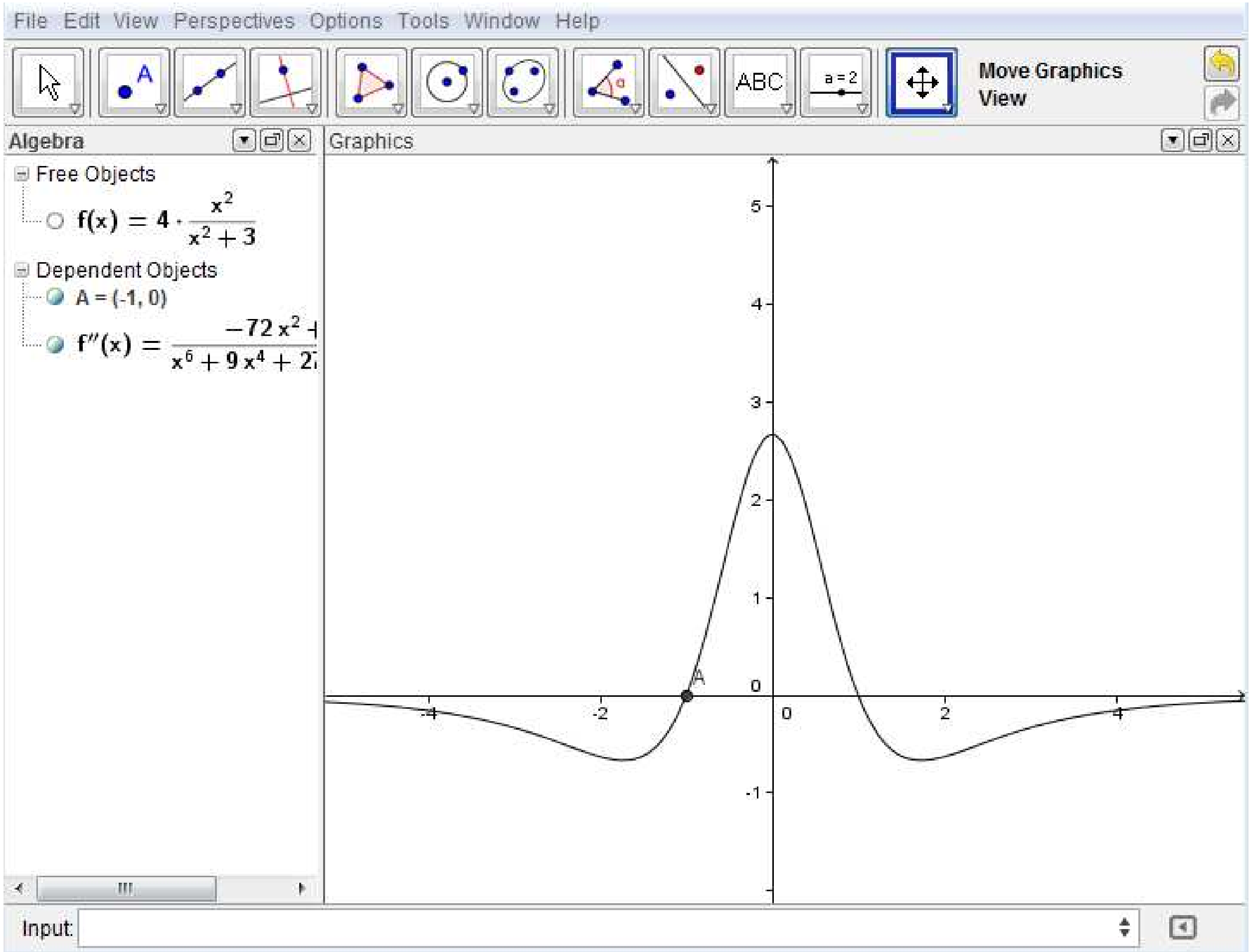
Then disable the function, so that only the second derivative is displayed.



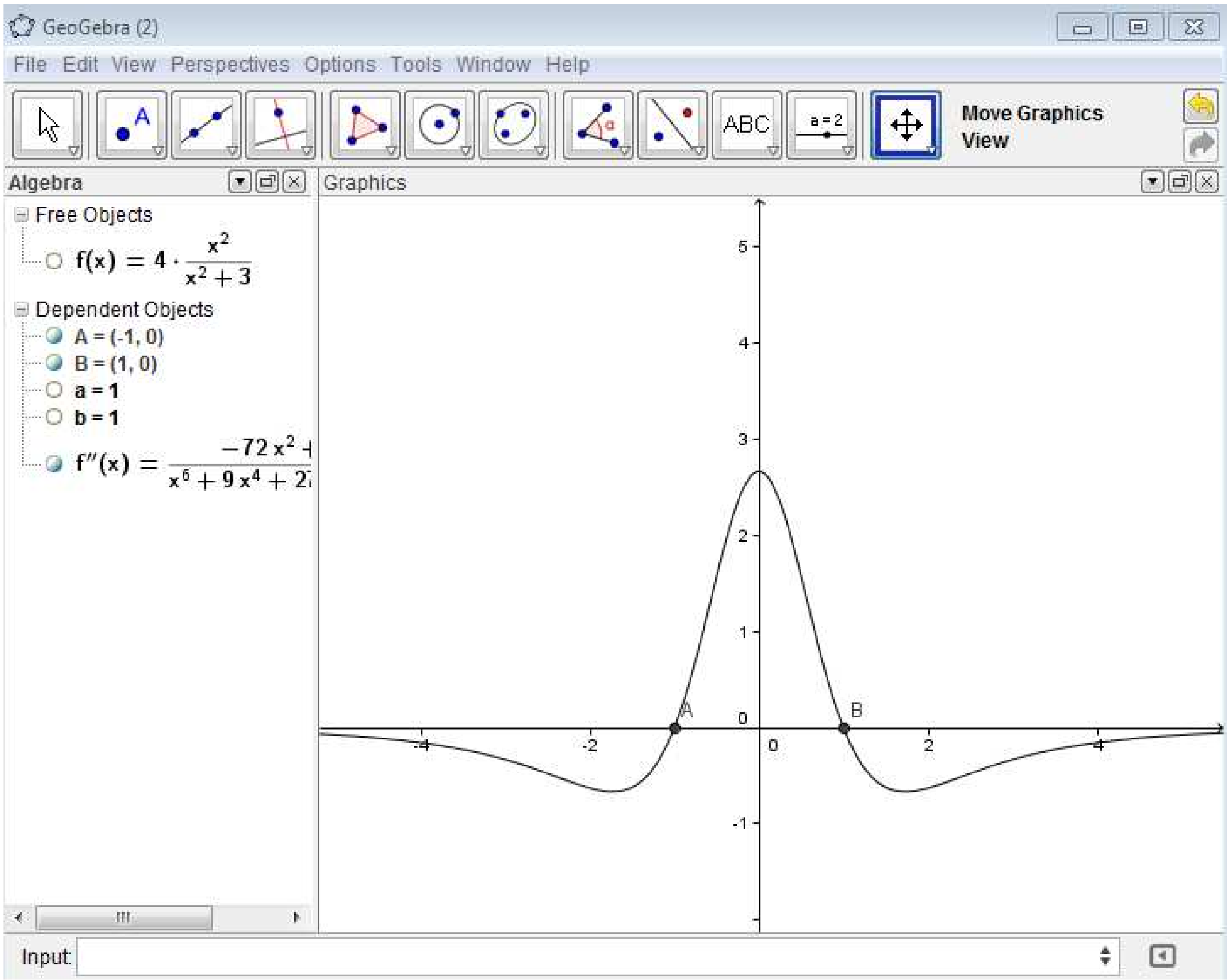
Now find the zeros of the second derivative. Type **Root** in the input line, and use f'', -2 and -0.5 inside the brackets.



Then press **Enter** on your keyboard.



The second derivative has a zero at the point $(-1, 0)$. Repeat the process to find the other zero. It is located at the point $(1, 0)$. Note that the function is defined at both of these values for *x* and the second derivative changes sign at these two points. To the left of -1, the derivative is negative and to the right, it is positive. Similarly, to the left of 1, the second derivative is positive and to the right, it is negative. These points meet the definition of an inflection points. To find the *y* values of the inflection point, evaluate the function at both values for *x*.



