Chapter 1

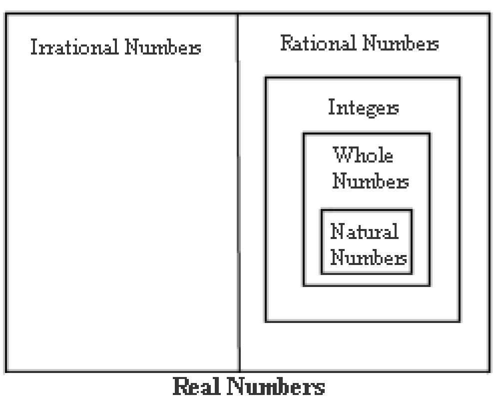
Introductory Information and Review

Section 1.1: Numbers

- Types of Numbers
- Order on a Number Line

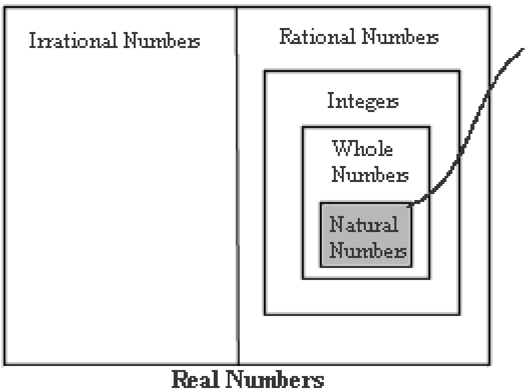
Types of Numbers

In this course, we will work with real numbers. The following diagram shows the types of numbers that form the set of real numbers.



Natural Numbers:

We begin with the right side of the diagram and describe the natural numbers.



The set of natural numbers is $\{1, 2, 3, 4, \ldots\}$.

A natural number is prime if it is greater than 1 and its only factors are itself and 1. A natural number greater than 1 that is not prime is called composite.

Example:
Classify each of the following natural numbers as either prime or composite.

(a) 37

(b) 18

Solution:

(a) Since 37 is a natural number greater than 1 and its only factors are 1 and itself, we see that 37 is a prime number.

(b) Since 18 is a natural number greater than 1 and the factors of 18 are 1, 2, 3, 6, 9, and 18, we see that 18 is a composite number.

Even/Odd Natural Numbers:

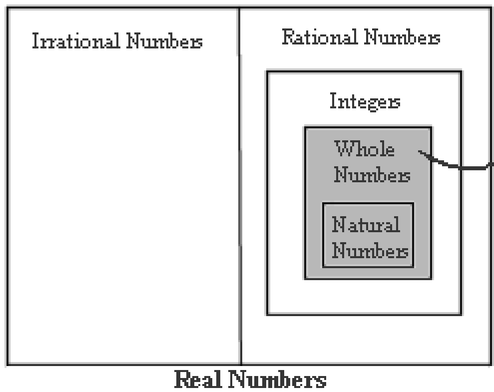
An even natural number is one that can be written in the form $2n$, where n is a natural number. For example, 38 is an even natural number: $38 = 2 \cdot 19$.

The set of even natural number is $\{2, 4, 6, 8, 10, \ldots\}$.

An odd natural number is one that can be written in the form $2n-1$, where n is a natural number. For example, 37 is an odd natural number: $37 = 2 \cdot 19 - 1$.

The set of odd natural number is $\{1, 3, 5, 7, 9, \ldots\}$.

Whole Numbers:



The set of whole numbers is $\{0, 1, 2, 3, \ldots\}$.

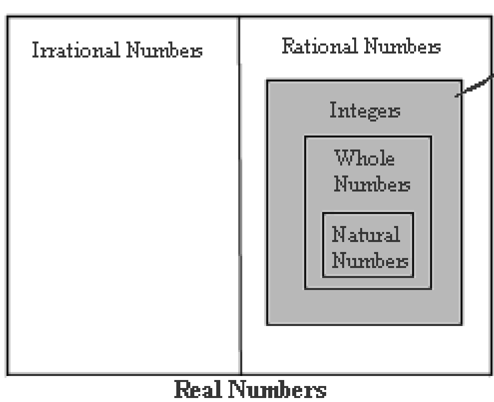
Every natural number is also a whole number.

$$\overbrace{0,\quad \underbrace{1, 2, 3, \ldots}_{\text{natural numbers}}}^{\text{whole numbers}}$$

Example:
True or False: Every whole number is also a natural number.

Solution:
False: 0 is a whole number, but 0 is not a natural number since it does not belong to the set $\{1, 2, 3, \ldots\}$.

Integers:



The set of integers is $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

Every whole number is also an integer.

$$\overbrace{\ldots,-3,-2,-1,\underbrace{0,1,2,3,\ldots}_{\text{whole numbers}}}^{\text{integers}}$$

Example:
True or False: Every integer is also a natural number.

Solution:
False: -12 is an integer, but -12 is not a natural number since it does not belong to the set $\{1,2,3,\ldots\}$.

Even/Odd Integers:

The integers can be qualified as even or odd:

$\{\ldots,-6,-4,-2,0,2,4,6,\ldots\}$ is the set of even integers; even integers are divisible by 2.

$\{\ldots,-5,-3,-1,1,3,5,\ldots\}$ is the set of odd integers; odd integers are not divisible by 2.

Example:
Classify each of the following integers as either even or odd.

(a) 38

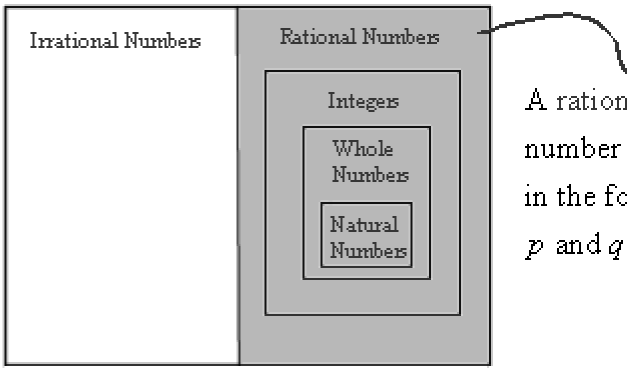
(b) -15

Solution:

(a) The integer 38 is even since it belongs in the set $\{\ldots,-6,-4,-2,0,2,4,6,\ldots\}$.

(b) The integer -15 is odd since it belongs in the set $\{\ldots,-5,-3,-1,1,3,5,\ldots\}$.

Rational Numbers:



Real Numbers

A rational number is a number that can be put in the form p/q, where p and $q \neq 0$ are integers.

Every integer is also a rational number. Each integer m can be expressed as the ratio of integers by writing $m = \dfrac{m}{1}$.

Terminating decimals and repeating decimals are rational numbers since they can be written as a ratio of integers.

Example:
Express each of the following rational numbers as a ratio of integers.

(a) $4\dfrac{2}{3}$

(b) 1.31

(c) $2.\overline{3}$

(d) 29

Solution:

(a) $4\dfrac{2}{3}$ is an example of a mixed number. It is the sum of the whole number 4 and the proper fraction $\dfrac{2}{3}$. (A proper fraction is one in which the numerator is less than the denominator.)

We can write a mixed number as an improper fraction. (An improper fraction is one in which the numerator is greater than or equal to the denominator.)

$$4\frac{2}{3} = \frac{(3)(4)+2}{3} = \frac{12+2}{3} = \frac{14}{3}$$

(b) 1.31 is an example of a terminating decimal.

$$1.31 = 1\frac{31}{100} = \frac{131}{100}$$

(c) $2.\overline{3}$ is an example of a repeating decimal. $2.\overline{3}$ can also be written as 2.33333… . (The three dots are used to indicate that the digits repeat in the pattern indicated.)

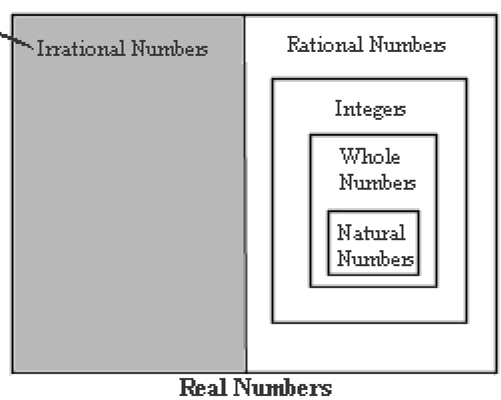
$$2.\overline{3} = 2\frac{1}{3} = \frac{7}{3}$$

(d) $29 = \frac{29}{1}$

Irrational Numbers:

An irrational number cannot be expressed as the ratio of integers.

All nonterminating, nonrepeating decimals are irrational numbers.



Examples of irrational numbers are π, e, $\sqrt{2}$, and $\sqrt{3}$. (The square root of any prime number is an irrational number.)

Approximate values for the four irrational numbers given above are shown below:

$$\pi \approx 3.142$$
$$e \approx 2.718$$
$$\sqrt{2} \approx 1.414$$
$$\sqrt{3} \approx 1.732$$

(The symbol $\approx$ is read "is approximately equal to.")

Real Numbers:

A real number is a number that is either rational or irrational (but not both). Every real number can be written as either a terminating decimal, a repeating decimal, or a nonterminating, nonrepeating decimal.

A real number x is positive if it is greater than 0. This is written as $x > 0$. A real number y is negative if it is less than 0. This is written as $y < 0$. The real number 0 is neither negative nor positive.

Example:
Circle all of the words that can be used to describe the each of the numbers below.

(a) 26 (b) $\sqrt{29}$

Even, Odd, Positive, Negative, Prime, Composite,
Natural, Whole, Integer, Rational, Irrational, Real

Solution:
(a) 26 is a natural number since it belongs to the set $\{1, 2, 3, \ldots\}$.

Thus, 26 is also a whole number, an integer, a rational number, and a real number.

Since 26 is a natural number greater than 1 and the factors of 26 are 1, 2, 13, and 26, we see that 26 is a composite number.

Since $26 > 0$, we see that 26 is positive.

The integer 26 is even since it belongs in the set $\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$.

The results are circled below:

Even, Odd, Positive, Negative,
Prime, Composite, Natural,
Whole, Integer, Rational,
Irrational, Real

(b) Since 29 is a prime number, we see that $\sqrt{29}$ is an irrational number. Thus, $\sqrt{29}$ is also a real number.

Note that a real number is either rational or irrational but not both.

Since $\sqrt{29} > \sqrt{25} = 5 > 0$, we see that $\sqrt{29}$ is positive.

The results are circled below:

Even, Odd, Positive, Negative,
Prime, Composite, Natural,
Whole, Integer, Rational,
Irrational, Real

Note About Division Involving Zero:

For each number $m \neq 0$, we have $\frac{0}{m} = 0$. Division by 0 is undefined.

Additional Example 1:
State whether each of the following numbers is prime, composite, or neither. If composite, then list all the factors of the number.

(a) 29

(b) 42

(c) -14

Solution:
The set of natural numbers is $\{1, 2, 3, 4, \ldots\}$. Prime numbers and composite numbers must be natural numbers. A natural number is prime if it is greater than 1 and its only factors are itself and 1. Otherwise, the natural number is composite.

(a) 29 is a natural number since it belongs to the set $\{1, 2, 3, 4, \ldots\}$. The only factors of 29 are itself and 1. Therefore, 29 is a prime number.

(b) 42 is a natural number since it belongs to the set $\{1, 2, 3, 4, \ldots\}$. Note that $42 = 1 \cdot 42 = 2 \cdot 21 = 6 \cdot 7 = 14 \cdot 3$. Thus, the factors of 42 are 1, 2, 3, 6, 7, 14, 21, and 42. Therefore, 42 is a composite number.

(c) Prime numbers and composite numbers must belong to the set of natural numbers. -14 is a not a natural number since it does not belong to the set $\{1, 2, 3, 4, \ldots\}$. Therefore, -14 is neither prime nor composite.

Additional Example 2:

Circle all of the words that can be used to describe each of the numbers below.

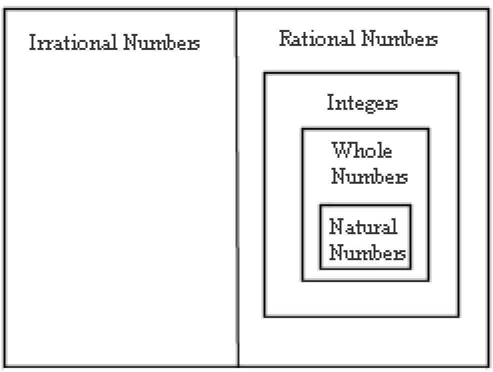
(a) 10 (b) -13

Even, Odd, Positive, Negative, Prime, Composite,
Natural, Whole, Integer, Rational, Irrational, Real

Solution:

We will use the diagram and the definitions given below:

The diagram below illustrates how the natural numbers, whole numbers, integers, rational numbers, and irrational numbers form the set of real numbers.



Real Numbers

<u>Natural Numbers:</u>
The set of natural numbers is $\{1, 2, 3, 4, \ldots\}$.

<u>Whole Numbers:</u>
The set of whole numbers is $\{0, 1, 2, 3, \ldots\}$.

<u>Integers:</u>
The set of integers is $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

<u>Prime/Composite Numbers:</u>
A natural number is prime if it is greater than 1 and its only factors are itself and 1; otherwise, it is composite.

<u>Positive/Negative Numbers:</u>
A real number x is positive if it is greater than 0. This is written as $x > 0$. A real number y is negative if it is less than 0. This is written as $y < 0$. The real number 0 is neither negative nor positive.

Even/Odd Numbers:

The integers can be qualified as even or odd:

$\{\ldots,-6,-4,-2,0,2,4,6,\ldots\}$ is the set of even integers; even integers are divisible by 2.

$\{\ldots,-5,-3,-1,1,3,5,\ldots\}$ is the set of odd integers; odd integers are not divisible by 2.

Rational Numbers:

A number that can be written in the form p/q, where p and $q \neq 0$ are integers, is a rational number.

(a) 10 is a natural number since it belongs to the set $\{1, 2, 3, \ldots\}$.

From the diagram shown above, we see that 10 is also a whole number, an integer, a rational number, and a real number.

Since 10 is a natural number greater than 1 and the factors of 10 are 1, 2, 5, and 10, we see that 10 is a composite number.

