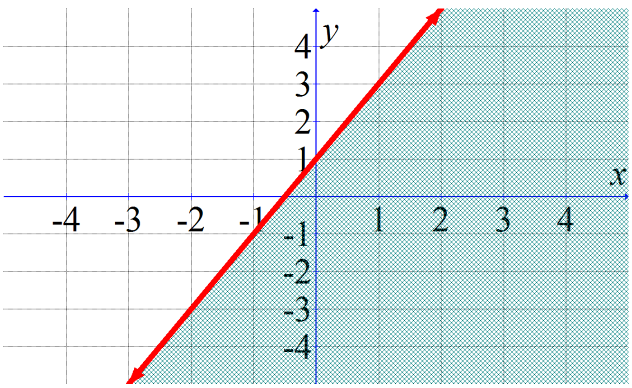
Section 1.4 – Graphs of Linear Inequalities

A Linear Inequality and its Graph

A **linear inequality** has the same form as a linear equation, except that the equal symbol is replaced with any one of $\leq$, $\geq$, $<$, or $>$.

The **solution set to an inequality** in two variables is the set of all ordered pairs that satisfies the inequality, and is best represented by its graph. The graph of a linear inequality is represented by a straight or dashed line and a shaded half-plane. An illustration is shown below.



Example 1: Without graphing, determine whether $(-3, -7)$ is a solution to $y > x - 4$.

Solution: Substitute $x = -3$ and $y = -7$ into the inequality and determine if the resulting statement is true or false.

$$y > x - 4$$

$$-7 \overset{?}{>} -3 - 4$$

$$-7 \not> -7$$

This statement $-7 > -7$ is false, so the point $(-3, -7)$ is not a solution to the inequality.

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Example 2: Without graphing, determine whether $(-1, 1)$ is a solution to $2x+10y \geq 5$.

Solution: Substitute $x=-1$ and $y=1$ into the inequality and determine if the resulting statement is true or false.

$$2x+10y \geq 5$$

$$2(-1)+10(1) \overset{?}{\geq} 5$$

$$-2+10 \overset{?}{\geq} 5$$

$$8 \geq 5$$

This statement $8 \geq 5$ is true, so the point $(-1, 1)$ is a solution to the inequality.

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Graphing a Linear Inequality in Two Variables

Next, we will graph linear inequalities in two variables. There are several steps, which are outlined below.

Steps for Graphing a Linear Inequality in Two Variables

1. Rewrite the inequality as an equation in order to graph the line.
2. Determine if the line should be solid or dashed. If the inequality symbol contains an equal sign (i.e. $\leq$ or $\geq$), graph a solid line. If the inequality symbol does not contain an equal sign (i.e. $<$ or $>$), graph a dashed line.
3. Determine which portion of the plane should be shaded. Choose a point not on the line, and plug it into the inequality.
4. If the test point satisfies the inequality, shade the half-plane containing this point. Otherwise, shade the other half-plane.

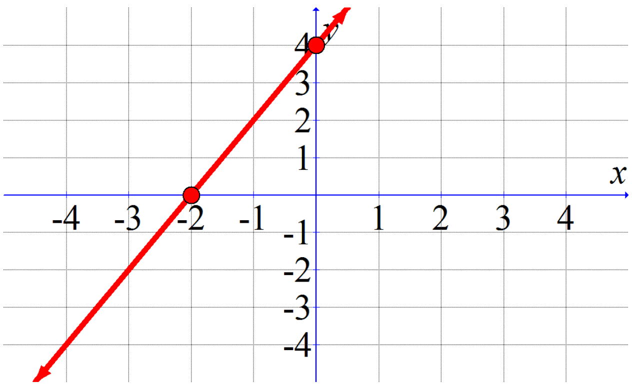
Example 3: Graph the inequality $-2x+y \leq 4$.

Solution: We first write the inequality as an equation, $-2x+y=4$. The line will be graphed as a solid line because the inequality in this problem is $\leq$, which includes the line. We can graph the line using *x*- and *y*-intercepts, or by putting it in slope-intercept form, $y=mx+b$.

We will choose to find the *x*- and *y*-intercepts of $-2x + y = 4$.

$-2x + y = 4$	$-2x + y = 4$
$-2x + 0 = 4$	$-2(0) + 4$
$-2x = 4$	$y = 4$
$x = -2$	

A solid line is drawn through the intercepts, which are located at $(-2,\ 0)$ and $(0,\ 4)$.



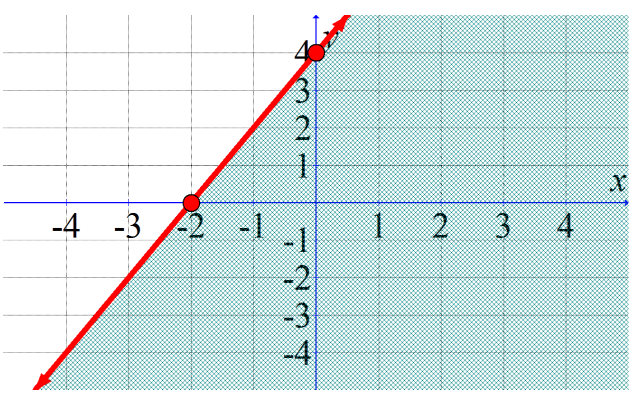
We now need to determine which portion of the plane should be shaded. To do this, we choose any test point not on the line, and substitute those coordinates into the inequality to determine if the resulting statement is true. We will choose the point $(0,\ 0)$.

$$-2x + y \le 4$$

$$-2(0) + 0 \overset{?}{\le} 4$$

$$0 \le 4$$

Since $0 \le 4$ is true, then $(0,\ 0)$ satisfies the inequality, and so we shade the half-plane containing $(0,\ 0)$.