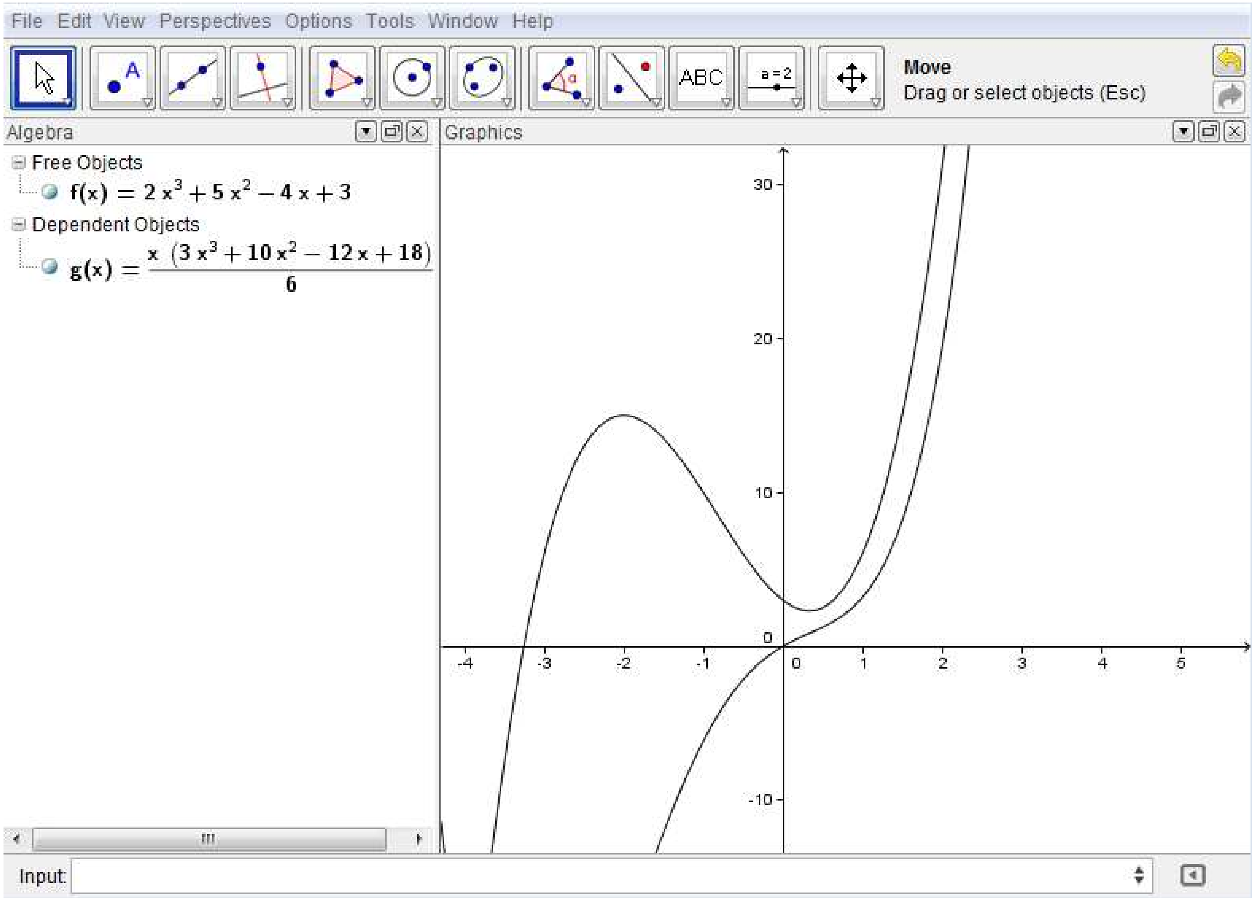
The inflection points are located at $(-1, 1)$ and $(1, 1)$.

***

You can use GeoGebra to find both indefinite and definite integrals.

Example 6: Find the indefinite integral: $\int \left(2x^3 + 5x^2 - 4x + 3\right)\, dx$

Solution: Start by entering the integrand in GeoGebra as function *f*. To find the indefinite integral, type **Integral** into the **Input** line or use the **Input Help** menu to locate **Integral**. Now type f(x) inside the brackets and press **Enter** on your keyboard. GeoGebra will return the indefinite integral in the Algebra window and will graph the antiderivative, along with the function.



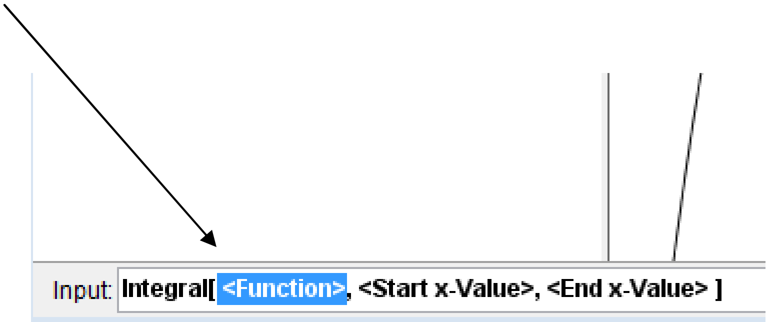
You may want to simplify this indefinite integral, term by term. Note that the antiderivative is called $g(x)$ and does not include the + C. The simplified version of the antiderivative is

$$\int \left(2x^3+5x^2-4x+3\right)dx = 0.5x^4+1.67x^3-2x^2+3x+C$$

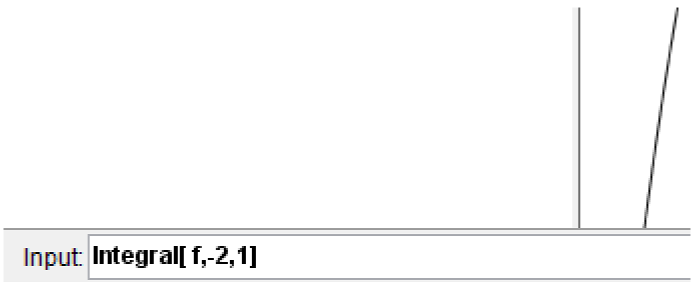
***

Example 5: Evaluate the definite integral: $\int_{-2}^{1}\left(2x^3+5x^2-4x+3\right)dx$

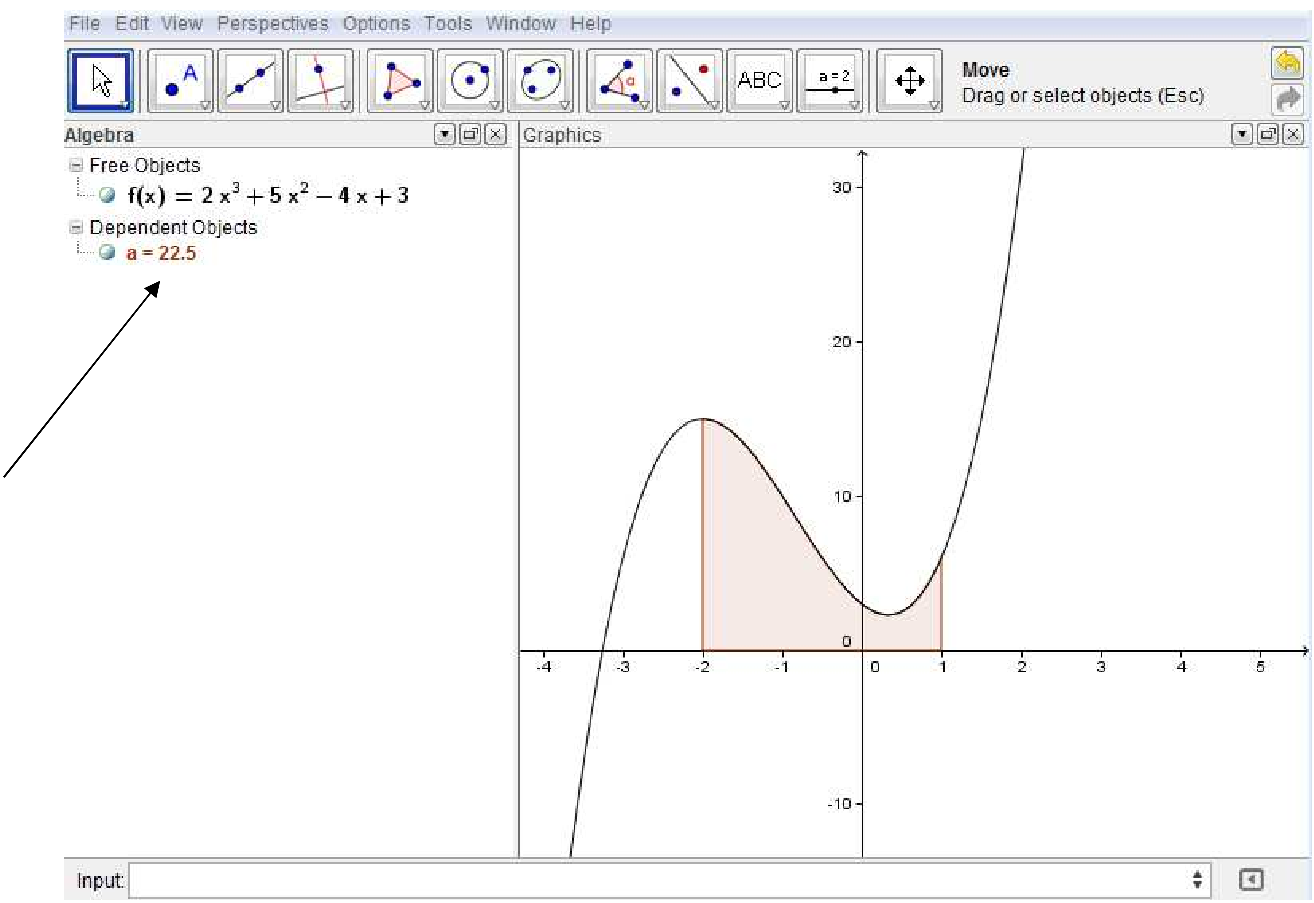
Solution: The integrand is the same as the function you worked with on the previous example and is already in GeoGebra as function *f*. Now you want to find the definite integral. This has an additional step – you must tell GeoGebra what the limits of integration are. You'll first insert the **Integral** function into the **Input** line. You can also use the **Input Help** menu to locate the command.



Now use the tab key and type f, -2, 1 inside the brackets.



Then press **Enter** on your keyboard.