Since $10 > 0$, we see that 10 is positive.

The integer 10 is even since it belongs in the set $\{\ldots,-6,-4,-2,0,2,4,6,\ldots\}$.

The results are circled below:

Even, Odd, Positive, Negative,
Prime, Composite, Natural,
Whole, Integer, Rational,
Irrational, Real

(b) -13 is an integer since it belongs to the set $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$.
From the diagram shown above, -13 is also a rational number and a real number.

Note that -13 does not belong to the following sets of numbers:

$\{0,1,2,3,\ldots\}$ (whole numbers)

$\{1, 2, 3,\ldots\}$ (natural numbers)

Since $-13 < 0$, we see that -13 is negative.

The integer -13 is odd since it belongs in the set $\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$.

The results are circled below:

Even, Odd, Positive, Negative,
Prime, Composite, Natural,
Whole, Integer, Rational,
Irrational, Real

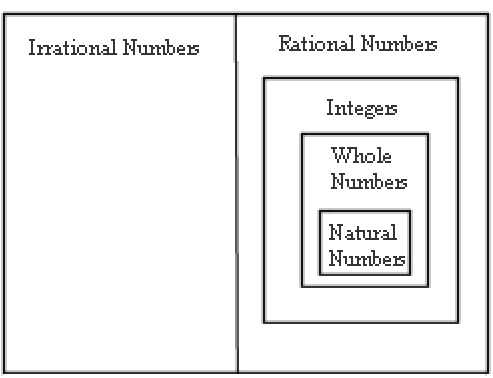
Additional Example 3:
Circle all of the words that can be used to describe each of the numbers below.

(a) $\frac{3}{8}$ (b) $\sqrt{7}$

Even, Odd, Positive, Negative, Prime, Composite,
Natural, Whole, Integer, Rational, Irrational, Real

Solution:
We will use the diagram and the definitions given below:

The diagram below illustrates how the natural numbers, whole numbers, integers, rational numbers, and irrational numbers form the set of real numbers.



Real Numbers

<u>Natural Numbers:</u>
The set of natural numbers is $\{1, 2, 3, 4, \ldots\}$.

<u>Whole Numbers:</u>
The set of whole numbers is $\{0, 1, 2, 3, \ldots\}$.

Integers:

The set of integers is $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

Prime/Composite Numbers:

A natural number is prime if it is greater than 1 and its only factors are itself and 1; otherwise, it is composite.

Positive/Negative Numbers:

A real number x is positive if it is greater than 0. This is written as $x > 0$. A real number y is negative if it is less than 0. This is written as $y < 0$. The real number 0 is neither negative nor positive.

Even/Odd Numbers:

The integers can be qualified as even or odd:

$\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$ is the set of even integers; even integers are divisible by 2.

$\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$ is the set of odd integers; odd integers are not divisible by 2.

Rational Numbers:

A number that can be written in the form p/q, where p and $q \neq 0$ are integers, is a rational number.

(a) Since 3 and 8 are integers, we see that $\frac{3}{8}$ is a rational number.

Note that $\frac{3}{8}$ does not belong to the following sets of numbers:

$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ (integers)

$\{0, 1, 2, 3, \ldots\}$ (whole numbers)

$\{1, 2, 3, \ldots\}$ (natural numbers)

From the diagram shown above, we see that $\frac{3}{8}$ is also a real number.

Since $\frac{3}{8} > 0$, we see that $\frac{3}{8}$ is positive.

The results are circled below:

Even, Odd, Positive, Negative,
Prime, Composite, Natural,
Whole, Integer, Rational,
Irrational, Real

(b) If m is a prime number, then $\sqrt{m}$ is an irrational number. Since 7 is a prime number, we see that $\sqrt{7}$ is an irrational number.

From the diagram shown above, we see that $\sqrt{7}$ is also a real number.

Note that a real number is either rational or irrational but not both.

Since $\sqrt{7} > \sqrt{4} = 2 > 0$, we see that $\sqrt{7}$ is positive.

The results are circled below:

Even, Odd, Positive, Negative,
Prime, Composite, Natural,
Whole, Integer, Rational,
Irrational, Real

Additional Example 4:

Which elements of $\left\{-10.2, -8, 0, \frac{7}{8}, 1.\overline{23}, \sqrt{17}, 23, 25\frac{1}{8}\right\}$ belong to the categories listed below?

(a) Natural, (b) Prime, (c) Composite, (d) Whole, (e) Integer, (f) Even, (g) Odd,
(h) Rational, (i) Irrational, (j) Real, (k) Positive, (l) Negative

Solution:

(a) Identify the elements in the given set that are natural numbers.

23 is a natural number since it belongs to the set $\{1, 2, 3, \ldots\}$.

(b/c) Decide which natural numbers are prime and which are composite.

23 is a prime number since it is a natural number greater than 1 and its only factors are itself and 1. There are no composite numbers in the given set.

(d) Identify the elements in the given set that are whole numbers.

0 and 23 are whole numbers since they belong to the set $\{0, 1, 2, 3, \ldots\}$.

(e) Identify the elements in the given set that are integers.

$-8, 0,$ and 23 are integers since they belong to the set $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

(f/g) Decide which integers are even and which are odd.

The integers -8 and 0 are even since they belong to the set

$$\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}.$$

The integer 23 is odd since it belongs to the set

$$\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}.$$

(h) Identify the elements in the given set that are rational numbers.

Recall that a number that can be written in the form p/q, where p and $q \neq 0$ are integers, is a rational number.

All terminating and all repeating decimals are rational numbers since they can be expressed as the ratio of two integers.

$-10.2, -8, 0, \frac{7}{8}, 1.\overline{23}, 23, 25\frac{1}{8}$
are rational numbers:

-10.2 (terminating decimal)

$$-10\frac{2}{10} = -\frac{102}{10} = \frac{-102}{10}$$

$$-8 = \frac{-8}{1}$$

$0 = \dfrac{0}{1}$

$1.\overline{23}$ (repeating decimal)

$1.\overline{23} = \dfrac{122}{99}$

$23 = \dfrac{23}{1}$

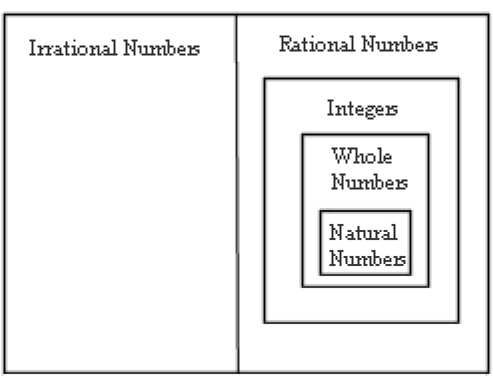
$25\dfrac{1}{8}$ (mixed number)

$25\dfrac{1}{8} = \dfrac{201}{8}$

(i) Identify the elements in the given set that are irrational numbers.

$\sqrt{17}$ is irrational since the square root of a prime number is irrational. Its decimal representation is nonterminating, nonrepeating.

(j) Identify the elements in the given set that are real numbers. Use the diagram below.



Real Numbers

From the diagram shown above, all of the numbers $-10.2, -8, 0, \dfrac{7}{8}, 1.\overline{23}$, $\sqrt{17}$, 23, and $25\dfrac{1}{8}$ are real numbers.

(k/l) Identify the elements in the given set that are negative (less than 0) and identify those elements that are positive (greater than 0). Recall that 0 is neither negative nor positive.

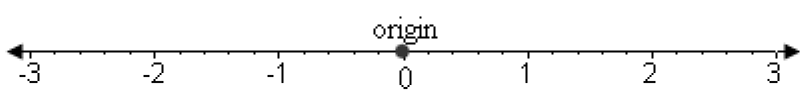
Those elements that are less than 0 and consequently negative are -10.2 and -8.

Those elements that are greater than 0 and consequently positive are $\frac{7}{8}, 1.\overline{23}, \sqrt{17}, 23,$ and $25\frac{1}{8}$.

Order on a Number Line

The Real Number Line:

We can graph real numbers on a number line. For each real number, there corresponds exacly one point on the line. Also, for each point on the line there corresponds exactly exactly one real number. This number is called the coordinate of the point. The point on the real number line whose coordinate is 0 is called the origin.



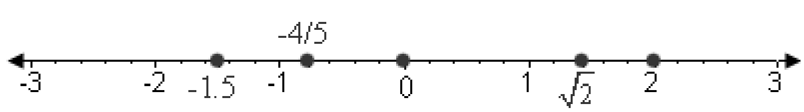
Points to the right of the origin have positive coordinates. Points to the left of the origin have negative coordinates.

Example:
Graph the following numbers on the number line:

$$-1.5, -\frac{4}{5}, 0, \sqrt{2}, 2$$

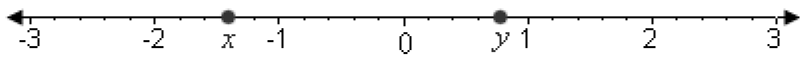
(Hint: Use the approximation: $\sqrt{2} \approx 1.41$.)

Solution:



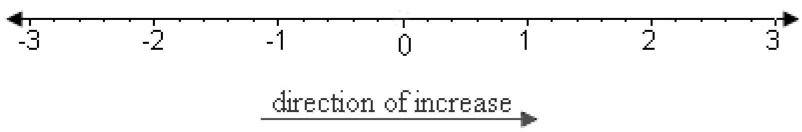
Inequality Symbols:

If a real number x is less than a real number y, then we write $x < y$. The inequality symbol $<$ is read "is less than." On a number line the graph of x is to the left of the graph of y.



$x < y$

The coordinates of points increase as we move left to right on the number line.



The following table describes additional inequality symbols.

Symbol	Meaning	Some Examples
$>$	is greater than	$0 > -4$, $\pi > 3$, $\sqrt{3} > \sqrt{2}$
$\geq$	is greater than or equal to	$2.\overline{4} \geq 2.4$, $\frac{3}{2} \geq 1.5$, $\sqrt{6} \geq \sqrt{5}$
$\leq$	is less than or equal to	$\sqrt{3} \leq 2$, $-1.2 \leq -\frac{6}{5}$, $\sqrt{5} \leq \sqrt{6}$

Example:
For each pair of real numbers, place one of the symbols $<$, $=$, or $>$ in the blank provided.

(a) $-\frac{7}{2}$ ___ $-3\frac{1}{2}$

(b) $\frac{1}{6}$ ___ $\frac{1}{5}$

(c) $\sqrt{5}$ ___ 2

(d) $\frac{5}{2}$ ___ 2.6

Solution:

(a) $-\frac{7}{2} = -3\frac{1}{2}$ Express $-\frac{7}{2}$ as a mixed number: $-\frac{7}{2} = -3\frac{1}{2}$

(b) $\frac{1}{6} < \frac{1}{5}$ Express $\frac{1}{5}$ and $\frac{1}{6}$ as decimals: $\frac{1}{5} = 0.2$ and $\frac{1}{6} = 0.1\overline{6}$

(c) $\sqrt{5} > 2$ Compare squareroots: $\sqrt{5} > \sqrt{4} = 2$

(d) $\frac{5}{2} < 2.6$ Express $\frac{5}{2}$ as a decimal: $\frac{5}{2} = 2\frac{1}{2} = 2.5.$

Example:
Find all natural numbers less than $\sqrt{35}$.

Solution:
Note that $5 = \sqrt{25} < \sqrt{35}$ and $\sqrt{35} < \sqrt{36} = 6$. Thus, $\sqrt{35}$ is between 5 and 6.

Thus, the natural numbers that are less than $\sqrt{35}$ are those natural numbers that are less than 6: 1, 2, 3, 4, 5 .

Example:
Find all even integers between -2.75 and 9.8.

Solution:
The integers between -2.75 and 9.8 must be greater than -2.75 and less than 9.8. They are $-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9.

The even integers on the list are $-2, 0, 2, 4, 6,$ and 8.

Additional Example 1:
List the prime numbers between 30 and 40.

Solution:
A prime number is a natural number greater than 1 whose only factors are itself and 1.

The set of natural numbers is $\{1, 2, 3, 4, \ldots\}$. The natural numbers between 30 and 40 are greater than 30 and less than 40:

31, 32, 33, 34, 35, 36, 37, 38, 39

Omit the even natural numbers between 30 and 40. They are not prime since they each have a factor of 2.

31, $\cancel{32}$, 33, $\cancel{34}$, 35, $\cancel{36}$, 37, $\cancel{38}$, 39

Find the factors of the odd natural numbers between 30 and 40.

Odd Natural Number	Factors
31	1, 31
33	1, 3, 11, 33
35	1, 5, 7, 35
37	1, 37
39	1, 3, 13, 39

Using the results in the table we see that the prime numbers between 30 and 40 are 31 and 37.

The numbers 32, 33, 34, 35, 36, 38, and 39 are composite numbers.

Additional Example 2:
List the composite numbers between $\sqrt{12}$ and 5π and list all the factors of these numbers. (Hint: Use the approximation: $\pi \approx 3.14$.)

Solution:
The given numbers are irrational.

Note that $3 = \sqrt{9} < \sqrt{12}$ and $\sqrt{12} < \sqrt{16} = 4$. Thus, $\sqrt{12}$ is between 3 and 4.

5π means $5 \cdot \pi$. From the given hint, $5\pi \approx 15.7$.

List the natural numbers between $\sqrt{12}$ and 5π. They must be greater than $\sqrt{12}$ and less than 5π.

The natural numbers between $\sqrt{12}$ and 5π are $4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14,$ and 15.

Omit the prime numbers from the list: $4, \cancel{5}, 6, \cancel{7}, 8, 9, 10, \cancel{11}, 12, \cancel{13}, 14, 15$

The remaining numbers in the list are composite numbers. Each of these natural numbers has factors other than itself and 1.

The composite numbers between $\sqrt{12}$ and 5π are $4, 6, 8, 9, 10, 12, 14$, and 15.

The factors are shown in the table below.

Composite Number	Factors
4	1, 2, 4
6	1, 2, 3, 6
8	1, 2, 4, 8
9	1, 3, 9
10	1, 2, 5, 10
12	1, 2, 3, 4, 6, 12
14	1, 2, 7, 14
15	1, 3, 5, 15

Additional Example 3:

List the composite numbers between $\sqrt{12}$ and 5π and list all the factors of these numbers. (Hint: Use the approximation: $\pi \approx 3.14$.)