The solution set is the half-plane lying on or below the line $-2x + y = 4$.

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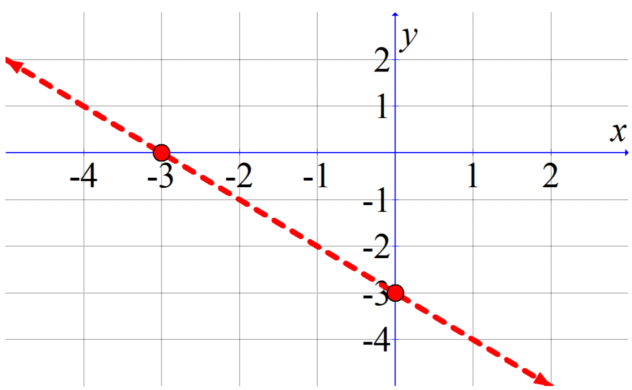
Example 4: Graph the inequality $x + y < -3$.

Solution: We first write the inequality as an equation, $x + y = -3$. The line will be graphed as a dashed line because the inequality in this problem is $<$, which does not include the line. We can graph the line using *x*- and *y*-intercepts, or by putting it in slope-intercept form, $y = mx + b$.

We will choose to find the *x*- and *y*-intercepts of $x + y = -3$.

$x + y = -3$	$x + y = -3$
$x + 0 = -3$	$0 + y = -3$
$x = -3$	$y = -3$

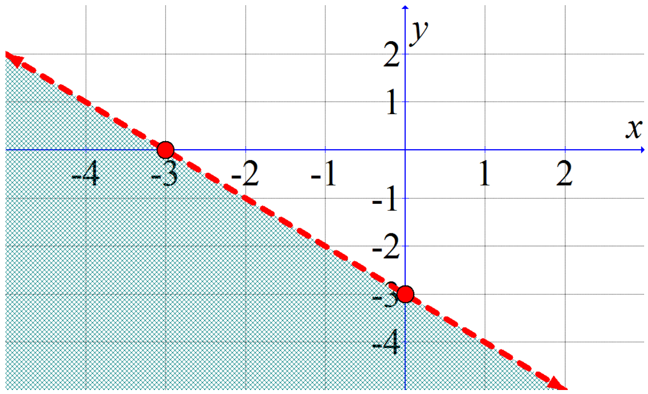
A dashed line is drawn through the intercepts, which are located at $(-3, 0)$ and $(0, -3)$.



We now need to determine which portion of the plane should be shaded. To do this, we choose any test point not on the line, and substitute those coordinates into the inequality to determine if the resulting statement is true. We will choose the point $(0, 0)$.

$$x + y < -3$$
$$0 + 0 < -3$$
$$0 \not< -3$$

Since $0 < -3$ is not true, then $(0, 0)$ does not satisfy the inequality, and so we shade the half-plane not containing $(0, 0)$.



The solution set is the half-plane lying below the line $x + y = -3$.

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There is a shortcut for graphing inequalities without using test points, provided that the inequality is written in the form $y < mx + b$, $y \le mx + b$, $y > mx + b$, or $y \ge mx + b$.

Graphing Inequalities Without Using Test Points

Inequality	**The solution is the half-plane lying:**
$y < mx + b$	below the line $y = mx + b$.
$y \le mx + b$	on or below the line $y = mx + b$.
$y > mx + b$	above the line $y = mx + b$.
$y \ge mx + b$	on or above the line $y = mx + b$.

If we think about the meaning of the chart above, we can remember its information without memorization. To say that $y < mx + b$ means that for every value of *x*, the *y*-values of the solution are lower than the *y*-value of the line, and therefore the shading occurs below the line. To say that $y > mx + b$ means that for every value of *x*, the *y*-values of the solution are higher than the *y*-value of the line, and therefore the shading occurs above the line. And as discussed in previous examples, an equals sign in the inequality means that the line is included in the solution.

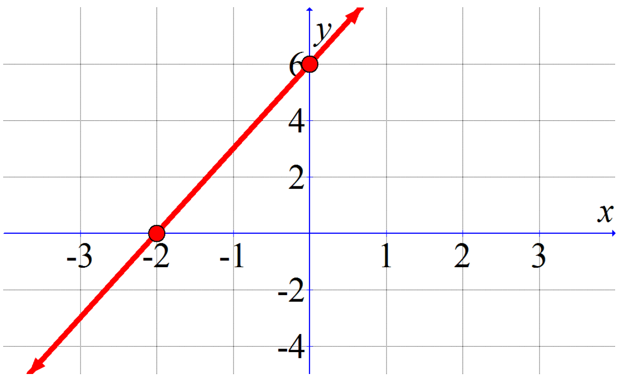
Example 5: Graph the inequality $y \ge 3x + 6$.