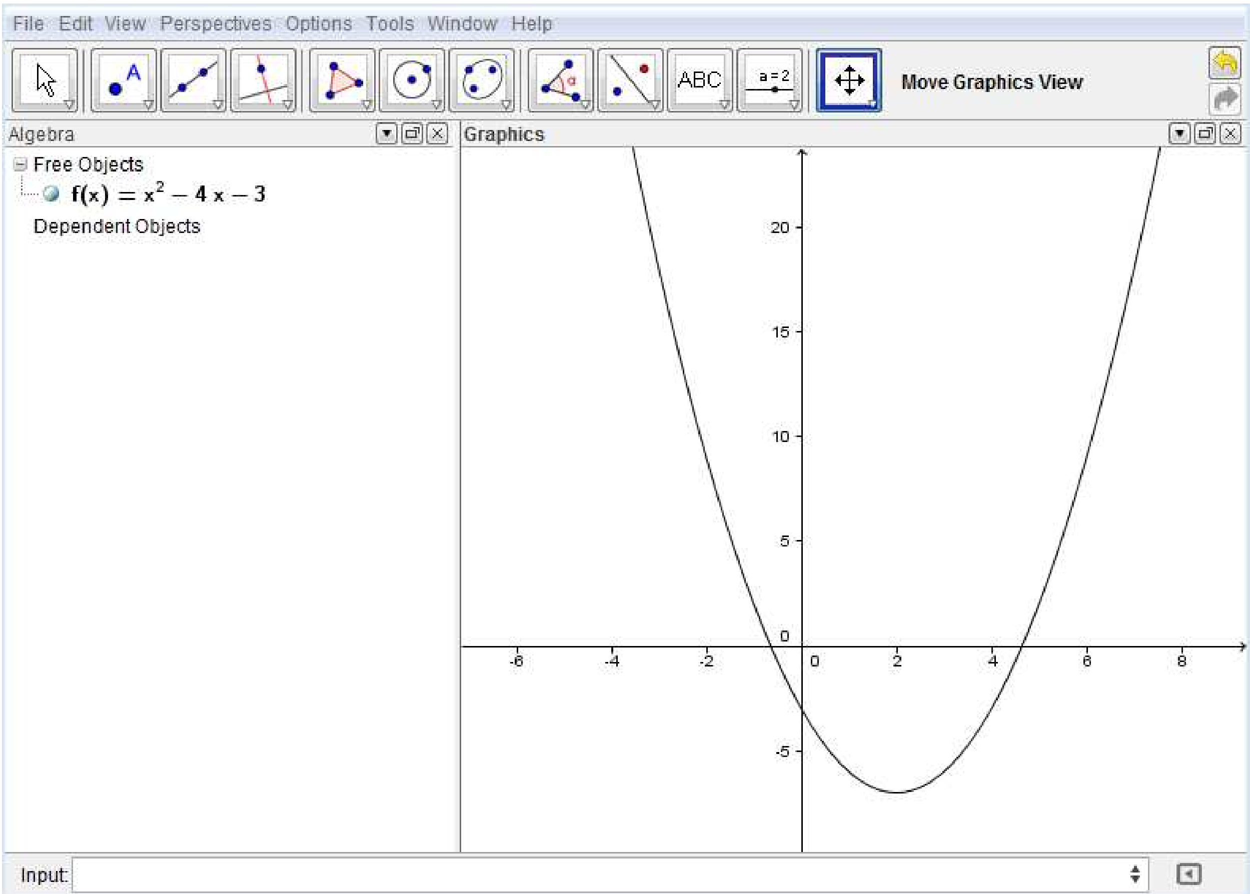
GeoGebra displays a shaded region, corresponding to the region described in the problem. The answer is given as c in the algebra window.

Here's the final answer: $\int_{-2}^{1} \left(2x^3 + 5x^2 - 4x + 3\right) dx = 22.5$.

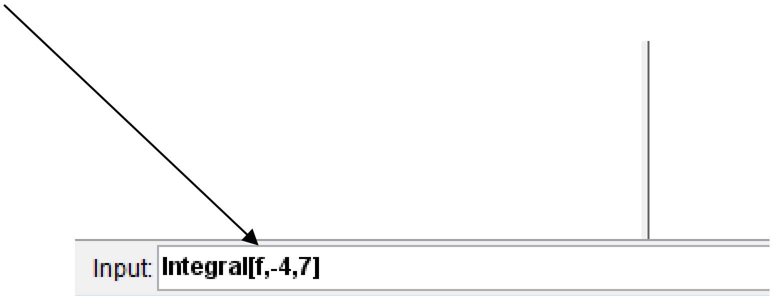
***

Example 6: Evaluate the definite integral: $\int_{-4}^{7} \left(x^2 - 4x - 3\right) dx$

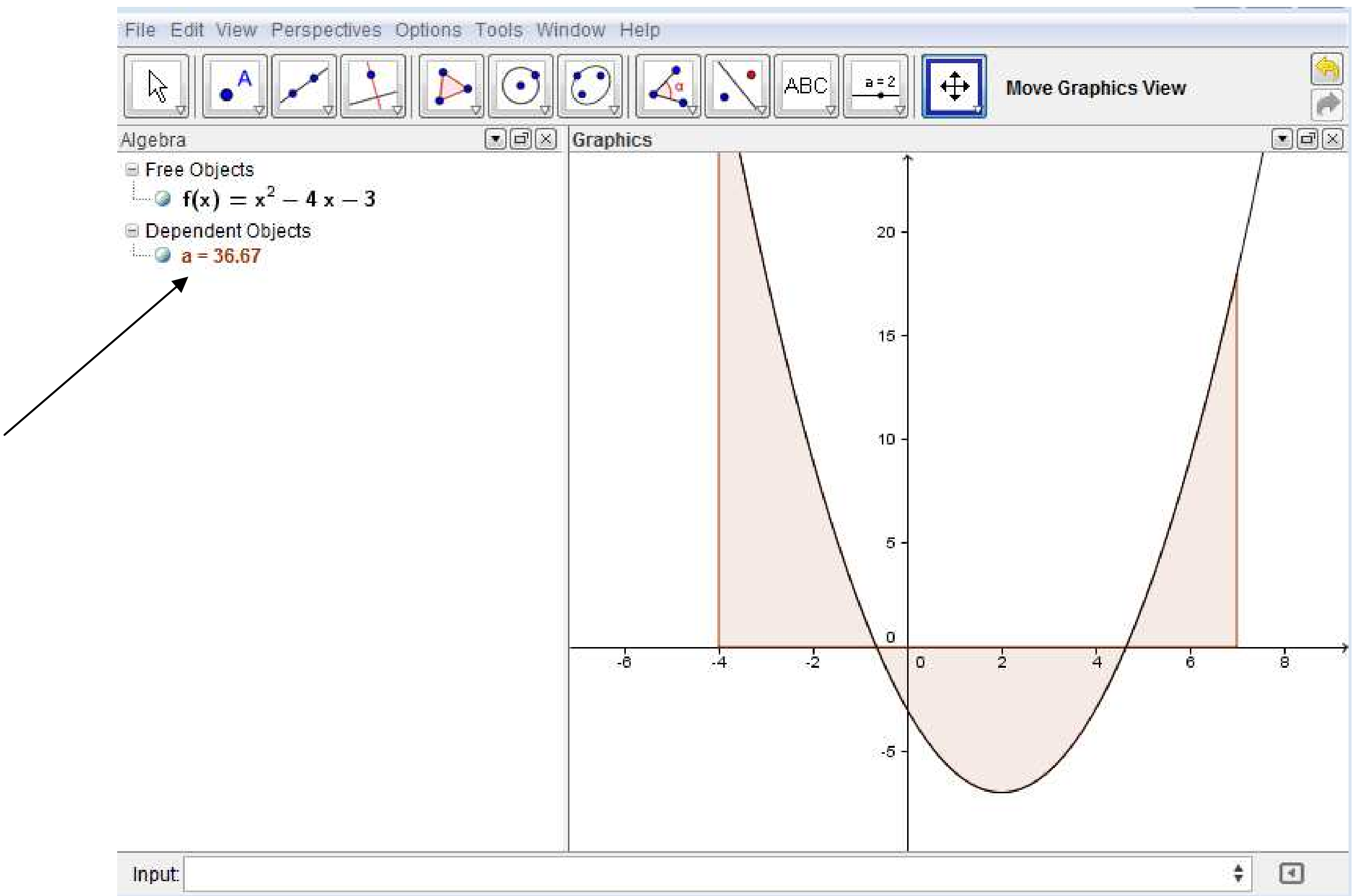
Solution: Start with a graph of the function.



Now enter the necessary commands in the **Input** line.



Press **Enter** on your keyboard.