Solution:

The given numbers are irrational.

Note that $3 = \sqrt{9} < \sqrt{12}$ and $\sqrt{12} < \sqrt{16} = 4$. Thus, $\sqrt{12}$ is between 3 and 4.

5π means $5 \cdot \pi$. From the given hint, $5\pi \approx 15.7$.

List the natural numbers between $\sqrt{12}$ and 5π. They must be greater than $\sqrt{12}$ and less than 5π.

The natural numbers between $\sqrt{12}$ and 5π are $4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14$, and 15.

Omit the prime numbers from the list: $4, \cancel{5}, 6, \cancel{7}, 8, 9, 10, \cancel{11}, 12, \cancel{13}, 14, 15$

The remaining numbers in the list are composite numbers. Each of these natural numbers has factors other than itself and 1.

The composite numbers between $\sqrt{12}$ and 5π are 4, 6, 8, 9, 10, 12, 14, and 15.

The factors are shown in the table below.

Composite Number	Factors
4	1, 2, 4
6	1, 2, 3, 6
8	1, 2, 4, 8
9	1, 3, 9
10	1, 2, 5, 10
12	1, 2, 3, 4, 6, 12
14	1, 2, 7, 14
15	1, 3, 5, 15

Additional Example 4:

Fill in the appropriate symbol from the set $\{<, >, =\}$.

(a) $\sqrt{11}$ ______ 11

(b) $-\sqrt{36}$ ______ -6

(c) $-\frac{1}{8}$ ______ $-\frac{1}{7}$

(d) 6.32 ______ $\frac{63}{10}$

Solution:

(a) Note that $\sqrt{11} < \sqrt{16} = 4$. Thus, $\sqrt{11} \ < \ 11$.

(b) Note that $\sqrt{36} = 6$. Thus, $-\sqrt{36} \ = \ -6$ since $-\sqrt{36} = (-1)\sqrt{36} = -6$.

(c) Represent $-\frac{1}{8}$ and $-\frac{1}{7}$ as decimals. Since both numbers are rational, the decimals will either be terminating or repeating.

$$-\frac{1}{7} = -0.\overline{142857}$$

$$-\frac{1}{8} = -0.125$$

Thus, $-\frac{1}{8} \;>\; -\frac{1}{7}$.

(d) Representing $\frac{63}{10}$ as a decimal, we see that $\frac{63}{10} = 6\frac{3}{10} = 6.3.$

Thus, $6.32 \;>\; \frac{63}{10}$.

Exercise Set 1.1: Numbers

State whether each of the following numbers is prime, composite, or neither. If composite, then list all the factors of the number.

1. **(a)** 8 **(b)** 5 **(c)** 1 **(d)** -7 **(e)** 12

2. **(a)** 11 **(b)** -6 **(c)** 15 **(d)** 0 **(e)** -2

Answer the following.

3. In (a)-(e), use long division to change the following fractions to decimals.

(a) $\frac{1}{9}$ **(b)** $\frac{2}{9}$ **(c)** $\frac{3}{9}$
(d) $\frac{4}{9}$ **(e)** $\frac{5}{9}$ *Note:* $\frac{3}{9}=\frac{1}{3}$

Notice the pattern above and use it as a shortcut in (f)-(m) to write the following fractions as decimals <u>without</u> performing long division.

(f) $\frac{6}{9}$ **(g)** $\frac{7}{9}$ **(h)** $\frac{8}{9}$
(i) $\frac{9}{9}$ **(j)** $\frac{10}{9}$ **(k)** $\frac{14}{9}$
(l) $\frac{25}{9}$ **(m)** $\frac{29}{9}$ *Note:* $\frac{6}{9}=\frac{2}{3}$

4. Use the patterns from the problem above to change each of the following decimals to either a proper fraction or a mixed number.

(a) $0.\overline{4}$ **(b)** $0.\overline{7}$ **(c)** $2.\overline{3}$
(d) $1.\overline{2}$ **(e)** $4.\overline{5}$ **(f)** $7.\overline{6}$

State whether each of the following numbers is rational or irrational. If rational, then write the number as a ratio of two integers. (If the number is already written as a ratio of two integers, simply rewrite the number.)

5. **(a)** 0.7 **(b)** $\sqrt{5}$ **(c)** $\frac{3}{7}$
(d) -5 **(e)** $\sqrt{16}$ **(f)** $0.\overline{3}$
(g) 12 **(h)** $\frac{2.3}{3.5}$ **(i)** e
(j) $-\sqrt{4}$ **(k)** 0.04004000400004...

6. **(a)** $\sqrt{\pi}$ **(b)** $0.\overline{6}$ **(c)** $\sqrt{8}$
(d) $\frac{1.3}{4.7}$ **(e)** $-\frac{4}{5}$ **(f)** $-\sqrt{9}$
(g) 3.1 **(h)** -10 **(i)** 0
(j) $\frac{7}{9}$ **(k)** 0.03003000300003…

Circle all of the words that can be used to describe each of the numbers below.

7. -9

Even	Odd	Positive	Negative
Prime	Composite	Natural	Whole
Integer	Rational	Irrational	Real
Undefined			

8. $0.\overline{7}$

Even	Odd	Positive	Negative
Prime	Composite	Natural	Whole
Integer	Rational	Irrational	Real
Undefined			

9. $\sqrt{2}$

Even	Odd	Positive	Negative
Prime	Composite	Natural	Whole
Integer	Rational	Irrational	Real
Undefined			

10. $-\frac{4}{7}$

Even	Odd	Positive	Negative
Prime	Composite	Natural	Whole
Integer	Rational	Irrational	Real
Undefined			

Answer the following.

11. Which elements of the set $\left\{-8,\ -2.1,\ -0.\overline{4},\ 0,\ \sqrt{7},\ \pi,\ \frac{15}{4},\ 5,\ 12\right\}$ belong to each category listed below?

(a) Even **(b)** Odd
(c) Positive **(d)** Negative
(e) Prime **(f)** Composite
(g) Natural **(h)** Whole
(i) Integer **(j)** Real
(k) Rational **(l)** Irrational
(m) Undefined

Exercise Set 1.1: Numbers

12. Which elements of the set
$\left\{-6.25,\ -4\tfrac{3}{4},\ -3,\ -\sqrt{5},\ -1,\ \tfrac{2}{5},\ 1,\ 2,\ 10\right\}$
belong to each category listed below?

(a) Even **(b)** Odd
(c) Positive **(d)** Negative
(e) Prime **(f)** Composite
(g) Natural **(h)** Whole
(i) Integer **(j)** Real
(k) Rational **(l)** Irrational
(m) Undefined

Fill in each of the following tables. Use "Y" for yes if the row name applies to the number or "N" for no if it does not.

13.

	$\frac{\sqrt{25}}{0}$	1	$5\frac{3}{10}$	-55	$13.\bar{3}$
Undefined					
Natural					
Whole					
Integer					
Rational					
Irrational					
Prime					
Composite					
Real					

14.

	2.36	$\frac{0}{5}=0$	$\frac{\sqrt{2}}{2}$	$\frac{2}{7}$	$\sqrt{9}=3$
Undefined					
Natural					
Whole					
Integer					
Rational					
Irrational					
Prime					
Composite					
Real					

Answer the following. If no such number exists, state "Does not exist."

15. Find a number that is both prime and even.

16. Find a rational number that is a composite number.

17. Find a rational number that is not a whole number.

18. Find a prime number that is negative.

19. Find a real number that is not a rational number.

20. Find a whole number that is not a natural number.

21. Find a negative integer that is not a rational number.

22. Find an integer that is not a whole number.

23. Find a prime number that is an irrational number.

24. Find a number that is both irrational and odd.

Answer True or False. If False, justify your answer.j

25. All natural numbers are integers.

26. No negative numbers are odd.

27. No irrational numbers are even.

28. Every even number is a composite number.

29. All whole numbers are natural numbers.

30. Zero is neither even nor odd.

31. All whole numbers are integers.

32. All integers are rational numbers.

33. All nonterminating decimals are irrational numbers.

34. Every terminating decimal is a rational number.

Answer the following.

35. List the prime numbers less than 10.

36. List the prime numbers between 20 and 30.

37. List the composite numbers between 7 and 19.

38. List the composite numbers between 31 and 41.

39. List the even numbers between $\sqrt{13}$ and $\sqrt{97}$.

40. List the odd numbers between $\sqrt{29}$ and $\sqrt{123}$.

Exercise Set 1.1: Numbers

Fill in the appropriate symbol from the set $\{<, >, =\}$.

41. $\sqrt{7}$ ______ 7

42. 3 ______ $\sqrt{3}$

43. $-\sqrt{7}$ ______ -7

44. -3 ______ $-\sqrt{3}$

45. $\sqrt{81}$ ______ 9

46. -5 ______ $-\sqrt{25}$

47. 5.32 ______ $\frac{53}{10}$

48. $\frac{7}{100}$ ______ $0.0\overline{7}$

49. $\frac{1}{3}$ ______ $\frac{1}{4}$

50. $\frac{1}{6}$ ______ $\frac{1}{5}$

51. $-\frac{1}{3}$ ______ $-\frac{1}{4}$

52. $-\frac{1}{6}$ ______ $-\frac{1}{5}$

53. $\sqrt{15}$ ______ 4

54. 7 ______ $\sqrt{49}$

55. -3 ______ $-\sqrt{9}$

56. $\sqrt{29}$ ______ 5

Answer the following.

57. Find the additive inverse of the following numbers. If undefined, write “undefined.”
(a) 3 **(b)** -4 **(c)** 1
(d) $-\frac{2}{3}$ **(e)** $2\frac{3}{7}$

58. Find the multiplicative inverse of the following numbers. If undefined, write “undefined.”
(a) 3 **(b)** -4 **(c)** 1
(d) $-\frac{2}{3}$ **(e)** $2\frac{3}{7}$

59. Find the multiplicative inverse of the following numbers. If undefined, write “undefined.”
(a) -2 **(b)** $\frac{5}{9}$ **(c)** 0
(d) $1\frac{3}{5}$ **(e)** -1

60. Find the additive inverse of the following numbers. If undefined, write “undefined.”
(a) -2 **(b)** $\frac{5}{9}$ **(c)** 0
(d) $1\frac{3}{5}$ **(e)** -1

61. Place the correct number in each of the following blanks:
(a) The sum of a number and its additive inverse is _____. (Fill in the correct number.)
(b) The product of a number and its multiplicative inverse is _____. (Fill in the correct number.)

62. Another name for the multiplicative inverse is the ____________________.

Order the numbers in each set from least to greatest and plot them on a number line.
(Hint: Use the approximations $\sqrt{2} \approx 1.41$ and $\sqrt{3} \approx 1.73$.)

63. $\left\{-1,\ -\sqrt{2},\ 0.\overline{4},\ \frac{0}{5},\ -\frac{9}{4},\ \sqrt{0.49}\right\}$

64. $\left\{-\sqrt{3}, 1, 0.65, \frac{2}{3}, -1.\overline{5}, \sqrt{0.64}\right\}$

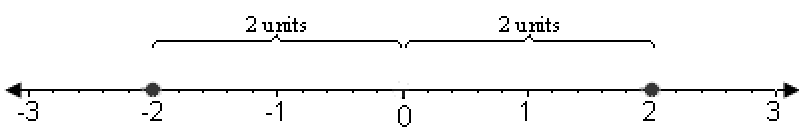
Section 1.2: Integers

- Operations with Integers

Operations with Integers

Absolute Value:

The numbers -2 and 2 are graphed on the number line below. Note that both of these numbers are at a distance of 2 units from 0, but their graphs are on opposite sides of the origin. The numbers are additive inverses of each other.



The absolute value of a real number is its distance from 0 on the number line. To indicate the absolute value of a real number x, we use the notation $|x|$.

For the numbers -2 and 2, we have $|-2| = |2| = 2$.

The absolute value of any real number x will never be negative by the following definition:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

In this section we examine the rules for addition, subtraction, multiplication, and division of integers. Recall that the set of integers is $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

Addition of Integers:

To find the sum of two integers, use the following rules:

(1) If the two integers have like signs, add their absolute values and keep the common sign.

(2) If the two integers have unlike signs, subtract their absolute values (the smaller from the larger) and keep the sign of the integer with the larger absolute value.

Example:
Evaluate each of the following.

(a) $-9+13$

(b) $-7+(-18)$

Solution:

(a) Since the given integers, -9 and 13, have unlike signs, we subtract their absolute values and keep the sign of the number with the larger absolute value.

Note that $\underbrace{|-9|=9, \quad |13|=13}_{\text{absolute values}},\quad \underbrace{13-9=4}_{\text{difference of absolute values}}$, and 13 has the larger absolute value.

Thus,

$$-9+13=4.$$

(b) Since the given integers, -7 and -18, have like signs, we add their absolute values and keep the common sign. In this case, the answer is negative.

Note that $\underbrace{|-7|=7, \quad |-18|=18}_{\text{absolute values}}$, and $\underbrace{7+18=25}_{\text{sum of absolute values}}$.

Thus,

$$-7+(-18)=-25.$$

Subtraction of Integers:

To find the difference of two integers m and n, change the subtraction to an equivalent addition by $m-n=m+(-n)$. Then follow the rules for addition.

Example:
Evaluate: $10 - 23$

Solution:
First change the subtraction to an equivalent addition:

$$10 - 23 = 10 + (-23)$$

Now, use the rules for adding integers for the problem $10 + (-23)$.

Since the integers 10 and -23 have unlike signs, we subtract their absolute values and keep the sign of the number with the larger absolute value.

Note that $\underbrace{|10| = 10, \quad |-23| = 23}_{\text{absolute values}}, \quad \underbrace{23 - 10 = 13}_{\substack{\text{difference of} \\ \text{absolute values}}},$ and -23 has the larger absolute value.

Thus,

$$10 - 23 = 10 + (-23) = -13.$$

Multiplication of Integers:

To find the product of two integers, use the following rules:

(1) If the two integers have like signs, multiply their absolute values. The answer will be positive.

(2) If the two integers have unlike signs, multiply their absolute values and then make the answer negative.