Solution: We first write the inequality as an equation, $y = 3x + 6$. The line will be graphed as a solid line because the inequality in this problem is $\ge$, which includes the line. We can graph the line using *x*- and *y*-intercepts, or by using the slope and *y*-intercept from slope-intercept form.

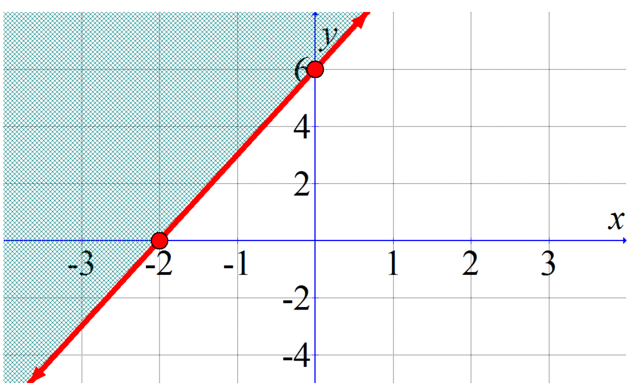
We will choose to find the *x*- and *y*-intercepts of $y = 3x + 6$.

$$y = 3x + 6 \qquad y = 3x + 6$$
$$0 = 3x + 6 \qquad y = 3(0) + 6$$
$$-3x = 6 \qquad y = 6$$
$$x = -2$$

A solid line is drawn through the intercepts, which are located at $(-2, 0)$ and $(0, 6)$.



We can now use our shortcut rather than selecting a test point. Since the inequality is of the form $y \geq 3x + 6$, we shade above the line.



The solution set is the half-plane lying on or above the line $y = 3x + 6$.

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Example 6: Graph the inequality $-12x - 3y > -9$.

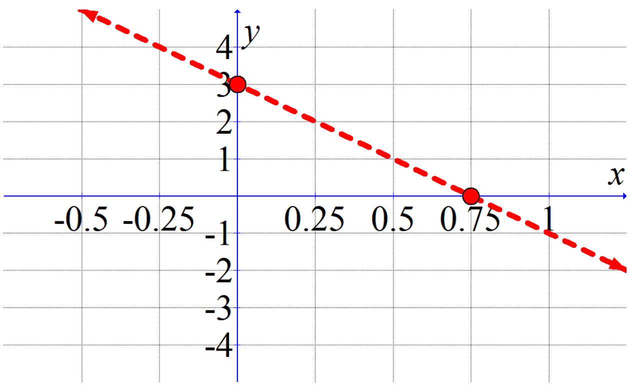
Solution: We first write the inequality as an equation, $-12x - 3y = -9$. The line will be graphed as a dashed line because the inequality in this problem is >, which does not include the line. We can graph the line using *x*- and *y*-intercepts, or by using the slope and *y*-intercept from slope-intercept form.

We will choose to find the *x*- and *y*-intercepts of $-12x-3y=-9$.

$$\begin{aligned}-12x-3y&=-9\\ -12x-3(0)&=-9\\ -12x&=-9\\ x&=\frac{3}{4}\end{aligned}$$

$$\begin{aligned}-12x-3y&=-9\\ -12(0)-3y&=-9\\ -3y&=-9\\ y&=3\end{aligned}$$

A dashed line is drawn through the intercepts, which are located at $\left(\frac{3}{4},\,0\right)$ and $(0,\,3)$.



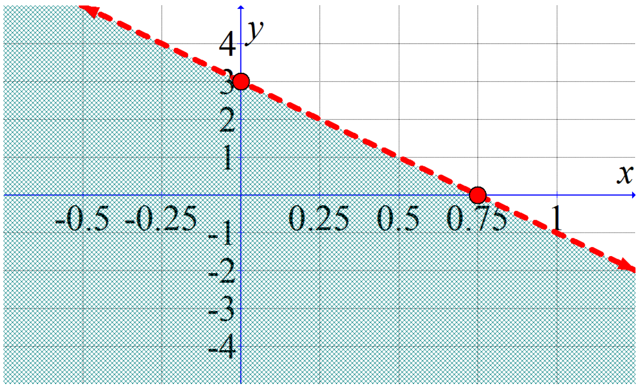
We then need to decide whether to shade above or below the line. Instead of choosing a test point, we can isolate the variable *y* on the left-hand side of $-12x-3y>-9$ and determine which half-plane to shade.

$$\begin{aligned}-12x-3y&>-9\\ -3y&>12x-9\end{aligned}$$

Next, we divide by -3. When dividing by a negative number, we need to reverse the inequality.

$$y<-4x+3$$

Since $y<-4x-3$, we shade below the line.



The solution set is the half-plane lying below the line $y=-4x+3$ or $-12x-3y=-9$.

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Solving Systems of Linear Inequalities

A **system of linear inequalities** is a set of two or more linear inequalities.

The **solution set to a system of linear inequalities** is the set of all ordered pairs that satisfies all of the inequalities. We solve these systems by graphing. To graph a system of linear inequalities, we graph each inequality (using techniques from previous examples in this section) and then find where all shaded regions intersect. The intersection represents the solution set to the system of inequalities.