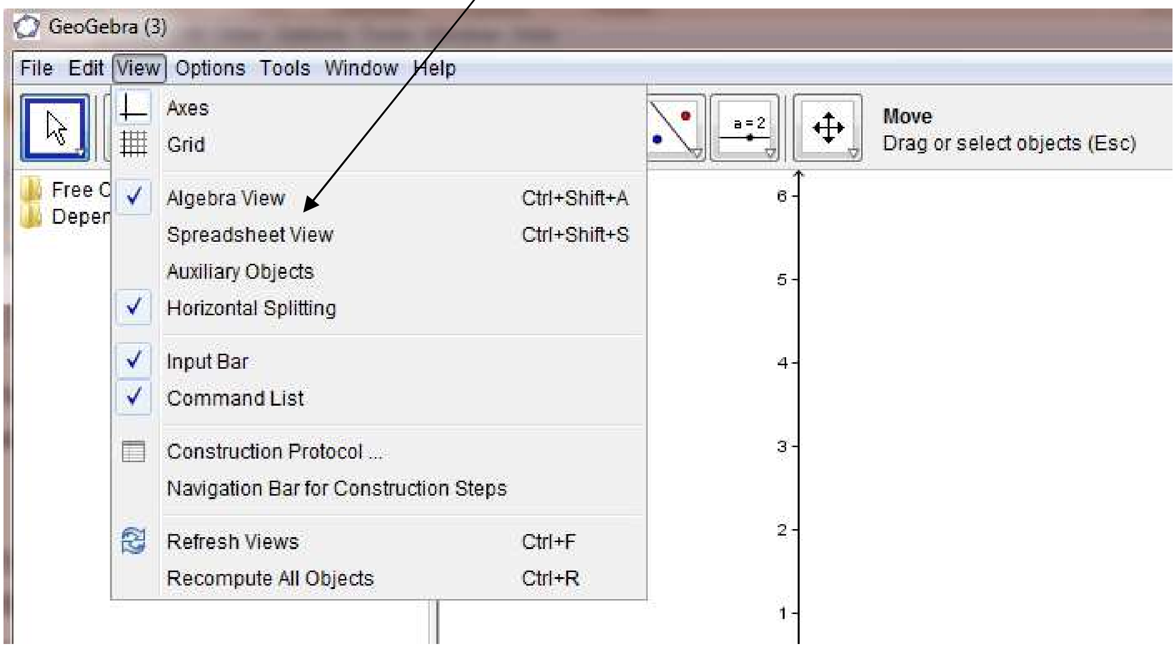
$$\int_{-4}^{7} \left(x^2 - 4x - 3\right)\, dx = 36.67$$

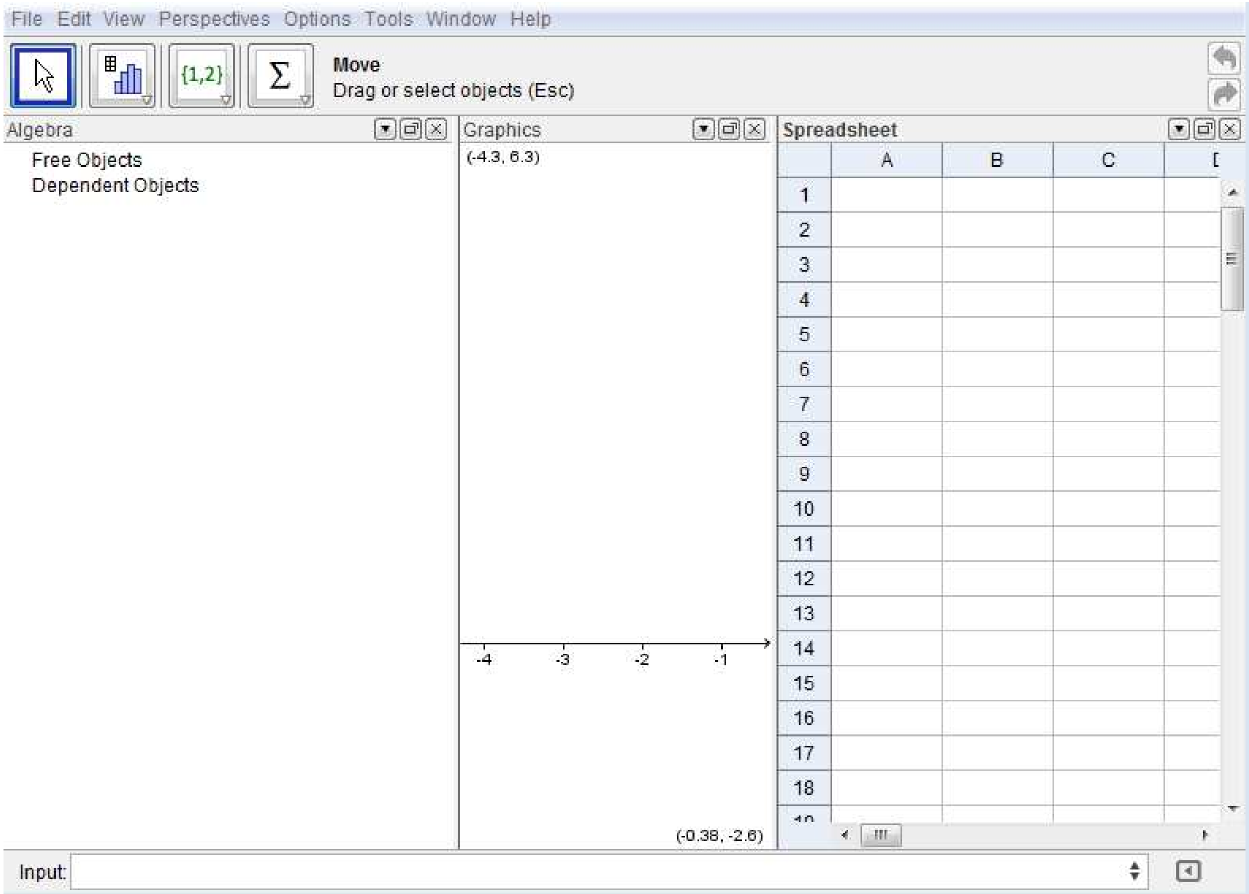
Note that this represents the "net" area of the three regions – that is the sum of the areas above the *x* axis minus the area below the *x* axis.

***

You can also use GeoGebra to find some regression equations. This provides a third option for finding regression equations.

To use GeoGebra for finding regression equations, start by clicking on the **View** menu and selecting **Spreadsheet View**.





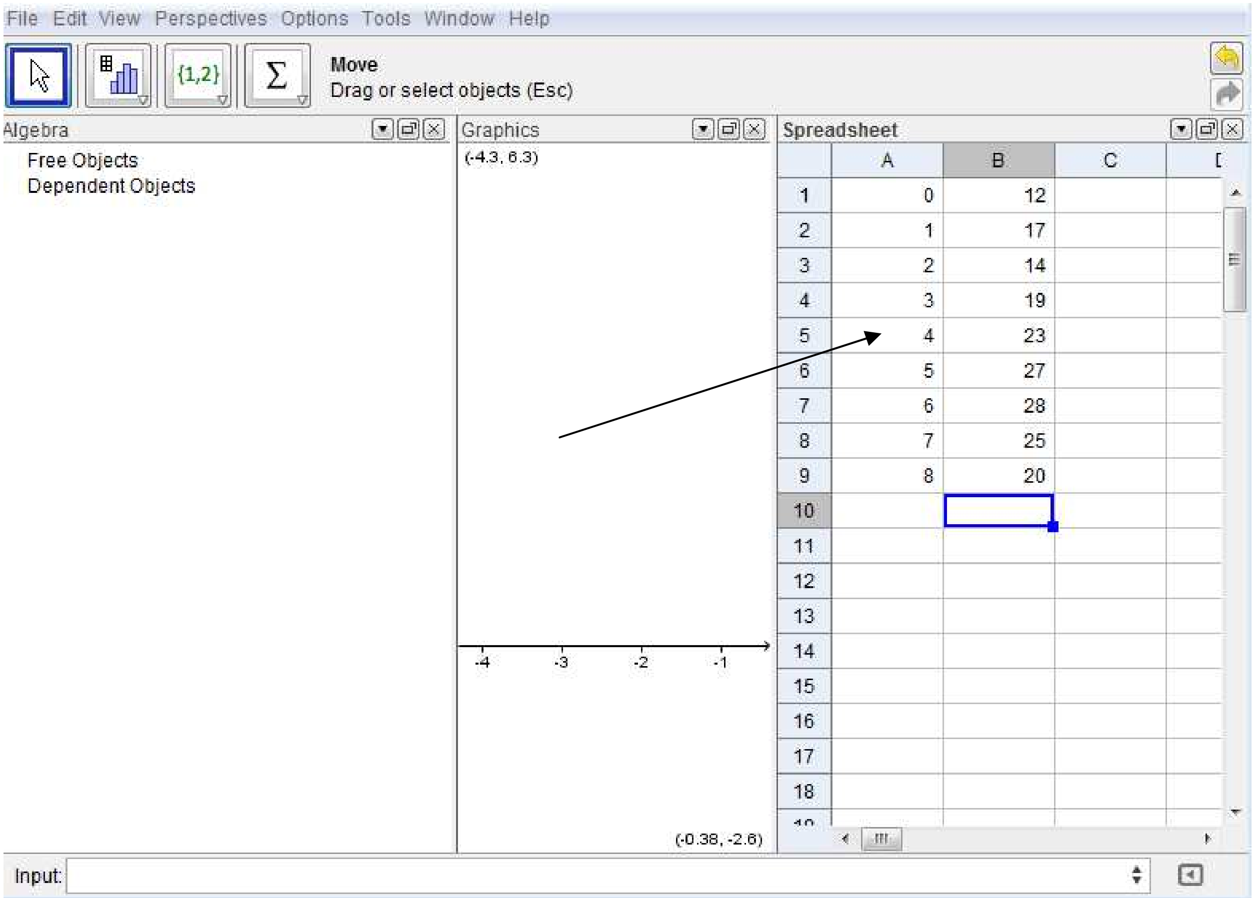
Now, in addition to the algebra window and the graph, you can see a spreadsheet. Note the bar at the bottom of the spreadsheet, which will allow you to move to the right on the spreadsheet to work with additional columns. You can also put the cursor on the line between the spreadsheet and the graph and click and drag to make the spreadsheet display more or fewer columns.