For each integer m, we have $m \cdot 0 = 0$.

Example:
Evaluate each of the following.

(a) $8(-12)$

(b) $-9(-6)$

Solution:
(a) Since the given integers, 8 and -12, have unlike signs, we multiply their absolute values and then make the answer negative.

Note that $\underbrace{|8| = 8, \quad |-12| = 12}_{\text{absolute values}}$, and $\underbrace{8 \cdot 12 = 96}_{\substack{\text{product of} \\ \text{absolute values}}}$.

Thus,

$$8(-12) = -96.$$

(b) Since the given integers, -9 and -6, have like signs, we multiply their absolute values. The answer is positive.

Note that $\underbrace{|-9| = 9, \quad |-6| = 6}_{\text{absolute values}}$, and $\underbrace{9 \cdot 6 = 54}_{\substack{\text{product of} \\ \text{absolute values}}}$.

Thus,

$$-9(-6) = 54.$$

Division of Integers:

To find the quotient of two integers, use the following rules:

(1) If the two integers have like signs, divide their absolute values. The answer will be positive.

(2) If the two integers have unlike signs, divide their absolute values and then make the answer negative.

For each integer $m \neq 0$, we have $\dfrac{0}{m} = 0$. Division by 0 is undefined.

Example:

Evaluate: $\dfrac{135}{-5}$

Solution:

Since the given integers, 135 and -5, have unlike signs, we divide their absolute values and then make the answer negative.

Note that $\underbrace{|135| = 135, \quad |-5| = 5}_{\text{absolute values}}$, and $\underbrace{\dfrac{135}{5} = 27}_{\substack{\text{quotient of} \\ \text{absolute values}}}$.

Thus,

$$\frac{135}{-5} = -27.$$

Additional Example 1:

Evaluate each of the following.

(a) $3+8$

(b) $2+(-7)$

(c) $-5+(-4)$

(d) $-6+9$

Solution:

(a) Since the given integers, 3 and 8, have like signs and both are positive, we simply add the integers together. The answer is positive.

Thus,

$$3+8=11.$$

(b) Since the given integers, 2 and -7, have unlike signs, we subtract their absolute values and keep the sign of the number with the larger absolute value.

Note that $\underbrace{|2| = 2, \quad |-7| = 7}_{\text{absolute values}}, \quad \underbrace{7-2=5}_{\substack{\text{difference of} \\ \text{absolute values}}}, \quad$ and -7 has the larger absolute value.

Thus,

$$2+(-7) = -5.$$

(c) Since the given integers, -5 and -4, have like signs, we add their absolute values and keep the common sign. The answer is negative.

Note that $\underbrace{|-5| = 5, \quad |-4| = 4}_{\text{absolute values}}, \quad$ and $\quad \underbrace{5+4=9}_{\substack{\text{sum of absolute} \\ \text{values}}}.$

Thus,

$$-5+(-4) = -9.$$

(d) Since the given integers, -6 and 9, have unlike signs, we subtract their absolute values and keep the sign of the number with the larger absolute value.

Note that $\underbrace{|-6| = 6, \quad |9| = 9}_{\text{absolute values}}$, $\underbrace{9 - 6 = 3}_{\substack{\text{difference of}\\ \text{absolute values}}}$, and 9 has the larger absolute value.

Thus,

$$-6 + 9 = 3.$$

Additional Example 2:

Evaluate each of the following.

(a) $5 - 7$

(b) $-2 - 6$

(c) $-5 - (-4)$

(d) $0 - 9$

Solution:

(a) First change the subtraction to an equivalent addition:

$$5 - 7 = 5 + (-7)$$

Now, use the rules for adding integers for the problem $5 + (-7)$.

Since the integers 5 and -7 have unlike signs, we subtract their absolute values and keep the sign of the number with the larger absolute value.

Note that $\underbrace{|5| = 5, \quad |-7| = 7}_{\text{absolute values}}$, $\underbrace{7 - 5 = 2}_{\substack{\text{difference of}\\ \text{absolute values}}}$, and -7 has the larger absolute value.

Thus,

$$5 - 7 = 5 + (-7) = -2.$$

(b) First change the subtraction to an equivalent addition:

$$-2 - 6 = -2 + (-6)$$

Now, use the rules for adding integers for the problem $-2 + (-6)$.

Since the integers -2 and -6 have like signs, we add their absolute values and keep the common sign. The answer is negative.

Note that $\underbrace{|-2| = 2, \quad |-6| = 6}_{\text{absolute values}}$, and $\underbrace{2+6=8}_{\text{sum of absolute values}}$.

Thus,

$$-2-6 = -2+(-6) = -8.$$

(c) First change the subtraction to an equivalent addition:

$$\begin{aligned} -5-(-4) &= -5+(-(-4)) \\ &= -5+4 \end{aligned}$$

Now, use the rules for adding integers for the problem $-5+4$.

Since the integers -5 and 4 have unlike signs, we subtract their absolute values and keep the sign of the number with the larger absolute value.

Note that $\underbrace{|-5| = 5, \quad |4| = 4}_{\text{absolute values}}$, $\underbrace{5-4=1}_{\text{difference of absolute values}}$, and -5 has the larger absolute value.

Thus,

$$\begin{aligned} -5-(-4) &= -5+(-(-4)) \\ &= -5+4 \\ &= -1. \end{aligned}$$

(d) Change the subtraction to an equivalent addition and add.

$$\begin{aligned} 0-9 &= 0+(-9) \\ &= -9 \end{aligned}$$

Additional Example 3:

Evaluate each of the following. If undefined, write "undefined."

(a) $8(-3)$

(b) $\dfrac{-9}{-3}$

(c) $\dfrac{0}{5}$

(d) $-7(-9)$

Solution:

(a) Since the given integers, 8 and -3, have unlike signs, we multiply their absolute values and then make the answer negative.

Note that $\underbrace{|8| = 8, \quad |-3| = 3}_{\text{absolute values}}$, and $\underbrace{8 \cdot 3 = 24}_{\substack{\text{product of} \\ \text{absolute values}}}$.

Thus,

$$8(-3) = -24.$$

(b) Since the given integers, -9 and -3, have like signs, we divide their absolute values. The answer is positive.

Note that $\underbrace{|-9| = 9, \quad |-3| = 3}_{\text{absolute values}}$, and $\underbrace{\frac{9}{3} = 3}_{\substack{\text{quotient of} \\ \text{absolute values}}}$.

Thus,

$$\frac{-9}{-3} = 3.$$

(c) For any integer $m \neq 0$ we have $\dfrac{0}{m} = 0$.

Thus,

$$\frac{0}{5} = 0.$$

(d) Since the given integers, -7 and -9, have like signs, we multiply their absolute values. The answer is positive.

Note that $\underbrace{|-7| = 7, \quad |-9| = 9}_{\text{absolute values}}$, and $\underbrace{7 \cdot 9 = 63}_{\substack{\text{product of} \\ \text{absolute values}}}$.

Thus,

$$-7(-9) = 63.$$

Additional Example 4:

Evaluate each of the following.

(a) $7(-3)(-2)$

(b) $3(0)$

(c) $-2(5)(6)$

(d) $\dfrac{15}{-5}$

Solution:

(a) We work from left to right using the rules for multiplying integers.

$7(-3)(-2) = -21(-2)$ Find the product of the first two integers 7 and -3. The result is negative since the integers have unlike signs.

$= 42$ Multiply the product in the first step by the last integer -2. The result is positive since the integers have like signs.

Thus,

$$7(-3)(-2) = -21(-2) = 42.$$

(b) For any integer m, we have $m \cdot 0 = 0$.

Thus,

$$3(0) = 0.$$

(c) We work from left to right using the rules for multiplying integers.

$-2(5)(6) = -10(6)$ Find the product of the first two integers -2 and 5. The result is negative since the integers have unlike signs.

$= -60$ Multiply the product in the first step by the last integer 6. The result is negative since the integers have unlike signs.

Thus,

$$-2(5)(6) = -10(6) = -60.$$

(d) Use the rules for dividing integers. The result is negative since the given integers, 15 and -5, have unlike signs.

Thus,

$$\frac{15}{-5} = -3.$$

Exercise Set 1.2: Integers

Evaluate the following.

1. **(a)** $3+7$ **(b)** $-3+(-7)$ **(c)** $-3+7$ **(d)** $3+(-7)$ **(e)** $-3+0$

2. **(a)** $8+5$ **(b)** $-8+5$ **(c)** $8+(-5)$ **(d)** $-8+(-5)$ **(e)** $0+(-5)$

3. **(a)** $0-4$ **(b)** $4-0$ **(c)** $0-(-4)$ **(d)** $-4-0$

4. **(a)** $6-0$ **(b)** $0-(-6)$ **(c)** $0-6$ **(d)** $-6-0$

5. **(a)** $-10-2$ **(b)** $-10-(-2)$ **(c)** $10-2$ **(d)** $-2-(-10)$ **(e)** $2-(-10)$ **(f)** $2-10$ **(g)** $-2-10$ **(h)** $10-(-2)$

6. **(a)** $-7-(-9)$ **(b)** $-7-9$ **(c)** $7-9$ **(d)** $9-(-7)$ **(e)** $-9-(-7)$ **(f)** $9-7$ **(g)** $7-(-9)$ **(f)** $-9-7$

Fill in the appropriate symbol from the set $\{<, >, =\}$ **.**

7. **(a)** $-1(4)$ _____ 0 **(b)** $-7(-2)$ _____ 0 **(c)** $5(-1)(-2)$ _____ 0 **(d)** $3(-1)(0)$ _____ 0

8. **(a)** $-3(-2)$ _____ 0 **(b)** $7(-1)$ _____ 0 **(c)** $-5(0)(-2)$ _____ 0 **(d)** $-2(-2)(-2)$ ____ 0

Evaluate the following. If undefined, write "Undefined."

9. **(a)** $6(0)$ **(b)** $\frac{6}{0}$ **(c)** $\frac{0}{-6}$ **(d)** $6(-1)$ **(e)** $6(1)$ **(f)** $6(-1)$ **(g)** $-6(-1)$ **(h)** $\frac{-6}{-1}$ **(i)** $\frac{6}{-1}$ **(j)** $\frac{-6}{0}$ **(k)** $-6(-1)(-1)$ **(l)** $\frac{0}{6}$

10. **(a)** $-1(7)$ **(b)** $\frac{-7}{-1}$ **(c)** $7(-1)$ **(d)** $0(-7)$ **(e)** $-1(-7)$ **(f)** $\frac{0}{-7}$ **(g)** $\frac{-7}{1}$ **(h)** $\frac{0}{7}$ **(i)** $\frac{7}{0}$ **(j)** $7(-1)(-1)$ **(k)** $-7(0)(-1)$ **(l)** $\frac{-7}{0}$

11. **(a)** $-10(-2)$ **(b)** $\frac{-10}{-2}$ **(c)** $-10(2)$ **(d)** $\frac{10}{-2}$ **(e)** $\frac{-10}{2}$ **(f)** $\frac{10}{2}$

12. **(a)** $\frac{-6}{3}$ **(b)** $6(-3)$ **(c)** $\frac{-6}{-3}$ **(d)** $6(3)$ **(e)** $-6(-3)$ **(f)** $\frac{6}{-3}$

13. **(a)** $2(-3)(-4)$ **(b)** $(-2)(-3)(-4)$ **(c)** $-1(-2)(-3)(-4)$ **(d)** $-1(2)(-3)(-4)$

14. **(a)** $3(-2)(5)$ **(b)** $-3(-2)(5)$ **(c)** $-3(-2)(-1)(5)$ **(d)** $-3(-2)(-2)(-5)$

15. **(a)** $8-2$ **(b)** $-8+(-2)$ **(c)** $-8(-2)$ **(d)** $\frac{-8}{2}$ **(e)** $-8-(-2)$ **(f)** $(-8)(0)$ **(g)** $-8(-1)$ **(h)** $-8-1$ **(i)** $\frac{8}{-1}$ **(j)** $0-8$ **(k)** $2-(-8)$ **(l)** $\frac{0}{-8}$ **(m)** $\frac{-2}{-8}$ **(n)** $\frac{2}{0}$ **(o)** $-2+8$

16. **(a)** $\frac{12}{-3}$ **(b)** $-12(-3)$ **(c)** $-12-3$ **(d)** $-3+12$ **(e)** $0(-3)$ **(f)** $0-(-3)$ **(g)** $(-3)(12)$ **(h)** $\frac{12}{-1}$ **(i)** $\frac{-3}{0}$ **(j)** $\frac{-3}{12}$ **(k)** $-1+(-3)$ **(l)** $-1(12)$ **(m)** $\frac{0}{-3}$ **(n)** $-3-(-1)$ **(o)** $-3(1)$

Section 1.3: Fractions

- Greatest Common Divisor and Least Common Multiple
- Addition and Subtraction of Fractions
- Multiplication and Division of Fractions

Greatest Common Divisor and Least Common Multiple

Greatest Common Divisor:

Consider the natural numbers 18 and 24. The table below shows the factors of each of these numbers.

Number	Factors							
18	1	2	3	6	9	18		
24	1	2	3	4	6	8	12	24

As we see from the table, there are factors that are shared by both 18 and 24. Of these common factors, the greatest one is 6. It is called the GCD (greatest common divisor) of 18 and 24, the largest natural number that divides both 18 and 24.

In general, if m and n are natural numbers with $m < n$, then the GCD of m and n is the largest natural number that divides both both m and n.

The GCD of m and n must be a natural number between and including 1 and m.

In the example above, $m = 18$ and $n = 24$. The GCD is 6, which is a natural number between 1 and 18.

A Method for Finding the GCD:

Factor 18 into a product of prime factors.

$$\begin{aligned} 18 &= 2 \cdot 9 \\ &= 2 \cdot 3 \cdot 3 \end{aligned}$$

Factor 24 into a product of prime factors.

$$\begin{aligned} 24 &= 4 \cdot 6 \\ &= 2 \cdot 2 \cdot 2 \cdot 3 \end{aligned}$$

Arrange the prime factors in a table, aligning common factors in the same column.

2	3	3		
2	3		2	2

The GCD is the product of the prime factors that are shared by both numbers.

2	3	3		
2	3		2	2

The GCD is $2 \cdot 3 = 6$.