In this textbook, each system of inequalities will be preceded by a single left curly brace, as shown in the examples below. Not all textbooks follow this convention, but it is a way to group the inequalities together and to quickly identify a system of inequalities.

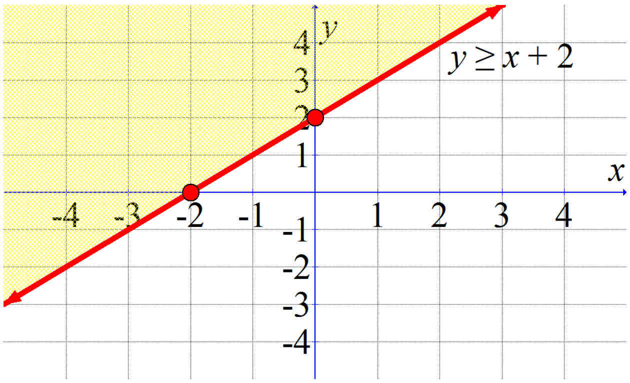
Example 7: Graph the following system of linear inequalities.

$$\begin{cases} y \geq x+2 \\ y \geq -x-2 \end{cases}$$

Solution: We first write the inequality $y \geq x+2$ as an equation, $y=x+2$. The line will be graphed as a solid line because the inequality is $\geq$, which includes the line. We will graph the line using *x*- and *y*-intercepts.

$$\begin{aligned} y&=x+2 \\ 0&=x+2 \\ x&=-2 \end{aligned} \qquad\qquad \begin{aligned} y&=x+2 \\ y&=0+2 \\ y&=2 \end{aligned}$$

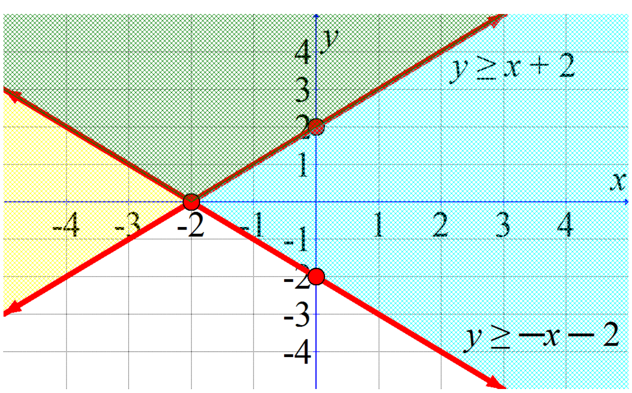
A solid line is drawn through the intercepts, which are located at $(-2, 0)$ and $(0, 2)$. Since the inequality is of the form $y \geq x+2$, we shade above the line.



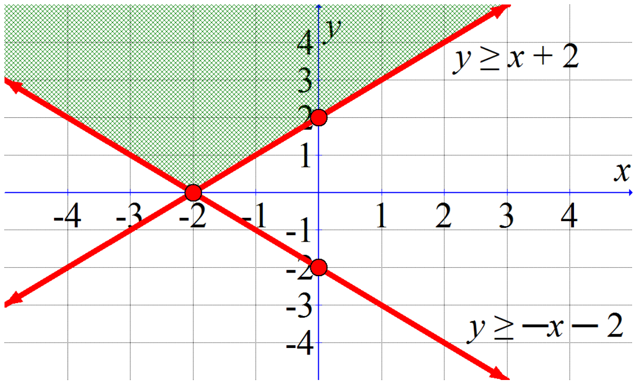
We now write the second inequality, $y \geq -x-2$, as an equation, $y = -x-2$. The line will be graphed as a solid line because the inequality is $\geq$, which includes the line. We will graph the line using *x*- and *y*-intercepts.

$y = x-2$	$y = x-2$
$0 = -x-2$	$y = 0-2$
$x = -2$	$y = -2$

A solid line is drawn through the intercepts, which are located at $(-2, 0)$ and $(0, -2)$. Since the inequality is of the form $y \geq -x-2$, we shade above the line, shown in the colors aqua and green below. (The green region is where the aqua shading overlaps with the previous yellow shading.)



The solution set of the system is the intersection of the two shaded half-planes, shown in green below, along with any points on the boundary of the shaded region.



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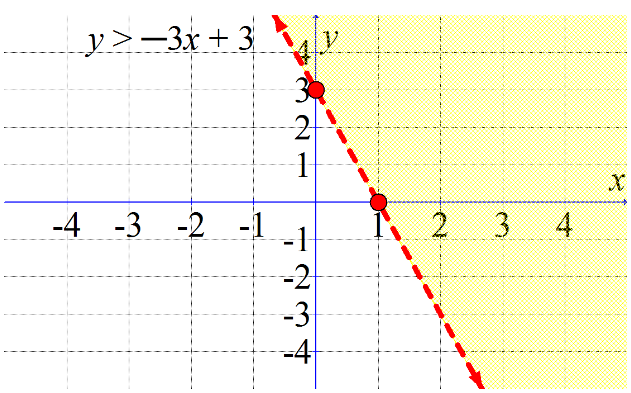
Example 8: Graph the following system of linear inequalities.

$$\begin{cases} y > -3x+3 \\ 2x - y > 4 \end{cases}$$

Solution: We first write the inequality $y > -3x+3$ as an equation, $y = -3x+3$. The line will be graphed as a dashed line because the inequality is $>$, which does not include the line. We will graph the line using *x*- and *y*-intercepts.