As with Excel, you will enter your information in the columns that are given. For ease, use column A for x values and column B for y values.

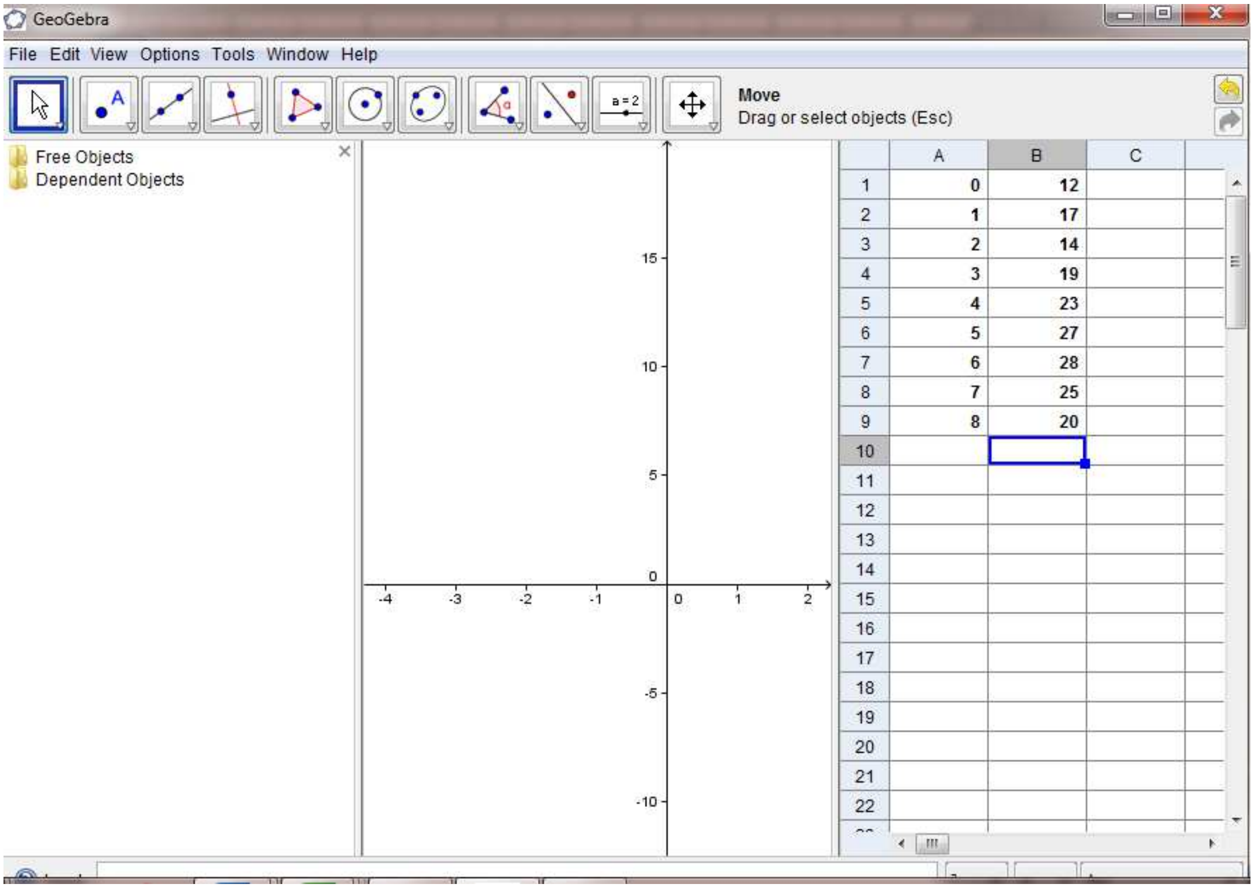
Example 7: Use GeoGebra to find the line of best fit and the polynomial function of degree four that best fits this data. Find r^2 for the line of best fit and R^2 for the quartic function.

x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
y	12	17	14	19	23	27	28	25	20

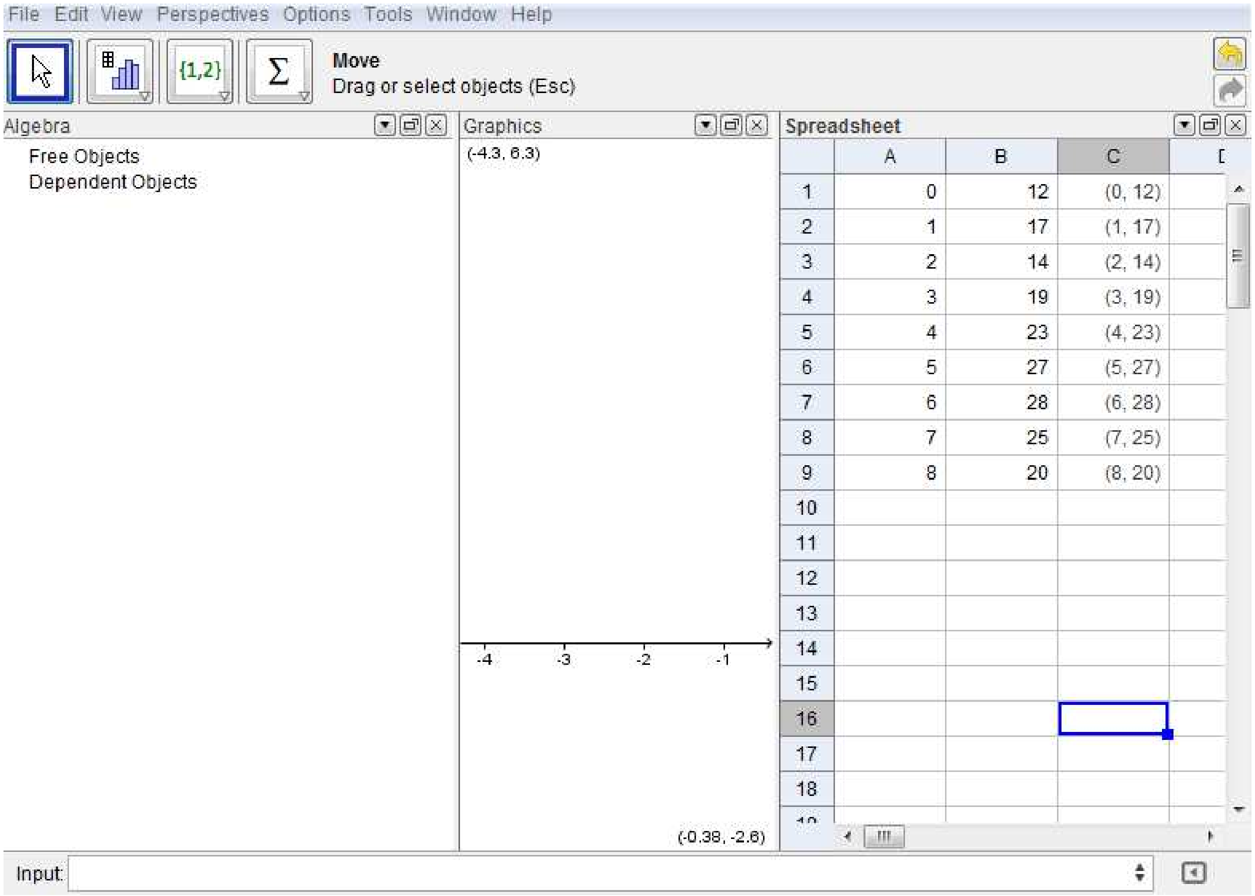
Solution: Start by entering the data in columns A and B of the spreadsheet in GeoGebra.