Least Common Multiple:

Consider the natural numbers 18 and 24. The table below shows a partial list of multiples of each of these numbers.

The first line is obtained by $1 \cdot 18 = 18$, $2 \cdot 18 = 36$, $3 \cdot 18 = 54$, $4 \cdot 18 = 72$, and so on. The list can continue indefinitely in this manner.

Number	Multiples										
18	18	36	54	72	90	108	126	144	162	180	198	216
24	24	48	72	96	120	144	168	192	216			

As we see from the table, there are multiples that are shared by both 18 and 24. They are 72, 144, and 216. (We could continue the list to find other common multiples.) Of these common multiples, the least one is 72. It is called the LCM (least common multiple) of 18 and 24, the smallest natural number that is a multiple of both 18 and 24.

In general, if m and n are natural numbers with $m < n$, then the LCM of m and n is the smallest natural number that is a multiple both both m and n.

The LCM of m and n must be a natural number between and including n and mn.

In the example above, $m = 18$ and $n = 24$. The LCM is 72, which is a natural number between 24 and $18 \cdot 24 = 432$.

A Method for Finding the LCM:

Factor both 18 and 24 into a product of primes and arrange the prime factors in a table, aligning common factors in the same column.

There are 5 columns in the table. The LCM is the product of the 5 prime factors that appear in these columns.

2	3	3		
2	3		2	2

The LCM is $2 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 72$.

Example:

Find the GCD (greatest common divisor) and LCM (least common multiple) of 20 and 28.

Solution:

Factor 20 into a product of prime factors.

$$\begin{aligned} 20 &= 4 \cdot 5 \\ &= 2 \cdot 2 \cdot 5 \end{aligned}$$

Factor 28 into a product of prime factors.

$$\begin{aligned} 28 &= 4 \cdot 7 \\ &= 2 \cdot 2 \cdot 7 \end{aligned}$$

Arrange the prime factors in a table, aligning common factors in the same column.

2	2	5	
2	2		7

The GCD is the product of the prime factors that are shared by both numbers.

2	2	5	
2	2		7

The GCD is $2 \cdot 2 = 4$.

There are 4 columns in the table. The LCM is the product of the 4 prime factors that appear in these columns.

2	2	5	
2	2		7

The LCM is
$2 \cdot 2 \cdot 5 \cdot 7 = 140.$

Additional Example 1:
Find the GCD (greatest common divisor) and the LCM (least common multiple) of 24 and 30.

Solution:
Factor 24 into a product of prime numbers.

$$\begin{aligned} 24 &= 4 \cdot 6 \\ &= 2 \cdot 2 \cdot 2 \cdot 3 \end{aligned}$$

Factor 30 into a product of prime numbers.

$$\begin{aligned} 30 &= 6 \cdot 5 \\ &= 2 \cdot 3 \cdot 5 \end{aligned}$$

Arrange the prime factors in a table, aligning common factors in the same column.

2	2	2	3	
2			3	5

The GCD is the product of the prime factors that are shared by both numbers.

2	2	2	3	
2			3	5

The GCD is $2 \cdot 3 = 6$.

There are 5 columns in the table. The LCM is the product of the 5 prime factors that appear in these columns.

2	2	2	3	
2			3	5

The LCM is $2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 120$.

Additional Example 2:

Find the GCD (greatest common divisor) and the LCM (least common multiple) of 90 and 105.

Solution:

Factor 90 into a product of prime numbers.

$$\begin{aligned} 90 &= 9 \cdot 10 \\ &= 3 \cdot 3 \cdot 2 \cdot 5 \end{aligned}$$

Factor 105 into a product of prime numbers.

$$\begin{aligned} 105 &= 5 \cdot 21 \\ &= 5 \cdot 3 \cdot 7 \end{aligned}$$

Arrange the prime factors in a table, aligning common factors in the same column.

2	3	3	5	
	3		5	7

The GCD is the product of the prime factors that are shared by both numbers.

2	3	3	5	
	3		5	7

The GCD is $3 \cdot 5 = 15$.

There are 5 columns in the table. The LCM is the product of the 5 prime factors that appear in these columns.

2	3	3	5	
	3		5	7

The LCM is $2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 = 630$.

Additional Example 3:
Find the GCD (greatest common divisor) and the LCM (least common multiple) of 12, 18, and 24.

Solution:
Factor 12 into a product of prime numbers.

$$\begin{aligned} 12 &= 4 \cdot 3 \\ &= 2 \cdot 2 \cdot 3 \end{aligned}$$

Factor 18 into a product of prime numbers.

$$\begin{aligned} 18 &= 2 \cdot 9 \\ &= 2 \cdot 3 \cdot 3 \end{aligned}$$

Factor 24 into a product of prime numbers.

$$\begin{aligned} 24 &= 4 \cdot 6 \\ &= 2 \cdot 2 \cdot 2 \cdot 3 \end{aligned}$$

Arrange the prime factors in a table, aligning common factors in the same column.

2	2	3		
2		3	3	
2	2	3		2

The GCD is the product of the prime factors that are shared by all numbers.

2	2	3		
2		3	3	
2	2	3		2

The GCD is $2 \cdot 3 = 6$.

There are 5 columns in the table. The LCM is the product of the 5 prime factors that appear in these columns.

2	2	3		
2		3	3	
2	2	3		2

The LCM is $2 \cdot 2 \cdot 3 \cdot 3 \cdot 2 = 72$.

Additional Example 4:

Find the GCD (greatest common divisor) and the LCM (least common multiple) of 18, 36, and 60.

Solution:

Factor 18 into a product of prime numbers.

$$\begin{aligned} 18 &= 2\cdot 9 \\ &= 2\cdot 3\cdot 3 \end{aligned}$$

Factor 36 into a product of prime numbers.

$$\begin{aligned} 36 &= 4\cdot 9 \\ &= 2\cdot 2\cdot 3\cdot 3 \end{aligned}$$

Factor 60 into a product of prime numbers.

$$\begin{aligned} 60 &= 6\cdot 10 \\ &= 2\cdot 3\cdot 2\cdot 5 \end{aligned}$$

Arrange the prime factors in a table, aligning common factors in the same column.

2	3	3		
2	3	3	2	
2	3		2	5

The GCD is the product of the prime factors that are shared by all numbers.

2	3	3		
2	3	3	2	
2	3		2	5

The GCD is $2\cdot 3 = 6$.

There are 5 columns in the table. The LCM is the product of the 5 prime factors that appear in these columns.

2	3	3		
2	3	3	2	
2	3		2	5

The LCM is $2\cdot 3\cdot 3\cdot 2\cdot 5 = 180$.

Addition and Subtraction of Fractions

The rules for adding, subtracting, multiplying, and dividing integers that were presented in Section 1.2 extend to all real numbers. In this section, we will use those rules in performing the operations of addition and subtraction of fractions.

Addition and Subtraction of Fractions with Like Denominators:

To add (or subtract) two fractions whose denominators are the same, add (or subtract) the numerators and keep the common denominator.

If $a, b,$ and c are real numbers and $c \neq 0$, then

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \quad \text{and} \quad \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$$

Example:
Evaluate each of the following and give all results in simplest form.

(a) $\frac{1}{12}+\frac{5}{12}$

(b) $\frac{3}{8}-\frac{5}{8}$

(c) $3\frac{1}{6}-4\frac{5}{6}$

Solution:

(a) To add two fractions with a common denominator, begin by writing the sum of the numerators over the common denominator.

$$\frac{1}{12}+\frac{5}{12}=\frac{1+5}{12}$$

$$=\frac{6}{12} \qquad \text{Perform the additon in the numerator.}$$

$$=\frac{\overset{1}{\cancel{2}}\cdot\overset{1}{\cancel{3}}}{\underset{1}{\cancel{2}}\cdot 2\cdot\underset{1}{\cancel{3}}} \qquad \text{Divide out common factors in numerator and denominator.}$$

$$=\frac{1}{2} \qquad \text{Simplify.}$$

(b) To subtract two fractions with a common denominator, begin by writing the difference of the numerators over the common denominator.

$$\frac{3}{8}-\frac{5}{8}=\frac{3-5}{8}$$

$$=\frac{3+(-5)}{8}$$ Rewrite the subraction in the numerator as addition.

$$=\frac{-2}{8}$$ Perform the addition in the numerator.

$$=-\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{2}}\cdot 4}$$ Divide out common factors in numerator and denominator.

$$=-\frac{1}{4}$$ Simplify.

(c) We begin by writing the mixed numbers as improper fractions.

$$3\frac{1}{6}-4\frac{5}{6}=\frac{19}{6}-\frac{29}{6}$$

$$=\frac{19-29}{6}$$ Write the difference of the numerators over the common denominator.

$$=\frac{19+(-29)}{6}$$ Rewrite the subtraction in the numerator as addition.

$$=\frac{-10}{6}$$ Perform the addition in the numerator.

$$=-\frac{\overset{1}{\cancel{2}}\cdot 5}{\underset{1}{\cancel{2}}\cdot 3}$$ Divide out common factors in numerator and denominator.

$$=-\frac{5}{3}$$ Simplify.

Addition and Subtraction of Fractions with Unlike Denominators:

To add (or subtract) two fractions whose denominators are not the same, we must rewrite each fraction so that they have a common denominator. The smallest such denominator is called the least common denominator (LCD). The method of finding the LCM of the denominators will produce the LCD.

Example:

Evaluate $\frac{3}{8}+\frac{5}{28}$ and give the result in simplest form.

Solution:

We must rewrite the given fractions so that they have a common denominator. Find the LCM of the denominators 8 and 28 to find the least common denominator.

$8=2\cdot 4=2\cdot 2\cdot 2$ and $28=4\cdot 7=2\cdot 2\cdot 7$

2	2	2	
2	2		7

The least common denominator is $2\cdot 2\cdot 2\cdot 7=56$.

Express each fraction as an equivalent fraction with a denominator of 56.

For the first fraction, we need to multiply 8 by 7 since 8(7)=56. We also must multiply the numerator by 7.

For the second fraction, we need to multiply 28 by 2 since 28(2)=56. We also must multiply the numerator by 2.

$$\frac{3}{8}+\frac{5}{28}=\frac{3\cdot 7}{8\cdot 7}+\frac{5\cdot 2}{28\cdot 2}$$

$$=\frac{21}{56}+\frac{10}{56}$$ Perform the multiplications.

$$=\frac{21+10}{56}$$ Write the sum of the numerators over the common denominator.

$$=\frac{31}{56}$$ Simplify.

Additional Example 1:

Evaluate each of the following and write all answers in simplest form.

(a) $\frac{3}{8}+\frac{1}{8}$

(b) $\frac{2}{9}-\frac{8}{9}$

Solution:

(a) To add two fractions with a common denominator, begin by writing the sum of the numerators over the common denominator.

$$\frac{3}{8}+\frac{1}{8}=\frac{3+1}{8}$$

$$=\frac{4}{8}$$ Perform the additon in the numerator.

$$=\frac{\cancel{4}^{1}}{\cancel{4}_{1}\cdot 2}$$ Divide out common factors in numerator and denominator.

$$=\frac{1}{2}$$ Simplify.

(b) To subtract two fractions with a common denominator, begin by writing the difference of the numerators over the common denominator.

$$\frac{2}{9}-\frac{8}{9}=\frac{2-8}{9}$$

$$=\frac{2+(-8)}{9}$$ Rewrite the subraction in the numerator as addition.

$$=\frac{-6}{9}$$ Perform the addition in the numerator.

$$=-\frac{2\cdot\cancel{3}^{1}}{\cancel{3}_{1}\cdot 3}$$ Divide out common factors in numerator and denominator.

$$=-\frac{2}{3}$$ Simplify.

Additional Example 2:

Evaluate each of the following and write all answers in simplest form.

(a) $3\frac{5}{12}-4\frac{1}{12}$

(b) $2\frac{3}{10}+4\frac{1}{10}$

Solution:

(a) We begin by writing the mixed numbers as improper fractions.

$$3\frac{5}{12}-4\frac{1}{12}=\frac{41}{12}-\frac{49}{12}$$

$$=\frac{41-49}{12}$$ Write the difference of the numerators over the common denominator.

$$=\frac{41+(-49)}{12}$$ Rewrite the subtraction in the numerator as addition.

$$=\frac{-8}{12}$$ Perform the addition in the numerator.

$$=-\frac{\overset{1}{\cancel{2}}\cdot\overset{1}{\cancel{2}}\cdot 2}{\underset{1}{\cancel{2}}\cdot\underset{1}{\cancel{2}}\cdot 3}$$ Divide out common factors in numerator and denominator.

$$=-\frac{2}{3}$$ Simplify.

(b) We begin by writing the mixed numbers as improper fractions.

$$2\frac{3}{10}+4\frac{1}{10}=\frac{23}{10}+\frac{41}{10}$$

$$=\frac{23+41}{10}$$ Write the sum of the numerators over the common denominator.

$$=\frac{64}{10}$$ Perform the addition in the numerator.

$$=\frac{\overset{1}{\cancel{2}}\cdot 32}{\underset{1}{\cancel{2}}\cdot 5}$$ Divide out common factors in numerator and denominator.

$$=\frac{32}{5}$$ Simplify.

$$=6\frac{2}{5}$$ Write the result as a mixed number.

Additional Example 3:

Evaluate each of the following and write all answers in simplest form.

(a) $\frac{3}{10}+\frac{4}{15}$

(b) $\frac{11}{14}-\frac{17}{21}$

Solution:

(a) We must rewrite the given fractions so that they have a common denominator. Find the LCM of the denominators 10 and 15 to find the least common denominator.

$10 = 2 \cdot 5$ and $15 = 3 \cdot 5$

2	5	
	5	3

The least common denominator is $2 \cdot 5 \cdot 3 = 30$.

Express each fraction as an equivalent fraction with a denominator of 30.

For the first fraction, we need to multiply 10 by 3 since $10 \cdot 3 = 30$. We also must multiply the numerator by 3.