$y = -3x+3$	$y = -3x+3$
$0 = -3x+3$	$y = -3(0)+3$
$3x = 3$	$y = 3$
$x = 1$	

A dashed line is drawn through the intercepts, which are located at $(1, 0)$ and $(0, 3)$. Since the inequality is of the form $y > -3x+3$, we shade above the line.



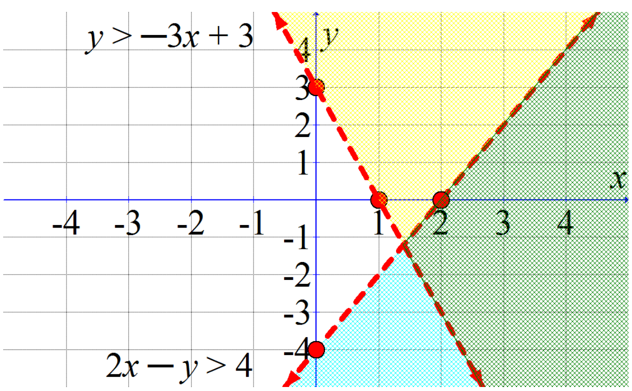
We now write the second inequality, $2x - y > 4$, as an equation, $2x - y = 4$. The line will be graphed as a dashed line because the inequality is $>$, which does not include the line. We will graph the line using *x*- and *y*-intercepts.

$2x - y = 4$	$2x - y = 4$
$2x - 0 = 4$	$2(0) - y = 4$
$2x = 4$	$-y = 4$
$x = 2$	$y = -4$

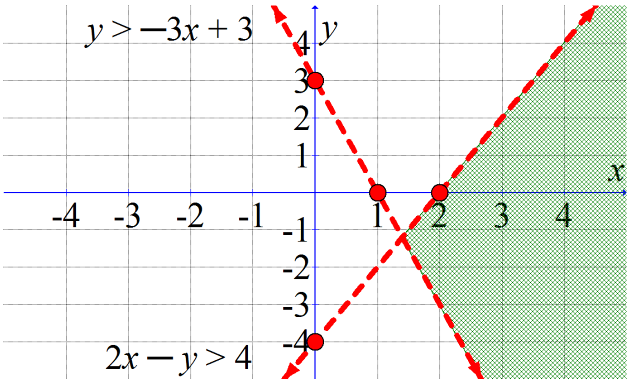
A dashed line is drawn through the intercepts, which are located at $(2, 0)$ and $(0, -4)$. We then need to decide whether to shade above or below the line. Instead of choosing a test point, we can isolate the variable *y* on the left-hand side of $2x - y > 4$ and determine which half-plane to shade. (Remember that when dividing by a negative number, we need to reverse the inequality.)

$$2x - y > 4$$
$$-y > -2x + 4$$
$$y < 2x - 4$$

Since the inequality is of the form $y < 2x - 4$, we shade below the line, shown in the colors aqua and green below. (The green region is where the aqua shading overlaps with the previous yellow shading.)



The solution set of the system is the intersection of the two shaded half-planes, shown in green below. Note that the solution set does not include the two boundary lines, since they are dashed instead of solid.



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Example 9: Graph the following system of linear inequalities.

$$\begin{cases} -12x+3y>-6 \\ -3x-y\geq 3 \end{cases}$$

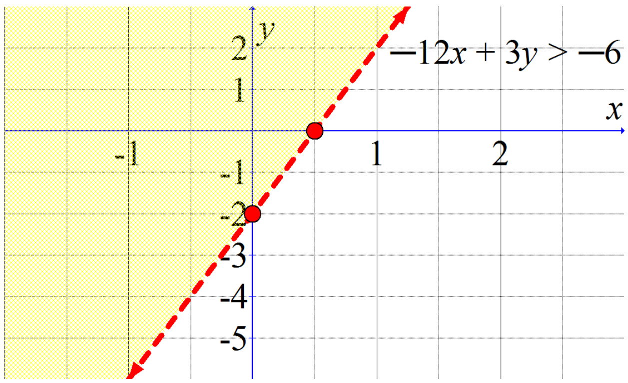
Solution: We first write the inequality $-12x+3y>-6$ as an equation, $-12x+3y=-6$. The line will be graphed as a dashed line because the inequality in this problem is $>$, which does not include the line. We will graph the line using x- and y-intercepts.

$$\begin{aligned} -12x+3y&=-6 \\ -12x+3(0)&=-6 \\ -12x&=-6 \\ x&=\frac{1}{2} \end{aligned} \qquad \begin{aligned} -12x+3y&=-6 \\ -12(0)+3y&=-6 \\ 3y&=-6 \\ y&=-2 \end{aligned}$$

A dashed line is drawn through the intercepts, which are located at $\left(\frac{1}{2}, 0\right)$ and $(0, 2)$. We then need to decide whether to shade above or below the line. Instead of choosing a test point, we can isolate the variable y on the left-hand side of $-12x+3y>-6$ and determine which half-plane to shade.

$$\begin{aligned} -12x+3y&>-6 \\ 3y&>12x-6 \\ y&>4x-2 \end{aligned}$$

Since the inequality is of the form $y>4x-2$, we shade above the line.



