Math 1313
Chapter 2 – Section 2.2

The Gauss-Jordan Elimination Method

When a system of linear equations involves a small number of equations and the number of variables is small, the method of substitution is suitable. But for large systems of equations the method becomes difficult to manage.

A suitable technique for solving systems of linear equations of any size is the **Gauss-Jordan Elimination Method**.

A **matrix** is a rectangular array of numbers. These numbers are called entries of the matrix.

A lot amount of work can be saved by using matrices to solve systems of linear equations. We will eliminate writing the variables and write only the coefficients and constants.

Given the following system of equations:

$$8x + 9y - 3z = 1$$
$$9x - 5z = 13$$
$$-7x - y + 4z = -10$$

This system can be represented by a matrix.

The **coefficient matrix** is $\begin{pmatrix} 8 & 9 & -3 \\ 9 & 0 & -5 \\ -7 & -1 & 4 \end{pmatrix}$.

The **augmented matrix** is $\left(\begin{array}{ccc|c} 8 & 9 & -3 & 1 \\ 9 & 0 & -5 & 13 \\ -7 & -1 & 4 & -10 \end{array}\right)$.

Example 1: Write the augmented matrix corresponding to the given system of equations.

$8x - 10y = 4$

$-4y = 3$

Example 2: Write the system of equations corresponding to the given augmented matrix.

$$\left(\begin{array}{cc|c} -3 & 9 & 10 \\ 7 & -4 & 2 \end{array}\right)$$

Row-Reduced Form

1. Each row consisting entirely of zeros lies below any other row having nonzero entries.
2. The first nonzero entry in each row is 1 (called a **leading 1**).
3. In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading 1 in the upper row.
4. If a column contains a leading 1, then the other entries in that column are zeros.

Example 3: Determine which of the following matrices are in row-reduced from. If a matrix is not in row-reduced form, state which condition is violated.

a. $\left(\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & -5 \end{array}\right)$

b. $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \end{array}\right)$

c. $\left(\begin{array}{ccc|c} 1 & 8 & 0 & -20 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$

d. $\left(\begin{array}{ccc|c} 1 & -8 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & -4 \end{array}\right)$

The Gauss-Jordan elimination method is a suitable technique to solve a system of linear equations. The **row operations** of this method, along with notation, are:

1. Interchange any two row.
Notation: $R_i \leftrightarrow R_j$ to mean: Interchange row *i* with row *j*.

2. Replace any row by any nonzero constant multiple of itself.
Notation: cR_i to mean: Replace row *i* with *c* times row *i*.

3. Replace any row by the sum of that row and a constant multiple of any other row.
Notation: $R_i + aR_j$ to mean: Replace row *i* with the sum of row *i* and *a* times row *j*.

A column in a coefficient matrix is called a **unit column** if one of the entries in the column is a 1 and the other entries are zeros.

The sequence of row operations that transform a column in an augmented matrix into a unit column is called **pivoting the matrix about the element that turned into a 1**.

For example, given the following augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right).$$

The row operations that transformed the first column into the unit column $\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}$ in the following matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{array}\right)$$

is called pivoting the matrix about the element 2.

Example 4: Pivot the matrix about the circled element.

$$\left(\begin{array}{cc|c} 2 & 9 & 2 \\ 12 & \textcircled{6} & 36 \end{array}\right)$$

The Gauss-Jordan Elimination Method

1. Write the augmented matrix corresponding to the linear system.
2. Interchange rows, if necessary, to obtain an augmented matrix in which the first entry in the first row is nonzero. Then pivot the matrix about this entry.
3. Interchange the second row with any row below it, if necessary, to obtain an augmented matrix in which the second entry in the second row is nonzero. Pivot the matrix about this entry.
4. Continue until the final matrix is in row-reduced from.

Example 5: Solve the system of linear equations using the Gauss-Jordan elimination method.

a. $$\begin{aligned} 2x - 4y &= -14 \\ 3x + 2y &= 3 \end{aligned}$$

b. $$\begin{aligned} 3x + y + 2z &= 31 \\ x + y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Example 6: You invested a total of \$38,000 in two municipal bonds – Bond A and Bond B, that have a yield of 4% and 6% interest per year, respectively. The interest you earned from the bonds was \$1,930. How much did you invest in each bond?

Example 7: A popular play at a certain performance hall sold 1,000 tickets on opening night. The seats in section A, the best section, sold for \$80 each, each seat in the middle section, Section B, sold for \$60, and each seat in Section C, the farthest section, sold for \$50. The combined number of tickets sold for Sections A and B exceeded twice the number of Section C tickets sold by 400. The total receipts for the performance were \$62,800. Determine how many tickets of each type were sold.