For the second fraction, we need to multiply 15 by 2 since $15 \cdot 2 = 30$. We also must multiply the numerator by 2.

$$\begin{aligned}
\frac{3}{10} + \frac{4}{15} &= \frac{3 \cdot 3}{10 \cdot 3} + \frac{4 \cdot 2}{15 \cdot 2} \\
&= \frac{9}{30} + \frac{8}{30} && \text{Perform the multiplications.} \\
&= \frac{9+8}{30} && \text{Write the sum of the numerators over the common denominator.} \\
&= \frac{17}{30} && \text{Simplify.}
\end{aligned}$$

(b) We must rewrite the given fractions so that they have a common denominator. Find the LCM of the denominators 14 and 21 to find the least common denominator.

$14 = 2 \cdot 7$ and $21 = 3 \cdot 7$

2	7	
	7	3

The least common denominator is $2 \cdot 7 \cdot 3 = 42$.

Express each fraction as an equivalent fraction with a denominator of 42.

For the first fraction, we need to multiply 14 by 3 since $14 \cdot 3 = 42$. We also must multiply the numerator by 3.

For the second fraction, we need to multiply 21 by 2 since $21 \cdot 2 = 42$. We also must multiply the numerator by 2.

$$\frac{11}{14} - \frac{17}{21} = \frac{11 \cdot 3}{14 \cdot 3} - \frac{17 \cdot 2}{21 \cdot 2}$$

$$= \frac{33}{42} - \frac{34}{42}$$ Perform the multiplications.

$$= \frac{33 - 34}{42}$$ Write the difference of the numerators over the common denominator.

$$= \frac{33 + (-34)}{42}$$ Write the subtraction in the numerator as addition.

$$= \frac{-1}{42} = -\frac{1}{42}$$ Perform the addition in the numerator.

Additional Example 4:

Evaluate the following and write the answer in simplest form. (If the answer is a mixed number/improper fraction, then write the answer as a mixed number.)

$$3\frac{1}{5} + 5\frac{1}{6}$$

Solution:

We begin by rewriting each mixed number as an improper fraction.

$$3\frac{1}{5} + 5\frac{1}{6} = \frac{16}{5} + \frac{31}{6}$$

We must rewrite the given fractions so that they have a common denominator. Find the LCM of the denominators 5 and 6 to find the least common denominator.

5 is prime and $6 = 2 \cdot 3$.

5		
	2	3

The least common denominator is $5 \cdot 2 \cdot 3 = 30$.

Express each fraction as an equivalent fraction with a denominator of 30.

For the first fraction, we need to multiply 5 by 6 since $5 \cdot 6 = 30$. We also must multiply the numerator by 6.

For the second fraction, we need to multiply 6 by 5 since $6 \cdot 5 = 30$. We also must multiply the numerator by 5.

$$
\begin{aligned}
3\frac{1}{5} + 5\frac{1}{6} &= \frac{16}{5} + \frac{31}{6} \\
&= \frac{16 \cdot 6}{5 \cdot 6} + \frac{31 \cdot 5}{6 \cdot 5} \\
&= \frac{96}{30} + \frac{155}{30} && \text{Perform the multiplications.} \\
&= \frac{96 + 155}{30} && \text{Write the sum of the numerators over the common denominator.} \\
&= \frac{251}{30} && \text{Perform the addition in the numerator.} \\
&= 8\frac{11}{30} && \text{Write the result as a mixed number.}
\end{aligned}
$$

Multiplication and Division of Fractions

The rules for adding, subtracting, multiplying, and dividing integers that were presented in Section 1.2 extend to all real numbers. In this section, we will use those rules in performing the operations of multiplication and division of fractions.

Multiplication of Fractions:

The multiply two fractions, place the product of the numerators over the produt of the denominators.

If $a, b, c,$ and d are real numbers and $c \neq 0$ and $d \neq 0$, then

$$\frac{a}{c} \cdot \frac{b}{d} = \frac{a \cdot b}{c \cdot d}.$$

Example:

Evaluate $\frac{7}{10} \cdot \frac{4}{21}$ and give the result in simplest form.

Solution:

To multiply two fractions, write the product of the numerators over the product of the denominators.

$$\frac{7}{10} \cdot \frac{4}{21} = \frac{7 \cdot 4}{10 \cdot 21}$$

$$= \frac{\overset{1}{\cancel{7}} \cdot \overset{1}{\cancel{2}} \cdot 2}{\underset{1}{\cancel{2}} \cdot 5 \cdot 3 \cdot \underset{1}{\cancel{7}}} \qquad \text{Divide out common factors in numerator and denominator.}$$

$$= \frac{2}{15} \qquad \text{Simplify.}$$

Division of Fractions:

To find the quotient of two fractions, multiply the first fraction by the reciprocal of the second fraction.

If a, b, c, and d are real numbers, and $b \neq 0$, $c \neq 0$, and $d \neq 0$, then

$$\frac{a}{c} \div \frac{b}{d} = \frac{a}{c} \cdot \frac{d}{b}$$

Two numbers are reciprocals of each other if their product is 1. For example, $\frac{5}{8}$ and $\frac{8}{5}$ are reciprocals since $\frac{5}{8} \cdot \frac{8}{5} = 1$.

Example:

Evaluate $\frac{2}{3} \div \frac{8}{9}$ and give the result in simplest form.

Solution:

To divide two fractions, multiply the first fraction by the reciprocal of the second fraction.

$$\frac{2}{3} \div \frac{8}{9} = \frac{2}{3} \cdot \frac{9}{8}$$

$$= \frac{2 \cdot 9}{3 \cdot 8}$$ Follow the rule for multiplying fractions.

$$= \frac{\overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{3}} \cdot 3}{\underset{1}{\cancel{3}} \cdot \underset{1}{\cancel{2}} \cdot 2 \cdot 2}$$ Divide out common factors in numerator and denominator.

$$= \frac{3}{4}$$ Simplify.

Additional Example 1:

Evaluate each of the following and write all answers in simplest form.

(a) $\frac{2}{3} \cdot 6$

(b) $\frac{2}{3} \div 6$

Solution:

(a) Write 6 as $\frac{6}{1}$. Then to multiply two fractions, write the product of the numerators over the product of the denominators.

$$\frac{2}{3} \cdot 6 = \frac{2}{3} \cdot \frac{6}{1}$$

$$= \frac{2 \cdot 6}{3 \cdot 1}$$

$$= \frac{2 \cdot \overset{1}{\cancel{3}} \cdot 2}{\underset{1}{\cancel{3}} \cdot 1}$$ Divide out common factors in numerator and denominator.

$$= \frac{4}{1}$$ Simplify.

$$= 4$$ Simplify.

(b) Write 6 as $\frac{6}{1}$. Then to divide two fractions, multiply the first fraction by the reciprocal of the second fraction.

$$\begin{aligned}\frac{2}{3}\div 6 &= \frac{2}{3}\div\frac{6}{1}\\ &= \frac{2}{3}\cdot\frac{1}{6}\\ &= \frac{2\cdot 1}{3\cdot 6} && \text{Follow the rule for multiplying fractions.}\\ &= \frac{\overset{1}{\cancel{2}}\cdot 1}{3\cdot\underset{1}{\cancel{2}}\cdot 3} && \text{Divide out common factors in numerator and denominator.}\\ &= \frac{1}{9} && \text{Simplify.}\end{aligned}$$

Additional Example 2:

Evaluate each of the following and write all answers in simplest form. (If the answer is a mixed number/improper fraction, then write the answer as an improper fraction.)

(a) $\frac{27}{25}\cdot\frac{10}{33}$

(b) $\frac{3}{11}\div\frac{9}{44}$

Solution:

(a) To multiply two fractions, write the product of the numerators over the product of the denominators.

$$\begin{aligned}\frac{27}{25}\cdot\frac{10}{33} &= \frac{27\cdot 10}{25\cdot 33}\\ &= \frac{\overset{1}{\cancel{3}}\cdot 3\cdot 3\cdot 2\cdot\overset{1}{\cancel{5}}}{5\cdot\underset{1}{\cancel{5}}\cdot\underset{1}{\cancel{3}}\cdot 11} && \text{Divide out common factors in numerator and denominator.}\\ &= \frac{18}{55} && \text{Simplify.}\end{aligned}$$

(b) To divide two fractions, multiply the first fraction by the reciprocal of the second fraction.

$$\frac{3}{11} \div \frac{9}{44} = \frac{3}{11} \cdot \frac{44}{9}$$

$$= \frac{3 \cdot 44}{11 \cdot 9}$$ Follow the rule for multiplying fractions.

$$= \frac{\overset{1}{\cancel{3}} \cdot 2 \cdot 2 \cdot \overset{1}{\cancel{11}}}{\underset{1}{\cancel{11}} \cdot \underset{1}{\cancel{3}} \cdot 3}$$ Divide out common factors in numerator and denominator.

$$= \frac{4}{3}$$ Simplify.

Additional Example 3:

Evaluate each of the following and write all answers in simplest form.

(a) $-\frac{3}{8}\left(-\frac{4}{21}\right)$

(b) $\dfrac{-\frac{1}{8}}{\frac{3}{14}}$

Solution:

(a) To multiply two fractions, write the product of the numerators over the product of the denominators. Since the fractions have like signs, the answer is positive.

$$-\frac{3}{8}\left(-\frac{4}{21}\right) = \frac{3 \cdot 4}{8 \cdot 21}$$

$$= \frac{\overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{2}}}{2 \cdot \underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{3}} \cdot 7}$$ Divide out common factors in numerator and denominator.

$$= \frac{1}{14}$$ Simplify.

(b) To divide two fractions, multiply the first fraction by the reciprocal of the second fraction. Since the fractions have unlike signs, the answer is negative.

$$\frac{-\frac{1}{8}}{\frac{3}{14}} = -\left(\frac{1}{8} \cdot \frac{14}{3}\right)$$

$$= -\left(\frac{1 \cdot 14}{8 \cdot 3}\right)$$ Follow the rule for multiplying fractions.

$$= -\left(\frac{1 \cdot \cancel{2}^{1} \cdot 7}{\cancel{2}_{1} \cdot 2 \cdot 2 \cdot 3}\right)$$ Divide out common factors in numerator and denominator.

$$= -\frac{7}{12}$$ Simplify.

Additional Example 4:

Evaluate the following and write the answer in simplest form. (If the answer is a mixed number/improper fraction, then write the answer as a <u>mixed number</u>.)

$$\left(3\frac{1}{2}\right) \div \left(2\frac{4}{5}\right)$$

Solution:

We begin by writing each mixed number as an improper fraction. Then to divide two fractions, multiply the first fraction by the reciprocal of the second fraction.

$$\left(3\frac{1}{2}\right) \div \left(2\frac{4}{5}\right) = \frac{7}{2} \div \frac{14}{5}$$

$$= \frac{7}{2} \cdot \frac{5}{14}$$

$$= \frac{7 \cdot 5}{2 \cdot 14}$$ Follow the rule for multiplying fractions.

$$= \frac{\cancel{7}^{1} \cdot 5}{2 \cdot 2 \cdot \cancel{7}_{1}}$$ Divide out common factors in numerator and denominator.

$$= \frac{5}{4}$$ Simplify.

$$= 1\frac{1}{4}$$ Write as a mixed number.

Exercise Set 1.3: Fractions

For each of the following groups of numbers,
(a) Find their GCD (greatest common divisor).
(b) Find their LCM (least common multiple).

1. 6 and 8

2. 4 and 5

3. 7 and 10

4. 12 and 15

5. 14 and 28

6. 6 and 22

7. 8 and 20

8. 9 and 18

9. 18 and 30

10. 60 and 210

11. 16, 20, and 24

12. 15, 21, and 27

Change each of the following improper fractions to a mixed number.

13. (a) $\frac{9}{7}$ **(b)** $\frac{23}{5}$ **(c)** $\frac{19}{3}$

14. (a) $\frac{10}{3}$ **(b)** $\frac{17}{6}$ **(c)** $\frac{49}{9}$

15. (a) $-\frac{27}{4}$ **(b)** $-\frac{32}{11}$ **(c)** $-\frac{73}{10}$

16. (a) $-\frac{15}{13}$ **(b)** $-\frac{43}{8}$ **(c)** $-\frac{57}{7}$

Change each of the following mixed numbers to an improper fraction.

17. (a) $5\frac{1}{6}$ **(b)** $7\frac{4}{9}$ **(c)** $8\frac{2}{3}$

18. (a) $3\frac{1}{2}$ **(b)** $10\frac{7}{8}$ **(c)** $6\frac{3}{5}$

19. (a) $-2\frac{5}{7}$ **(b)** $-5\frac{2}{3}$ **(c)** $-12\frac{1}{4}$

20. (a) $-4\frac{1}{9}$ **(b)** $-11\frac{4}{5}$ **(c)** $-9\frac{3}{7}$

Evaluate the following. Write all answers in simplest form. (If the answer is a mixed number/improper fraction, then write the answer as a <u>mixed number</u>.)

21. (a) $\frac{2}{7}+\frac{1}{7}$ **(b)** $\frac{8}{11}+\frac{4}{11}-\frac{3}{11}$

22. (a) $\frac{3}{5}-\frac{1}{5}$ **(b)** $\frac{4}{9}+\frac{5}{9}-\frac{2}{9}$

23. (a) $8\frac{4}{5}-2\frac{1}{5}$ **(b)** $\frac{7}{3}-\frac{23}{3}$

24. (a) $\frac{3}{5}-\frac{21}{5}$ **(b)** $7\frac{6}{11}+5\frac{2}{11}$

25. (a) $5\frac{3}{4}-2\frac{1}{4}$ **(b)** $6\frac{3}{5}+7\frac{4}{5}$

26. (a) $9\frac{5}{7}-2\frac{3}{7}$ **(b)** $4-\frac{5}{11}$

27. (a) $7-\frac{2}{3}$ **(b)** $7\frac{3}{10}-3\frac{9}{10}$

28. (a) $6\frac{7}{12}+2\frac{11}{12}$ **(b)** $8\frac{1}{6}-2\frac{5}{6}$

Evaluate the following. Write all answers in simplest form. (If the answer is a mixed number/improper fraction, then write the answer as a <u>mixed number</u>.)