We now write the second inequality, $-3x - y \geq 3$, as an equation, $-3x - y = 3$. The line will be graphed as a solid line because the inequality is $\geq$, which includes the line. We will graph the line using *x*- and *y*-intercepts.

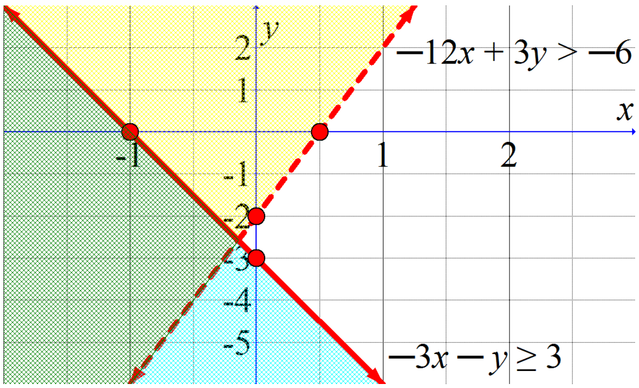
$-3x - y = -3$	$-3x - y = 3$
$-3x - 0 = 3$	$-3(0) - y = 3$
$-3x = 3$	$-y = 3$
$x = -1$	$y = -3$

A solid line is drawn through the intercepts, which are located at $(-1, 0)$ and $(0, -3)$.

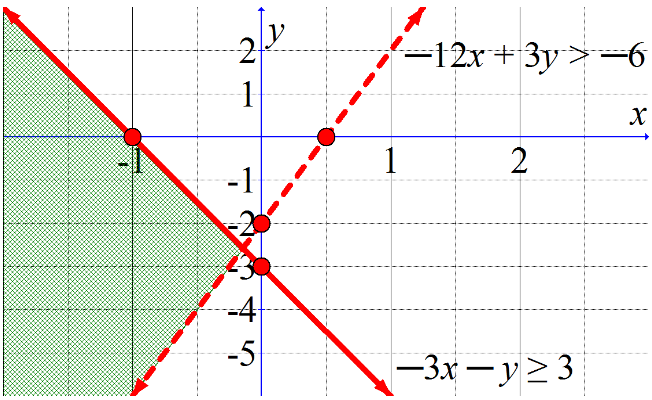
We then need to decide whether to shade above or below the line. Instead of choosing a test point, we can isolate the variable *y* on the left-hand side of $-3x - y \geq 3$ and determine which half-plane to shade. (Remember that when dividing by a negative number, we need to reverse the inequality.)

$$-3x - y \geq 3$$
$$-y \geq 3x + 3$$
$$y \leq -3x - 3$$

Since the inequality is of the form $y \leq -3x - 3$, we shade below the line, shown in the colors aqua and green below. (The green region is where the aqua shading overlaps with the previous yellow shading.)



The solution set of the system is the intersection of the two shaded half-planes, shown in green below, along with any points on the solid boundary of the intersection.



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A system of linear inequalities can include more than two inequalities. The next example shows the solution to a system with three inequalities. We can use similar methods to solve systems with even more inequalities.

Example 10: Graph the following system of linear inequalities.

$$\begin{cases} 2x+3y<9 \\ x \geq 2 \\ y \geq 0 \end{cases}$$

Solution: We first write the inequality $2x+3y<9$ as an equation, $2x+3y=9$. The line will be graphed as a dashed line because the inequality in this problem is $<$, which does not include the line. We will graph the line using *x*- and *y*-intercepts.

$$\begin{aligned} 2x+3y&=9 \\ 2x+3(0)&=9 \\ 2x&=9 \\ x&=\frac{9}{2} \end{aligned}$$

$$\begin{aligned} 2x+3y&=9 \\ 2(0)+3y&=9 \\ 3y&=9 \\ y&=3 \end{aligned}$$

A dashed line is drawn through the intercepts, which are located at $\left(\frac{9}{2}, 0\right)$ and $(0, 3)$. We then need to decide whether to shade above or below the line. Instead of choosing a test point, we can isolate the variable *y* on the left-hand side of $2x+3y<9$ and determine which half-plane to shade.

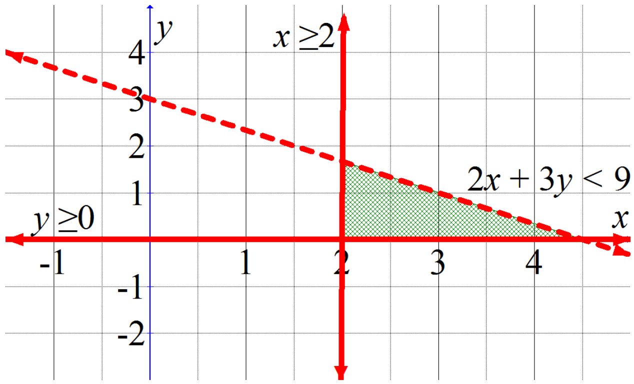
$$\begin{aligned} 2x+3y&<9 \\ 3y&<-2x+9 \\ y&<-\frac{2}{3}x+3 \end{aligned}$$

Since $y<-\frac{2}{3}x+3$, we shade below the dashed line.