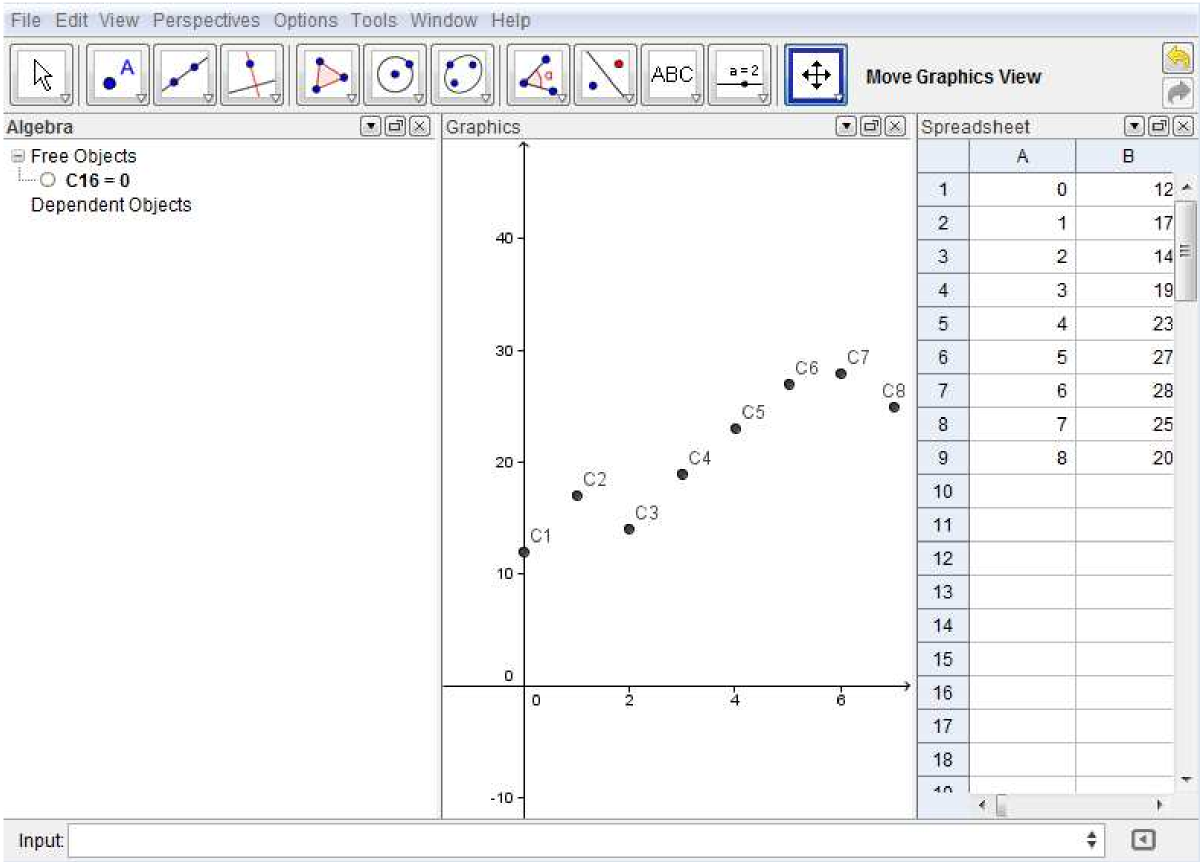
Now, you'll need to write these as ordered pairs. You'll need to be able to view column C, so put your cursor on the line between the spreadsheet and the graph and pull the spreadsheet to the left.



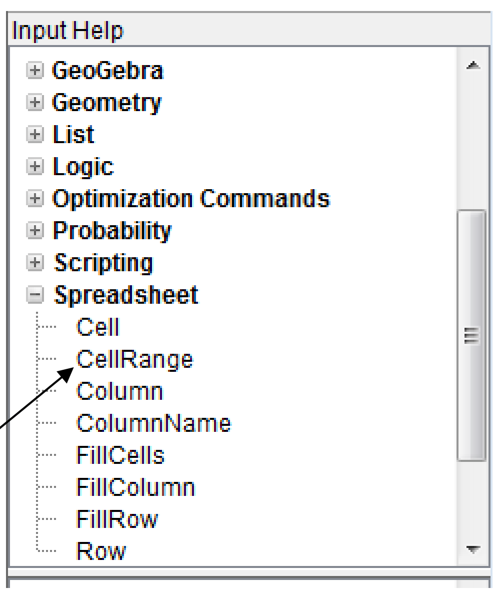
Put the cursor in cell C1 and type = (A1, B1). Press **Enter** on your keyboard. Click click and drag the box on the lower right corner of cell C1 to complete the list.



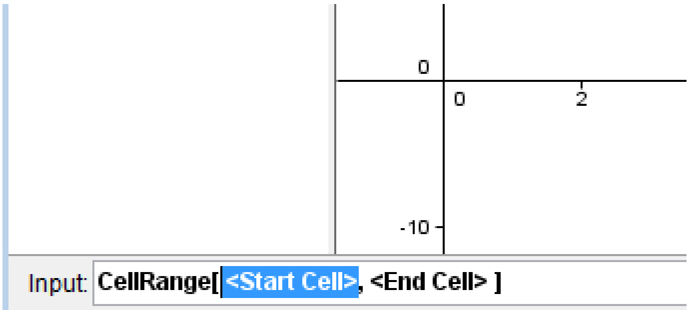
Note that the points appear in column C and on the graph. You may need to move the spreadsheet and resize the graph to be able to see the points.



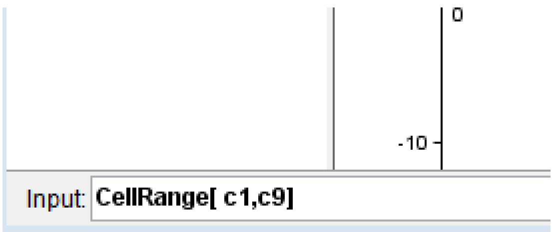
Now you want to find a line of best fit, or a linear regression model. You'll need to create a list of ordered pairs. To do this, you can use the **Input Help** menu to find the needed command on the **Spreadsheet** submenu. The command you need is "Cellrange".



You can also access command just by typing in the input line.

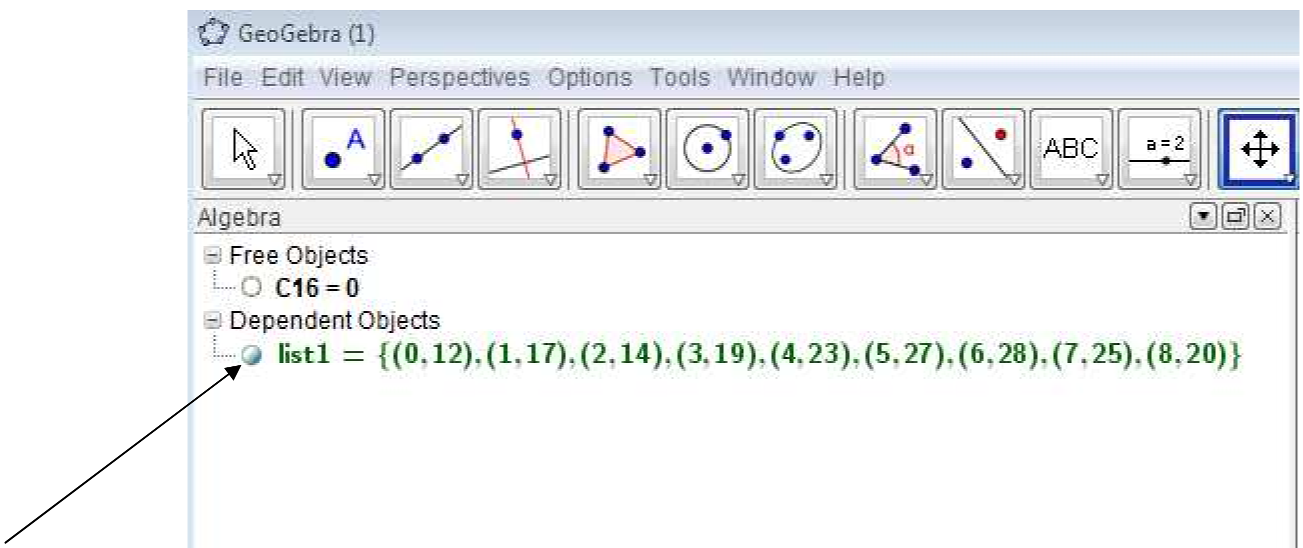


Enter the first and last cell names for the points inside the brackets, “c1, c9”.



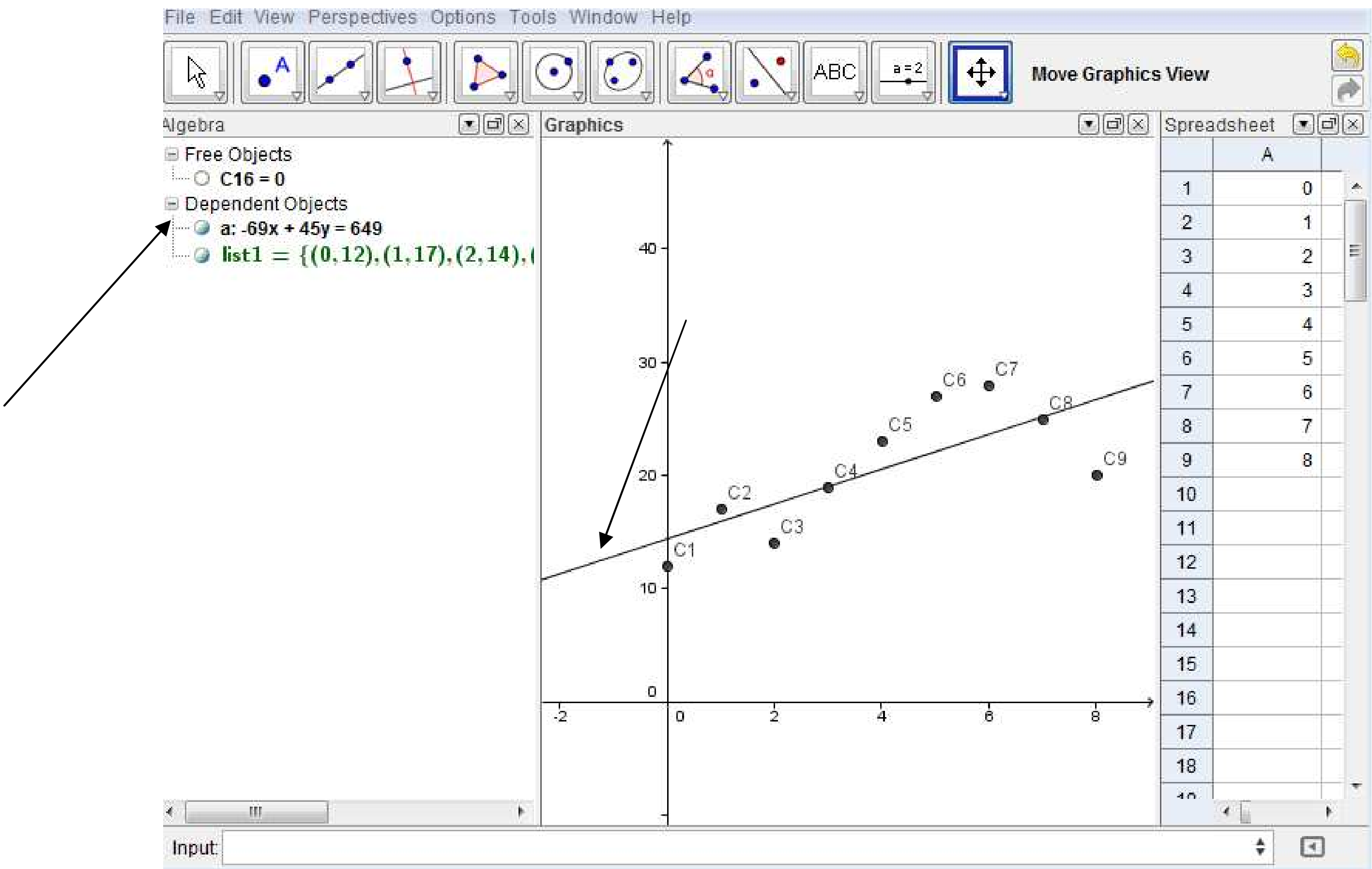
Then press **Enter** on your keyboard.