29. (a) $\frac{1}{4}+\frac{1}{2}$ **(b)** $\frac{1}{3}-\frac{1}{7}$

30. (a) $\frac{1}{8}-\frac{1}{10}$ **(b)** $\frac{1}{6}+\frac{1}{5}$

31. (a) $\frac{1}{4}+\frac{1}{5}-\frac{1}{6}$ **(b)** $\frac{2}{7}+\frac{3}{5}$

32. (a) $\frac{1}{2}+\frac{1}{7}-\frac{1}{5}$ **(b)** $-\frac{4}{11}+\frac{3}{7}$

33. (a) $\frac{1}{35}-\frac{1}{10}$ **(b)** $\frac{3}{4}+\frac{5}{6}$

Exercise Set 1.3: Fractions

34. **(a)** $\frac{1}{6}-\frac{1}{24}$ **(b)** $\frac{8}{15}+\frac{7}{12}$

35. **(a)** $4\tfrac{3}{7}+5\tfrac{1}{6}$ **(b)** $7\tfrac{7}{10}-5\tfrac{1}{2}$

36. **(a)** $10\tfrac{5}{7}-3\tfrac{1}{4}$ **(b)** $6\tfrac{1}{12}+4\tfrac{3}{8}$

37. **(a)** $7\tfrac{3}{5}+8\tfrac{4}{7}$ **(b)** $5\tfrac{4}{9}-1\tfrac{2}{3}$

38. **(a)** $7\tfrac{1}{4}-3\tfrac{5}{6}$ **(b)** $2\tfrac{7}{8}+9\tfrac{13}{24}$

39. **(a)** $5\tfrac{2}{15}-2\tfrac{7}{12}$ **(b)** $9\tfrac{7}{16}+2\tfrac{5}{6}$

40. **(a)** $7\tfrac{9}{10}+6\tfrac{5}{8}$ **(b)** $11\tfrac{5}{14}-\tfrac{3}{4}$

Evaluate the following. Write all answers in simplest form. (If the answer is a mixed number/improper fraction, then write the answer as an <u>improper fraction</u>.)

41. **(a)** $\frac{2}{9}+\frac{3}{4}$ **(b)** $\frac{4}{15}-\frac{8}{9}$

42. **(a)** $\frac{7}{16}-\frac{9}{10}$ **(b)** $\frac{11}{14}+\frac{17}{35}$

43. **(a)** $5+\tfrac{1}{3}$ **(b)** $7-\tfrac{2}{3}$

44. **(a)** $9-\tfrac{2}{5}$ **(b)** $6+\tfrac{2}{7}$

Evaluate the following. Write all answers in simplest form. (If the answer is a mixed number/improper fraction, then write the answer as an <u>improper fraction</u>.)

45. **(a)** $5\cdot\frac{1}{3}$ **(b)** $21\cdot\frac{5}{6}$ **(c)** $-16\cdot\frac{5}{4}$

46. **(a)** $8\cdot\frac{3}{7}$ **(b)** $24\cdot\frac{1}{18}$ **(c)** $-25\cdot\frac{11}{10}$

47. **(a)** $\frac{1}{7}\cdot\frac{25}{11}$ **(b)** $-\frac{10}{21}\cdot\left(-\frac{9}{8}\right)$ **(c)** $\frac{3}{20}\cdot\frac{16}{15}$

48. **(a)** $\frac{36}{25}\cdot\left(-\frac{1}{8}\right)$ **(b)** $\frac{8}{19}\cdot\frac{7}{3}$ **(c)** $\frac{1}{14}\cdot\frac{42}{5}$

49. **(a)** $5\div\frac{1}{20}$ **(b)** $\frac{8}{3}\div 4$ **(c)** $\frac{-10}{\frac{7}{5}}$

50. **(a)** $\frac{3}{11}\div 6$ **(b)** $-20\div\left(-\frac{8}{5}\right)$ **(c)** $22\div\frac{4}{9}$

51. **(a)** $\frac{12}{35}\div\frac{18}{7}$ **(b)** $\frac{-\frac{3}{5}}{-\frac{5}{9}}$ **(c)** $\frac{15}{16}\div\frac{5}{24}$

52. **(a)** $\frac{\frac{1}{4}}{\frac{5}{16}}$ **(b)** $-\frac{36}{5}\div\frac{9}{50}$ **(c)** $\frac{49}{24}\div\frac{35}{32}$

Evaluate the following. Write all answers in simplest form. (If the answer is a mixed number/improper fraction, then write the answer as a <u>mixed number</u>.)

53. **(a)** $\left(8\tfrac{4}{5}\right)\cdot\left(\tfrac{10}{77}\right)$ **(b)** $\left(1\tfrac{7}{8}\right)\cdot\left(\tfrac{9}{10}\right)$

54. **(a)** $\left(2\tfrac{2}{9}\right)\cdot\left(\tfrac{3}{4}\right)$ **(b)** $\left(3\tfrac{7}{16}\right)\cdot\left(\tfrac{4}{5}\right)$

55. **(a)** $\left(2\tfrac{1}{3}\right)\cdot\left(5\tfrac{1}{7}\right)$ **(b)** $\left(6\tfrac{3}{5}\right)\cdot\left(2\tfrac{3}{11}\right)$

56. **(a)** $\left(3\tfrac{1}{7}\right)\cdot\left(5\tfrac{1}{4}\right)$ **(b)** $\left(5\tfrac{3}{5}\right)\cdot\left(-2\tfrac{11}{12}\right)$

57. **(a)** $\left(5\tfrac{5}{8}\right)\div\left(2\tfrac{1}{4}\right)$ **(b)** $\left(-11\tfrac{1}{9}\right)\div\left(1\tfrac{17}{18}\right)$

58. **(a)** $\left(4\tfrac{4}{5}\right)\div\left(1\tfrac{5}{7}\right)$ **(b)** $\left(2\tfrac{5}{11}\right)\div\left(2\tfrac{1}{22}\right)$

Section 1.4: Exponents and Radicals

- Evaluating Exponential Expressions
- Square Roots

Evaluating Exponential Expressions

Let n be a natural number. Then the exponential expression x^n is defined by

$x^n = \underbrace{x \cdot x \cdot x \cdot \cdots \cdot x}_{n \text{ factors}}$. The expression is read as "x to the n^{th} power."

The number x is called the **base** and n is called the **exponent**. The exponent n gives the number of factors that the base x is used in a product.

Two Rules for Exponential Expressions:

Let n and m be natural numbers.

(1) Product rule: $x^m x^n = x^{m+n}$

If two exponential expressions with the same base are multiplied, keep the common base and add the exponents.

(2) Power rule: $\left(x^m\right)^n = x^{mn}$

If an exponential expression is raised to a power, keep the base and multiply the exponents.

Example:

Identify the base and exponent for each of the following exponential expressions. Then evaluate each expression.

(a) 9^2

(b) -9^2

Solution:

(a) The base is 9 and the exponent is 2. This tells us that the base of 9 is used as a factor 2 times in a product.

Thus,

$$\begin{aligned} 9^2 &= 9 \cdot 9 \\ &= 81. \end{aligned}$$

(b) The base is 9 and the exponent is 2. This tells us that the base of 9 is used as a factor 2 times in a product.

Thus,

$$\begin{aligned} -9^2 &= -9 \cdot 9 \\ &= -81. \end{aligned}$$

Example:

Write each of the following as a base and exponent. Do not evaluate.

(a) $3^4 \cdot 3^6$

(b) $\left(3^5\right)^2$

Solution:

(a) Use the product rule for exponential expressions. Keep the common base and find the sum of the exponents.

$$\begin{aligned} 3^4 \cdot 3^6 &= 3^{4+6} \\ &= 3^{10} \end{aligned}$$

(b) Use the power rule for exponential expressions. Keep the base and find the product of the exponents.

$$\begin{aligned} \left(3^5\right)^2 &= 3^{5(2)} \\ &= 3^{10} \end{aligned}$$

Additional Properties for Exponential Expressions:

Two Definitions:

Let n be a natural number. We have the following definitions:

(1) If $x \neq 0$, then $x^0 = 1$.

(2) (Negative exponents) If $x \neq 0$, then $x^{-n} = \dfrac{1}{x^n}$ and $x^n = \dfrac{1}{x^{-n}}$.

Quotient Rule for Exponential Expressions:

Quotient Rule: If m and n are natural numbers and $x \neq 0$, then $\dfrac{x^m}{x^n} = x^{m-n}$.

If two exponential expressions with the same base are divided, keep the common base and subtract the exponents.

From the definition of negative exponents, it follows that the Product Rule, the Power Rule, and the Quotient Rule hold for all exponents that are integers.

Exponential Expressions with Bases of Products:

If n is an integer, $(xy)^n = x^n y^n$.

Exponential Expressions with Bases of Fractions:

If n is an integer, $x \neq 0$, and $y \neq 0$, then $\left(\dfrac{x}{y}\right)^{-n} = \left(\dfrac{y}{x}\right)^n$.

Note that $\left(\dfrac{y}{x}\right)^n = \dfrac{y^n}{x^n}$.

Example:
Evaluate each of the following:

(a) 2^{-3} (b) $\dfrac{5^9}{5^6}$ (c) $\left(\dfrac{2}{5}\right)^{-3}$

Solution:

(a) If n is an integer and $x \neq 0$, then $x^{-n} = \dfrac{1}{x^n}$ and $\dfrac{1}{x^{-n}} = x^n$.

Using this rule, rewrite the given expression so that it contains a positive exponent. Then evaluate the resulting exponential expression.

$$\begin{aligned} 2^{-3} &= \frac{1}{2^3} \\ &= \frac{1}{2 \cdot 2 \cdot 2} \\ &= \frac{1}{8} \end{aligned}$$

(b) If m and n are integers and $x \neq 0$, then $\dfrac{x^m}{x^n} = x^{m-n}$.

Using this rule, rewrite the given expression and then evaluate.

$$\begin{aligned} \frac{5^9}{5^6} &= 5^{9-6} \\ &= 5^3 \\ &= 5 \cdot 5 \cdot 5 \\ &= 125 \end{aligned}$$

(c) If n is an integer, $x \neq 0$, and $y \neq 0$, then $\left(\dfrac{x}{y}\right)^{-n} = \left(\dfrac{y}{x}\right)^{n}$.

Using this rule, rewrite the given expression so that it contains a positive exponent. Then evaluate the resulting exponential expression.

$$\begin{aligned} \left(\frac{2}{5}\right)^{-3} &= \left(\frac{5}{2}\right)^{3} \\ &= \frac{5}{2} \cdot \frac{5}{2} \cdot \frac{5}{2} \\ &= \frac{5 \cdot 5 \cdot 5}{2 \cdot 2 \cdot 2} \\ &= \frac{125}{8} \end{aligned}$$

Additional Example 1:

Identify the base and exponent for each of the following exponential expressions. Then evaluate each expression.

(a) 6^3

(b) -6^2

(c) $\left(\frac{1}{6}\right)^3$

(d) $\left(-\frac{1}{6}\right)^2$

Solution:

(a) The base is 6 and the exponent is 3. This tells us that the base of 6 is used as a factor 3 times in a product.

Thus,

$$\begin{aligned} 6^3 &= 6 \cdot 6 \cdot 6 \\ &= 36 \cdot 6 \\ &= 216. \end{aligned}$$

(b) The base is 6 and the exponent is 2. This tells us that the base of 6 is used as a factor 2 times in a product.

Thus,

$$\begin{aligned} -6^2 &= -6 \cdot 6 \\ &= -36. \end{aligned}$$

(c) The base is $\frac{1}{6}$ and the exponent is 3. This tells us that the base of $\frac{1}{6}$ is used as a factor 3 times in a product.

Thus,

$$\begin{aligned} \left(\frac{1}{6}\right)^3 &= \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \\ &= \frac{1}{36} \cdot \frac{1}{6} \\ &= \frac{1}{216}. \end{aligned}$$

(d) The base is $-\frac{1}{6}$ and the exponent is 2. This tells us that the base of $-\frac{1}{6}$ is used as a factor 2 times in a product.

Thus,

$$\begin{aligned}\left(-\frac{1}{6}\right)^2 &= \left(-\frac{1}{6}\right)\left(-\frac{1}{6}\right)\\ &= \frac{1}{36}.\end{aligned}$$

Additional Example 2:

Write each of the following as a base and exponent. Do not evaluate.

(a) $4^3 \cdot 4^5$

(b) $4^{-2} \cdot 4^7$

(c) $\frac{4^8}{4^3}$

(d) $\left(4^2\right)^6$

Solution:

(a) Use the product rule for exponential expressions. Keep the common base and find the sum of the exponents.

$$\begin{aligned}4^3 \cdot 4^5 &= 4^{3+5}\\ &= 4^8\end{aligned}$$

(b) Use the product rule for exponential expressions. Keep the common base and find the sum of the exponents.

$$\begin{aligned}4^{-2} \cdot 4^7 &= 4^{-2+7}\\ &= 4^5\end{aligned}$$

(c) Use the quotient rule for exponential expressions. Keep the common base and subtract the exponents.

$$\begin{aligned}\frac{4^8}{4^3} &= 4^{8-3}\\ &= 4^5\end{aligned}$$

(d) Use the power rule for exponential expressions. Keep the base and find the product of the exponents.

$$\left(4^2\right)^6 = 4^{2(6)}$$
$$= 4^{12}$$

Additional Example 3:

Rewrite each expression so that it contains positive exponents rather than negative exponents, and then evaluate the expression.

(a) 8^{-2}

(b) $\dfrac{1}{7^{-3}}$

(c) $\left(\dfrac{4}{9}\right)^{-3}$

(d) $\left(-\dfrac{2}{3}\right)^{-2}$

Solution:

(a) If n is an integer and $x \neq 0$, then $x^{-n} = \dfrac{1}{x^n}$ and $\dfrac{1}{x^{-n}} = x^n$.

Using this rule, rewrite the given expression so that it contains a positive exponent. Then evaluate the resulting exponential expression.

$$8^{-2} = \frac{1}{8^2}$$
$$= \frac{1}{8 \cdot 8}$$
$$= \frac{1}{64}$$

(b) If n is an integer and $x \neq 0$, then $x^{-n} = \dfrac{1}{x^n}$ and $\dfrac{1}{x^{-n}} = x^n$.

Using this rule, rewrite the given expression so that it contains a positive exponent. Then evaluate the resulting exponential expression.

$$\begin{aligned}\frac{1}{7^{-3}} &= 7^3\\ &= 7\cdot 7\cdot 7\\ &= 343\end{aligned}$$

(c) If n is an integer, $x \neq 0$, and $y \neq 0$, then $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^{n}$.