We now want to graph $x \geq 2$. Remember that $x=2$ is a vertical line with *x*-intercept 2. Then think about the *x*-values on the graph to decide where to shade. (It may help to think about a number line.) To say that the *x*-values are greater than 2 means that they fall to the right of 2, so we shade to the right of the solid line $x=2$.

Finally, we want to graph $y \geq 0$. Remember that $y=0$ is a horizontal line with *y*-intercept 0, which means that it coincides with the *x*-axis. Since $y \geq 0$, we shade above the solid line $y=0$.

The solution set of the system is the intersection of the three shaded half-planes (shown in green below) and any point on the solid boundary of the intersection.



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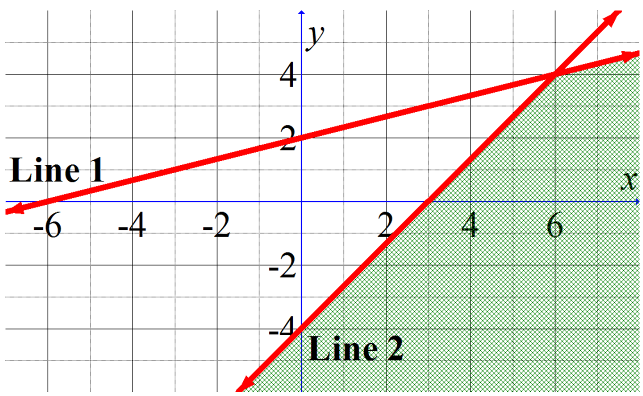
Next, we will reverse the process. Suppose that we are given a solution set and are asked to write the system of inequalities that generates that solution.

Example 11: Write the system of inequalities that corresponds to the following graph.



Solution: First, we need to find the equation of each line. We will use the slope and y-intercept, and write each line in the form $y = mx + b$.

By observing Line 1, we see that the y-intercept is 2, so $b = 2$. To determine the slope of Line 1, we can start at $(0, 2)$, move down 2 units and then to the left 6 units to get to the point $(-6, 0)$.

$$m = \frac{\text{rise}}{\text{run}} = \frac{-2}{-6} = \frac{1}{3}$$

Hence, the equation for Line 1 is $y = \dfrac{1}{3}x + 2$.

Next, we need to find the equation of Line 2. The y-intercept is -4, so $b=-4$. To determine the slope of Line 2, we can start at $(0,-4)$, move up 4 units and to the right 3 units to get to the point $(3,0)$.

$$m=\frac{\text{rise}}{\text{run}}=\frac{4}{3}$$

Therefore, the equation for Line 2 is $y=\frac{4}{3}x-4$.

We now need to determine the inequalities. The shaded region lies below Line 1 and the line is solid, so $y\le\frac{1}{3}x+2$. The shaded region lies below Line 2 and the line is solid, so $y\le\frac{4}{3}x-4$.

The green shaded region is determined entirely by Line 1 and Line 2. The system of inequalities is written below.

$$\begin{cases} y\le\frac{1}{3}x+2 \\ y\le\frac{4}{3}x-4 \end{cases}$$

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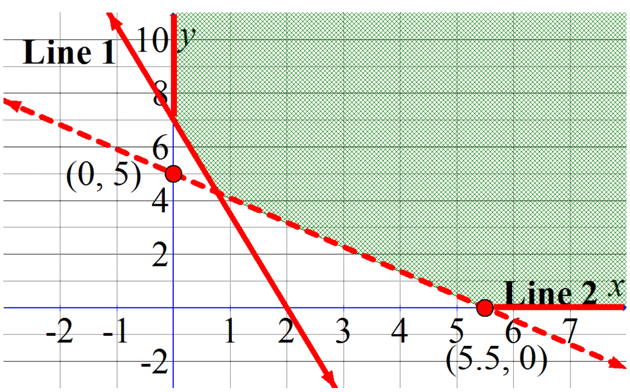
Example 12: Write the system of inequalities that corresponds to the following graph.



Solution: First, we need to find the equation of each line. We will use the slope and y-intercept, and write each line in the form $y=mx+b$.

Line 1 passes through the point $(0, 7)$, so $b = 7$. Since the line also passes through $(2, 0)$,
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-7}{2-0} = -\frac{7}{2}$.

The equation for Line 1 is $y = -\frac{7}{2}x + 7$.

Line 2 passes through the point $(0, 5)$, so $b = 5$. Since the line also passes through $(5.5, 0)$,
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-5}{5.5-0} = \frac{-5}{5.5} = \frac{-5}{\frac{11}{2}} = \frac{-5}{1} \cdot \frac{2}{11} = -\frac{10}{11}$

The equation for Line 2 is $y = -\frac{10}{11}x + 5$.

Also notice that the shaded region is bounded by the *y*-axis ($x = 0$) and the *x*-axis ($y = 0$).

We now need to determine the inequalities. The shaded region lies above Line 1 and the line is solid, so $y \geq -\frac{7}{2}x + 7$. The shaded region lies above Line 2 and the line is dashed, so $y > -\frac{10}{11}x + 5$. Since the shaded region is to the right of the solid *y*-axis and above the solid *x*-axis, we have two more inequalities involved: $x \geq 0$ and $y \geq 0$.

