

Math 1313
The Normal Distribution

Basic Information

All of the distributions of random variables we have discussed so far deal with distributions of finite discrete random variables. Their probability distributions are referred to as finite probability distributions.

Recall that the graph of this probability distribution is a bar graph.

In this section, we will look at a continuous random variable, so our probability distributions will now be continuous probability distributions. With a finite probability distribution, we can create a table and list each value of the RV and its corresponding probability. We can't do that with a continuous probability distribution. Instead, we will look at a function, called a probability density function.

The graph of a continuous probability density function looks something like this:

Properties of all probability density functions:

1. $f(x) > 0$ for all values of x
2. area between the curve and the x axis is 1

We will look at special probability density functions - **normal distributions** - which have these additional properties:

3. there's a peak at the mean
4. the curve is symmetric about the mean
5. 68.7% of the area is within one standard deviation of the mean (i.e., in the interval

95.45% of the area is within two standard deviations of the mean (i.e., in the interval

and 99.73% of the area is within three standard deviations of the mean (i.e., in the interval

6. The curve approaches the x axis as x extends indefinitely in either direction

We will begin by looking at **standard normal distributions** which have these additional properties:

7. mean is 0
8. standard deviation is 1

Notation: We will denote the random variable which gives us the standard normal distribution by Z .

Note that all of the following are equal, so we will adopt a uniform notation:

Using the Standard Normal Table

Next, we will learn to use the standard normal table to find probabilities:

The standard normal table gives the area between the curve and the x axis to the left of the line $x = z$. This area corresponds to the probability that Z is less than z or $P(Z < z)$.

We can use the table to solve two classes of problems. In the first class, we are trying to find a number in the body of the table, so we use the numbers on the left margin and at the top to find the number we need. Let's say we want to find $P(Z < 2.15)$. Look down the left margin until you see the number 2.1.

Once you have found 2.1, follow the numbers across that line in the body of the table until you are underneath 0.05. You should have located the number 0.9842. So, $P(Z < 2.15) = 0.9842$. This means that the probability that a randomly selected data point is less than 2.15 is 0.9842.

The table that I have reproduced here may look slightly different from the one you have, but the numbers are the same.



Probability Content from $-\infty$ to Z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952

We can use the properties listed above to answer many different types of questions about probabilities.

Example 1: Find $P(Z < 1.36)$

Examples 2 - 5

Example 2: Find $P(-1.25 < Z < 2.03)$

Example 3. Find $P(Z > 1.78)$

Example 4: Find $P(-.68 < Z < 1.41)$

Example 5: Find $P(Z > -0.6)$

Another Type of Problem

Sometimes, we'll be given $P(Z < z)$ and we'll want to find z . In other words, we'll be given what was the answer in previous problems. We'll look in the body of the table we find the number we need, then read off the value for z from the left margin and top.



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from $-\infty$ to Z**

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2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952

Example 6: Find z if $P(Z < z) = .8944$

Example 7: Find z if $P(Z < z) = .0401$

Example 8: Find z if $P(Z > z) = .9463$

Example 9: Find z if $P(Z > z) = .0132$

Example 10: Find z if $P(-z < Z < z) = .9812$

Example 11: Find z if $P(-z < Z < z) = .5408$

Example 12: Find z if $P(-z < Z < z) = .1820$

Example 13: Find z if $P(-z < Z < z) = .7458$

Working with the Normal Distribution

Next we need to look at what we'll do with a distribution that is normal, but not standard normal (that is, it meets the criteria to be a normal distribution but it has mean not equal to 0 and standard deviation not equal to 1).

We can convert any problem involving probability of a normally distributed RV to one with a standard normal RV, Z . This will allow us to easily find the probability using the standard normal table.

Here's how:

Let X be a normal random variable with $E(X) = \mu$ and standard deviation $= \sigma$. Then X can be converted to the standard normal random variable using the formula

To evaluate this new problem, we'll use the techniques of the above examples.

Example 14: Suppose X is a normally distributed random variable with $\mu = 50$ and $\sigma = 30$. Find $P(X < 95)$.

Example 15: Suppose X is a normally distributed random variable with $\mu = 85$ and $\sigma = 16$. Find $P(X > 54)$.

Example 16: Suppose X is a normally distributed random variable with a mean of 100 and standard deviation of 20. Find $P(85 < X < 110)$.