The ordered pairs will show up as “list1” in the algebra window.



Now find the line of best fit. Put the cursor in the **Input** line and type “Fitline”, Scroll up or use the **Statistics** submenu of **Input Help** to find **FitLine**.

Type “list1” inside the brackets. Then press **Enter** on your keyboard.

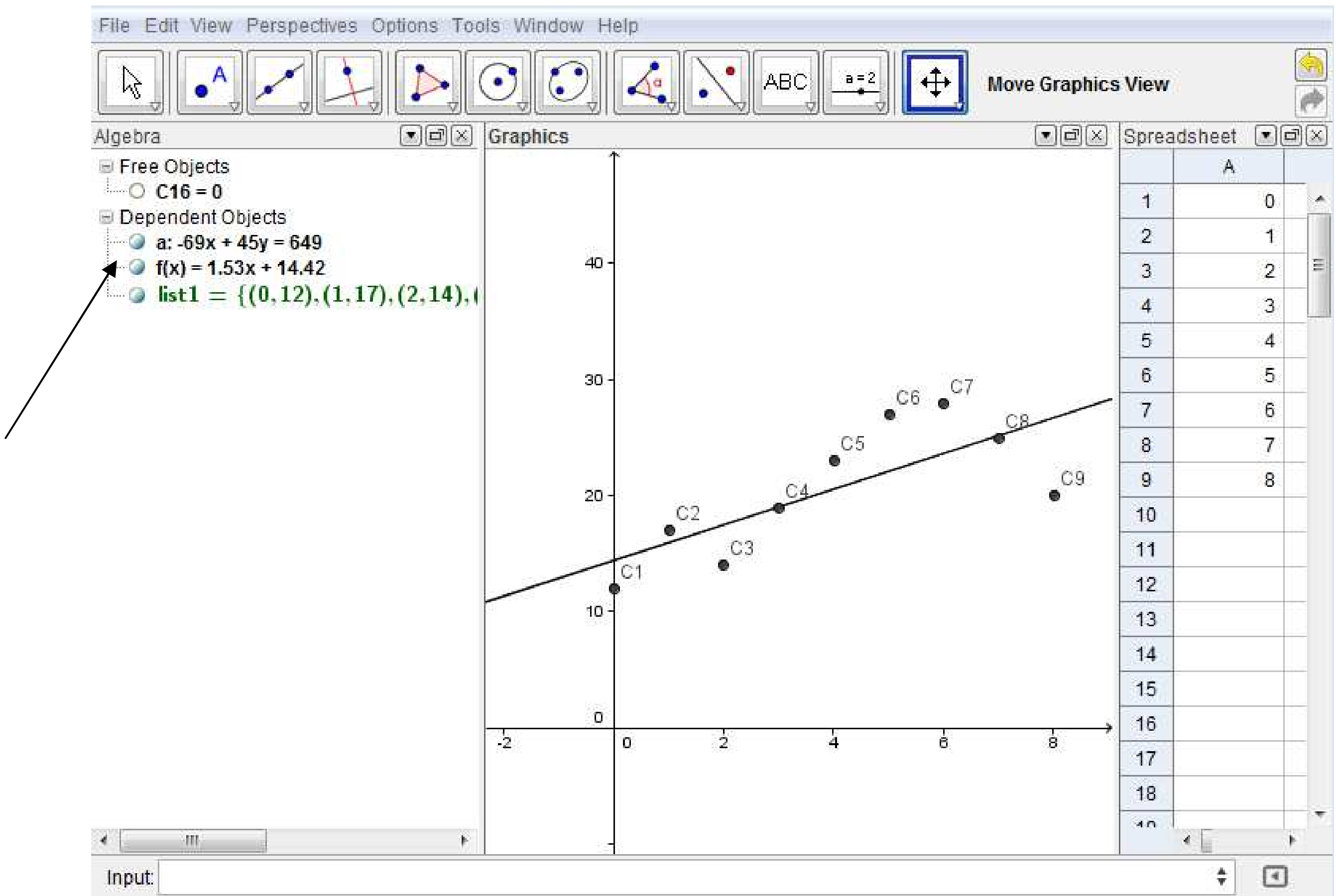


The line of best fit is given in standard form in the algebra window, $69x - 45y = -649$. The graph of the line appears in the Graphics window, along with the data points.

You could also find this equation in function form by finding a first degree polynomial, using the same list. To to this, put your cursor on the **Input** line and type “FitPoly”, or scroll up or down the list of commands until you find **FitPoly**.

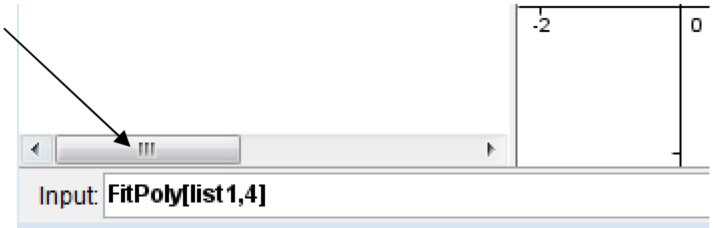
This command requires that you name the data list to use and also state the degree of the polynomial you wish to find. Since a line is a first degree polynomial, type "list1, 1" inside the brackets.

Then press **Enter** on your keyboard.

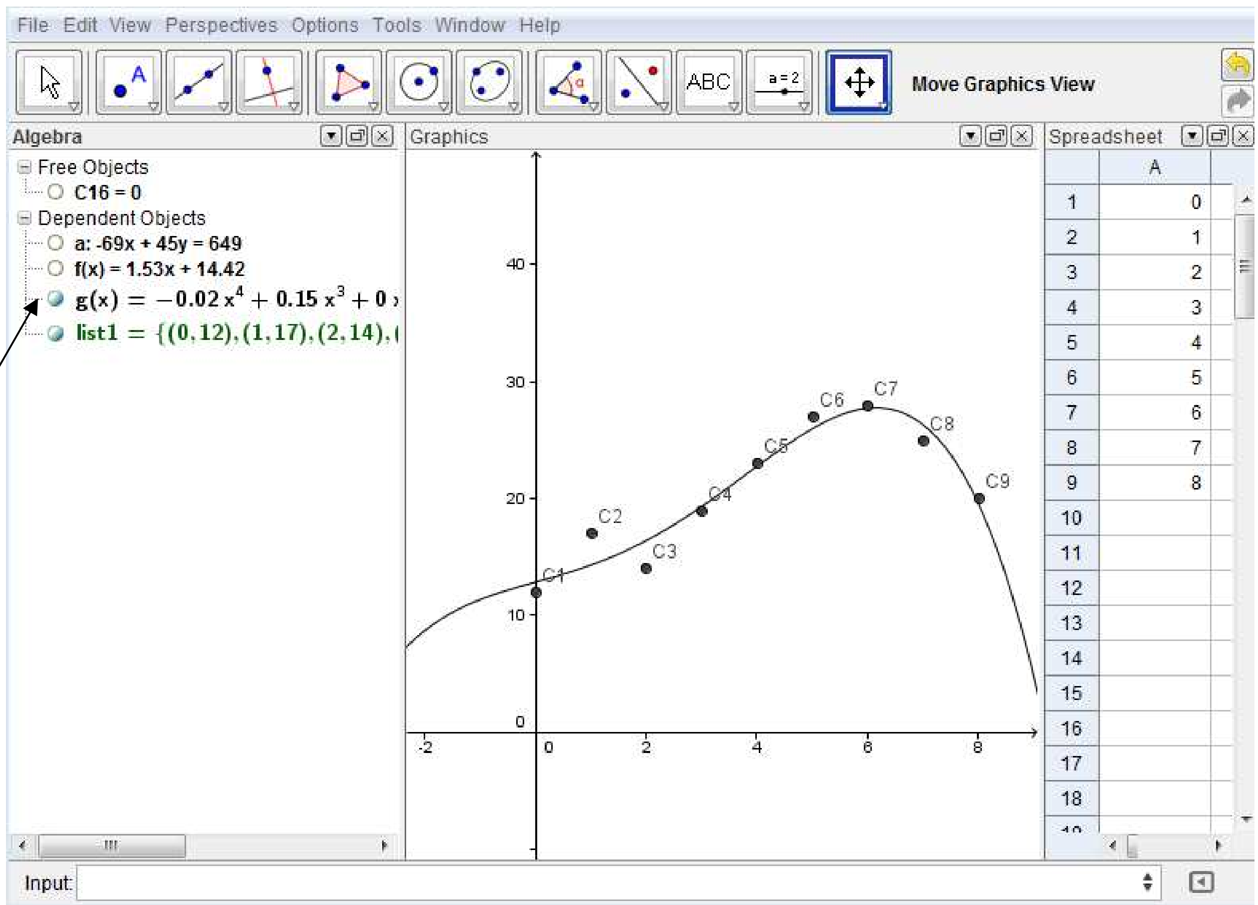


In function form, the line of best fit is $f(x) = 1.53x + 14.42$. Write $69x - 45y = -649$ in slope-intercept form and you'll see that the two equations are identical.