The system of inequalities is written below.

$$\begin{cases} y \geq -\frac{7}{2}x + 7 \\ y > -\frac{10}{11}x + 5 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

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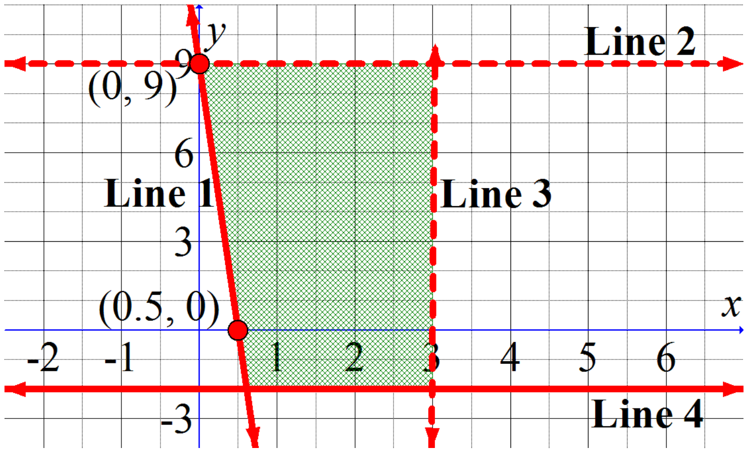
Example 13: Write the system of inequalities that corresponds to the following graph.



Solution: First, we need to find the equation of each line. We will use the slope and *y*-intercept, and write each line in the form $y = mx + b$.

Line 1 passes through the point $(0, 9)$, so $b = 9$. Since the line also passes through $(0.5, 0)$,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-9}{0.5-0} = \frac{-9}{\frac{1}{2}} = \frac{-9}{1} \cdot \frac{2}{1} = -18.$$

The equation for Line 1 is $y = -18x + 9$.

The equation for Line 2 is $y = 9$.

The equation for Line 3 is $x = 3$.

The equation for Line 4 is $y = -2$.

We now need to determine the inequalities:

The shaded region lies above Line 1 and the line is solid, so $y \geq -18x + 9$.

The shaded region lies below Line 2 and the line is dashed, so $y < 9$.

The shaded region lies to the left of Line 3 and the line is dashed, so $x < 3$.

The shaded region lies above Line 4 and the line is solid, so $y \geq -2$.

We can combine the inequalities for Line 2 and Line 4: $-2 \leq y < 9$

The system of inequalities is: $\begin{cases} y \geq -18x+9 \\ x < 3 \\ -2 \leq y < 9 \end{cases}$

The inequalities may be written in an another equivalent form. For example, we can also state the system of inequalities as: $\begin{cases} 18x+y \geq 9 \\ x < 3 \\ -2 \leq y < 9 \end{cases}$

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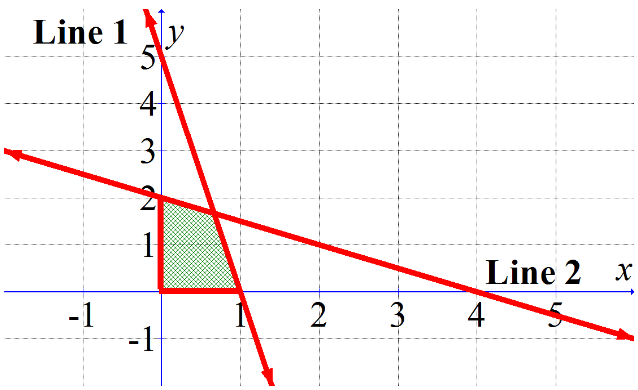
Example 14: Write the system of inequalities that corresponds to the following graph.



Solution: First, we need to find the equation of each line. We will use the slope and *y*-intercept, and write each line in the form $y = mx + b$.

Line 1 passes through the point $(0, 5)$, so $b = 5$. Since the line also passes through $(1, 0)$, $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0-5}{1-0} = -5$. The equation for Line 1 is $y = -5x + 5$.

Line 2 passes through the point $(0, 2)$, so $b = 2$. Since the line also passes through $(4, 0)$, $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0-2}{4-0} = \dfrac{-2}{4} = -\dfrac{1}{2}$. The equation for Line 2 is $y = -\dfrac{1}{2}x + 2$.

Also notice that the shaded region is bounded by the *y*-axis ($x = 0$) and the *x*-axis ($y = 0$).

We now need to determine the inequalities. The shaded region lies below Line 1 and the line is solid, so $y \leq -5x + 5$. The shaded region lies below Line 2 and the line is solid, so $y \leq -\dfrac{1}{2}x + 2$.

Since the shaded region is to the right of the solid *y*-axis and above the solid *x*-axis, we have two more inequalities involved: $x \ge 0$ and $y \ge 0$.

The system of inequalities is: $\begin{cases} y \le -5x+5 \\ y \le -\dfrac{1}{2}x+2 \\ x \ge 0 \\ y \ge 0 \end{cases}$

The inequalities may be written in another equivalent form. For example, the following systems of inequalities are equivalent to the one above:

$$\begin{cases} 5x+y \le 5 \\ \dfrac{1}{2}x+y \le 2 \\ x \ge 0 \\ y \ge 0 \end{cases} \qquad \begin{cases} 5x+y \le 5 \\ x+2y \le 4 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

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