The problem also asks you to find a fourth degree polynomial of best fit. To do this, you can repeat the steps for a first degree polynomial, but now choose degree 4 instead of degree 1. In the **Input** line, enter the command FitPoly, then type "list1, 4" in the brackets.

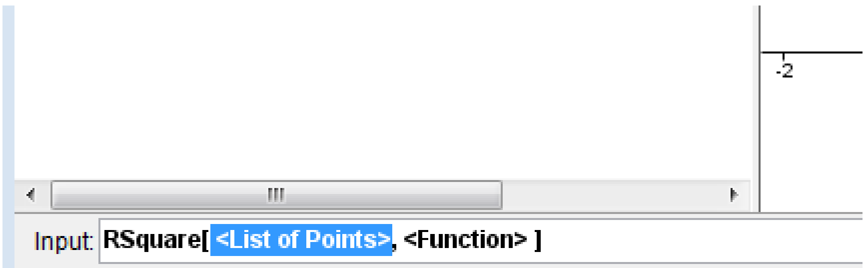


Enter on your keyboard.

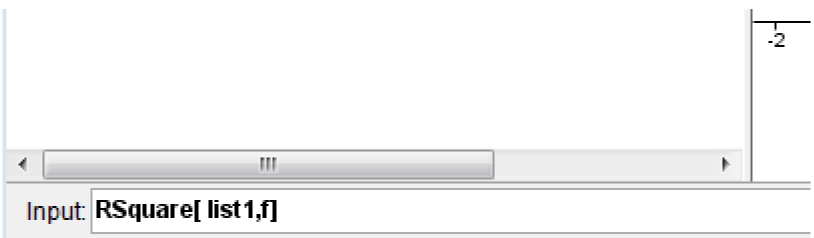


The quartic function of best fit is $g(x) = -0.02x^4 + 0.15x^3 + 1.33x + 12.83$. You can move the Graphics window to the right to view the entire function.

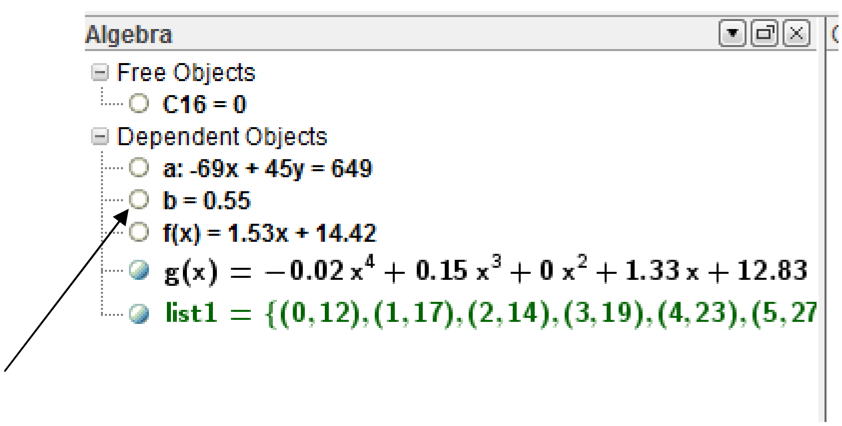
Finally, you can use GeoGebra to find the values for r^2 and R^2. To do this, you ask GeoGebra to compare a list of ordered pairs to a regression equation. The command is **Rsquare**, and you can just type it in the input line or find it on the **Statistics** submenu of the **Input Help** menu.



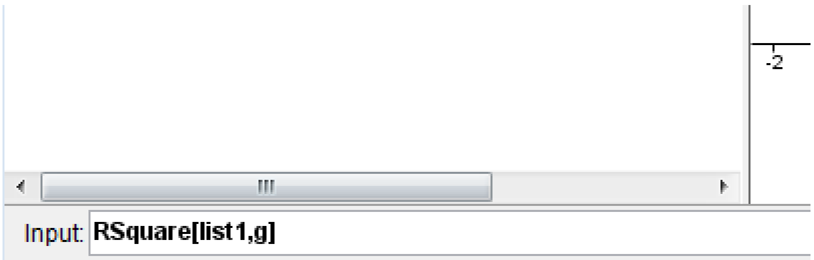
To find the r^2 value for the line of best fit, compare list1 with $f(x)$.



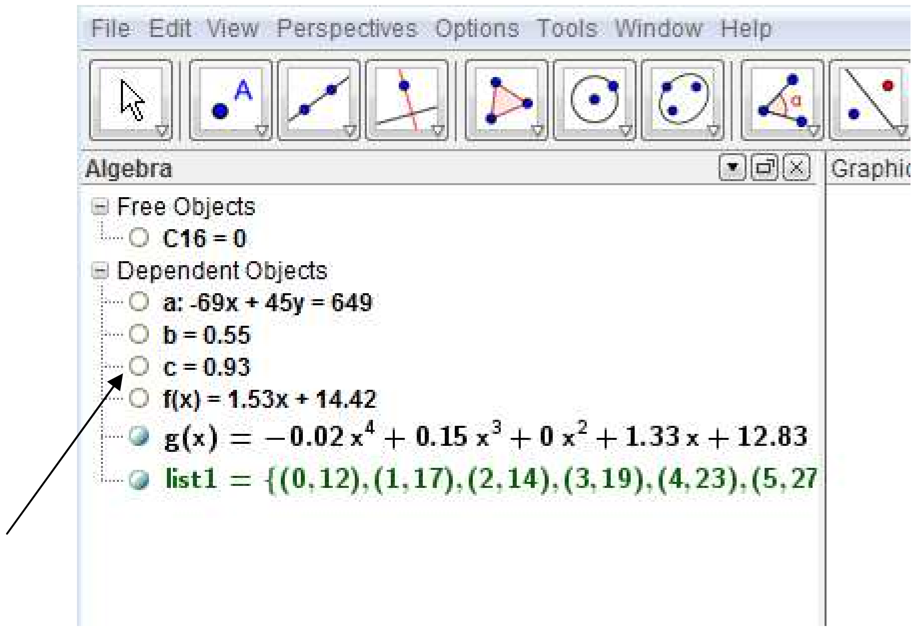
Now press **Enter** on your keyboard. The value for r^2 is listed as *b* in the Algebra window.



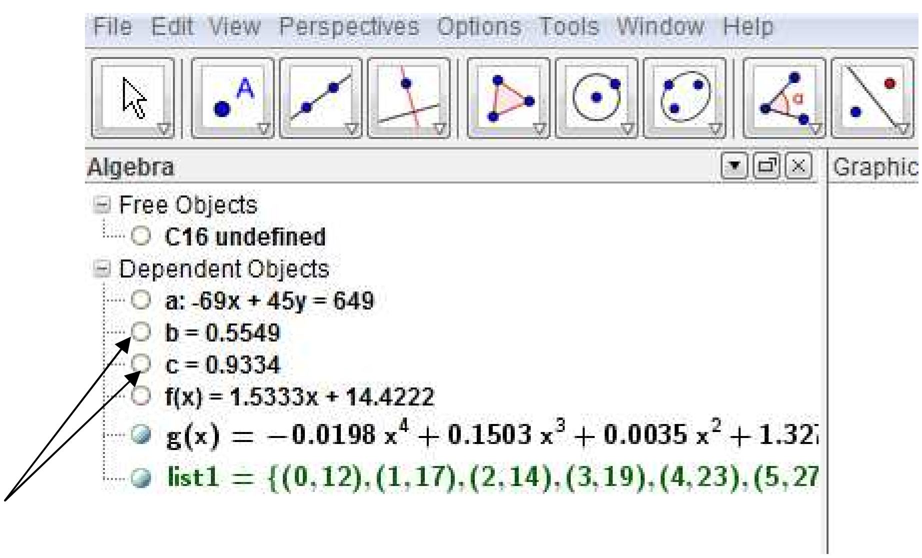
Now repeat the process to find the R^2 value for the quartic function, which is $g(x)$ in the Algebra window.



Press **Enter** and read off the value for R^2. It is listed as c in the Algebra window.



If you need to display R^2 to more decimal places, choose **Rounding** on the **Options** menu and select more decimal places.



Both values and all coefficients have increased to four decimal places in the last window.

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