Using this rule, rewrite the given expression so that it contains a positive exponent. Then evaluate the resulting exponential expression.

$$\begin{aligned}\left(\frac{4}{9}\right)^{-3} &= \left(\frac{9}{4}\right)^{3}\\ &= \frac{9}{4}\cdot\frac{9}{4}\cdot\frac{9}{4}\\ &= \frac{729}{64}\end{aligned}$$

(d) If n is an integer, $x \neq 0$, and $y \neq 0$, then $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^{n}$.

Using this rule, rewrite the given expression so that it contains a positive exponent. Then evaluate the resulting exponential expression.

$$\begin{aligned}\left(-\frac{2}{3}\right)^{-2} &= \left(-\frac{3}{2}\right)^{2}\\ &= \left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right)\\ &= \frac{9}{4}\end{aligned}$$

Square Roots

Definitions:

A number y is called a square root of a number x provided that $y^2 = x$. We see that -9 and 9 are square roots of 81 since $(-9)^2 = 9^2 = 81$. In general, if $x > 0$, then x has two square roots, one is negative and one is positive.

The prinicipal square root of x is the positive square root of x and is denoted by $\sqrt{x}$. The expression $\sqrt{x}$ is an example of a radical expression and is read "the square root of x."

Also, the principal square root of 0 is 0: $\sqrt{0} = 0$. Moreover, the square root of a negative number is not a real number.

In the example above, we have $\sqrt{81} = 9$ and $-\sqrt{81} = -9$.
In the expression $\sqrt{x}$, the symbol $\sqrt{\ }$ is called a radical sign and x is called the radicand.

Two Rules for Square Roots:

(1) Product Rule: If $x \geq 0$ and $y \geq 0$, then $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$.

(2) Quotient Rule: If $x \geq 0$ and $y > 0$, then $\sqrt{\dfrac{x}{y}} = \dfrac{\sqrt{x}}{\sqrt{y}}$.

Writing Radical Expressions in Simplest Radical Form:

The Product Rule and Quotient Rule for square roots can be used to write radical expressions in simplest radical form.

A square root is not in simplest radical form if the radicand contains a perfect square factor.

Examples of perfect integer squares are $36 = 6^2$, $64 = 8^2$, and $121 = 11^2$. (Their square roots are integers.) For example, $\sqrt{24}$ is not in simplest radical form since 24 contains a perfect square factor of 4.

Example:

Write $\sqrt{24}$ in simplest radical form.

Solution:

First, factor 24 into the product of two numbers so that one of the factors is the largest perfect square factor that divides 24: $\sqrt{24} = \sqrt{4 \cdot 6}$

Next, use the product rule for square roots: $\sqrt{4 \cdot 6} = \sqrt{4} \cdot \sqrt{6}$

Thus,

$$\begin{aligned}\sqrt{24} &= \sqrt{4 \cdot 6}\\ &= \sqrt{4} \cdot \sqrt{6}\\ &= 2\sqrt{6}. \qquad (\sqrt{4} = 2 \text{ since } 2^2 = 4 \text{ and } 2 \text{ is positive.})\end{aligned}$$

A square root is not in simplest radical form if the radicand contains a fraction. In addition, a radical expression is not in simplest radical form if there is a radical in the denominator.

For example, $\dfrac{1}{\sqrt{3}}$ is not in simplest radical form. In this case, we can *rationalize* the denominator by multiplying both numerator and denominator by $\sqrt{3}$ and use the definition of a square root to note that $\sqrt{3} \cdot \sqrt{3} = 3$.

In general, if $x \geq 0$, then $\sqrt{x} \cdot \sqrt{x} = x$.

Example:

Write $\sqrt{\dfrac{64}{5}}$ in simplest radical form.

Solution:

First, use the quotient property of square roots: $\sqrt{\dfrac{64}{5}} = \dfrac{\sqrt{64}}{\sqrt{5}}$

Then

$$\begin{aligned}
\sqrt{\frac{64}{5}} &= \frac{\sqrt{64}}{\sqrt{5}} \\
&= \frac{8}{\sqrt{5}} && \text{Simplify in the numerator.} \\
&= \frac{8\cdot\sqrt{5}}{\sqrt{5}\cdot\sqrt{5}} && \text{Multiply numerator and denominator by } \sqrt{5}. \\
&= \frac{8\sqrt{5}}{5}. && \text{Simplify. (Note that if } x \ge 0, \text{ then } \sqrt{x}\cdot\sqrt{x} = x.)
\end{aligned}$$

Exponential Form:

By the following definition, a square root can be written in exponential form.

$$\text{If } x \ge 0, \text{ then } x^{\frac{1}{2}} = \sqrt{x}.$$

For example, $36^{\frac{1}{2}} = \sqrt{36} = 6$.

Additional Example 1:

Write each expression in simplest radical form.

(a) $\sqrt{49}$

(b) $(100)^{\frac{1}{2}}$

(c) $\sqrt{28}$

(d) $\left(\dfrac{25}{81}\right)^{\frac{1}{2}}$

Solution:

(a) $\sqrt{49}$ is the principal square root of 49. Note that $7^2 = 49$ and 7 is positive.
Thus,

$$\sqrt{49} = 7.$$

(b) First, rewrite the expression in radical form: $(100)^{\frac{1}{2}} = \sqrt{100}$

$\sqrt{100}$ is the principal square root of 100. Note that $10^2 = 100$ and 10 is positive.

Thus,

$$\begin{aligned}(100)^{\frac{1}{2}} &= \sqrt{100}\\ &= 10.\end{aligned}$$

(c) First, factor 28 into the product of two numbers so that one of the factors is the largest perfect square that divides 28: $\sqrt{28} = \sqrt{4 \cdot 7}$

Next, use the product rule for square roots: $\sqrt{4 \cdot 7} = \sqrt{4} \cdot \sqrt{7}$

Thus,

$$\begin{aligned}\sqrt{28} &= \sqrt{4 \cdot 7}\\ &= \sqrt{4} \cdot \sqrt{7}\\ &= 2\sqrt{7}.\end{aligned}$$

(d) First, rewrite the expression in radical form: $\left(\frac{25}{81}\right)^{\frac{1}{2}} = \sqrt{\frac{25}{81}}$

Next, use the quotient rule for square roots: $\sqrt{\frac{25}{81}} = \frac{\sqrt{25}}{\sqrt{81}}$

Thus,

$$\begin{aligned}\left(\frac{25}{81}\right)^{\frac{1}{2}} &= \sqrt{\frac{25}{81}}\\ &= \frac{\sqrt{25}}{\sqrt{81}}\\ &= \frac{5}{9}.\end{aligned}$$

Additional Example 2:

Write each expression in simplest radical form.

(a) $\frac{7}{\sqrt{3}}$

(b) $\sqrt{\frac{4}{5}}$

Solution:

(a) First, multiply numerator and denominator by $\sqrt{3}$. Then simplify by noting that if $x \geq 0$, $\sqrt{x} \cdot \sqrt{x} = x$.

Thus,

$$\frac{7}{\sqrt{3}} = \frac{7 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$$
$$= \frac{7\sqrt{3}}{3}.$$

(b) $\sqrt{\frac{4}{5}} = \frac{\sqrt{4}}{\sqrt{5}}$ Use the quotient rule for square roots.

$= \frac{2}{\sqrt{5}}$ Simplify in the numerator.

$= \frac{2 \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}}$ Multiply numerator and denominator by $\sqrt{5}$.

$= \frac{2\sqrt{5}}{5}$ Simplify. Note that if $x \geq 0$, then $\sqrt{x} \cdot \sqrt{x} = x$.

Additional Example 3:

Write each expression in simplest radical form.

(a) $-\frac{\sqrt{18}}{6}$

(b) $-\frac{4}{\sqrt{12}}$

Solution:

(a) First, factor 18 into the product of two numbers so that one of the factors is the largest perfect square that divides 18: $-\frac{\sqrt{18}}{6} = -\frac{\sqrt{9 \cdot 2}}{6}$

Then

$$-\frac{\sqrt{18}}{6} = -\frac{\sqrt{9 \cdot 2}}{6}$$

$= -\frac{\sqrt{9} \cdot \sqrt{2}}{6}$ Use the product rule for square roots.

$= -\frac{3\sqrt{2}}{6}$ Simplify in the numerator.

$$= -\frac{\overset{1}{\cancel{3}}\sqrt{2}}{2\cdot\underset{1}{\cancel{3}}}$$ Divide out common factors.

$$= -\frac{\sqrt{2}}{2}.$$ Simplify.

(b) First, factor 12 into the product of two numbers so that one of the factors is the largest perfect square that divides 12: $-\frac{4}{\sqrt{12}} = -\frac{4}{\sqrt{4\cdot 3}}$

Then

$$-\frac{4}{\sqrt{12}} = -\frac{4}{\sqrt{4\cdot 3}}$$

$$= -\frac{4}{\sqrt{4}\cdot\sqrt{3}}$$ Use the product rule for square roots.

$$= -\frac{4}{2\sqrt{3}}$$ Simplify in the denominator.

$$= -\frac{4\cdot\sqrt{3}}{2\sqrt{3}\cdot\sqrt{3}}$$ Multiply numerator and denominator by $\sqrt{3}$.

$$= -\frac{4\sqrt{3}}{2\cdot 3}$$ Simplify. (Note that if $x \geq 0$, $\sqrt{x}\cdot\sqrt{x} = x$.)

$$= -\frac{\overset{1}{\cancel{2}}\cdot 2\cdot\sqrt{3}}{\underset{1}{\cancel{2}}\cdot 3}$$ Divide out common factors.

$$= -\frac{2\sqrt{3}}{3}.$$ Simplify.

Exercise Set 1.4: Exponents and Radicals

Write each of the following products instead as a base and exponent. (For example, $6 \cdot 6 = 6^2$)

1. **(a)** $7 \cdot 7 \cdot 7$ **(b)** $10 \cdot 10$
(c) $8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8$ **(d)** $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

2. **(a)** $9 \cdot 9 \cdot 9$ **(b)** $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
(c) $5 \cdot 5 \cdot 5 \cdot 5$ **(d)** $17 \cdot 17$

Fill in the appropriate symbol from the set $\{<, >, =\}$.

3. -7^2 ______ 0

4. $(-9)^4$ ______ 0

5. $(-8)^6$ ______ 0

6. -8^6 ______ 0

7. -10^2 ______ $(-10)^2$

8. -10^3 ______ $(-10)^3$

Evaluate the following.

9. **(a)** 3^1 **(b)** 3^2 **(c)** 3^3
(d) -3^1 **(e)** -3^2 **(f)** -3^3
(g) $(-3)^1$ **(h)** $(-3)^2$ **(i)** $(-3)^3$
(j) 3^0 **(k)** -3^0 **(l)** $(-3)^0$
(m) 3^4 **(n)** -3^4 **(o)** $(-3)^4$

10. **(a)** 5^0 **(b)** $(-5)^0$ **(c)** -5^0
(d) 5^1 **(e)** $(-5)^1$ **(f)** -5^1
(g) 5^2 **(h)** $(-5)^2$ **(i)** -5^2
(j) 5^3 **(k)** $(-5)^3$ **(l)** -5^3
(m) 5^4 **(n)** $(-5)^4$ **(o)** -5^4

11. **(a)** $(0.5)^2$ **(b)** $\left(\frac{1}{5}\right)^2$ **(c)** $\left(-\frac{1}{9}\right)^2$

12. **(a)** $(0.03)^2$ **(b)** $\left(\frac{1}{3}\right)^4$ **(c)** $-\left(\frac{1}{12}\right)^2$

Write each of the following products instead as a base and exponent. (Do not evaluate; simply write the base and exponent.) No answers should contain negative exponents.

13. **(a)** $5^2 \cdot 5^6$ **(b)** $5^{-2} \cdot 5^6$

14. **(a)** $3^8 \cdot 3^5$ **(b)** $3^8 \cdot 3^{-5}$

15. **(a)** $\dfrac{6^9}{6^2}$ **(b)** $\dfrac{6^9}{6^{-2}}$

16. **(a)** $\dfrac{7^9}{7^5}$ **(b)** $\dfrac{7^9}{7^{-5}}$

17. **(a)** $\dfrac{4^7 \cdot 4^3}{4^8}$ **(b)** $\dfrac{4^{11} \cdot 4^{-3}}{4^8 \cdot 4^{-5}}$

18. **(a)** $\dfrac{8^{12}}{8^5 \cdot 8^4}$ **(b)** $\dfrac{8^{-4} \cdot 8^9}{8^4 \cdot 8^{-1}}$

19. **(a)** $\left(7^3\right)^6$ **(b)** $\left(\left(5^2\right)^4\right)^3$

20. **(a)** $\left(3^2\right)^4$ **(b)** $\left(\left(2^3\right)^5\right)^4$

Rewrite each expression so that it contains positive exponent(s) rather than negative exponent(s), and then evaluate the expression.

21. **(a)** 5^{-1} **(b)** 5^{-2} **(c)** 5^{-3}

22. **(a)** 3^{-1} **(b)** 3^{-2} **(c)** 3^{-3}

23. **(a)** 2^{-3} **(b)** 2^{-5}

24. **(a)** 7^{-2} **(b)** 10^{-4}

25. **(a)** $\left(\frac{1}{5}\right)^{-1}$ **(b)** $\left(\frac{2}{3}\right)^{-1}$

26. **(a)** $\left(\frac{1}{7}\right)^{-1}$ **(b)** $\left(\frac{6}{5}\right)^{-1}$

27. **(a)** -5^{-2} **(b)** $(-5)^{-2}$

28. **(a)** $(-8)^{-2}$ **(b)** -8^{-2}

Exercise Set 1.4: Exponents and Radicals

Evaluate the following.

29. (a) $\dfrac{-2^3}{2^8}$ **(b)** $\dfrac{2^{-2}}{2^{-6}}$

30. (a) $\dfrac{5^{-1}}{5^2}$ **(b)** $\dfrac{-5^{-1}}{5^{-3}}$

31. (a) $\left(\left(2^3\right)^0\right)^2$ **(b)** $\left(\left(2^{-3}\right)^{-1}\right)^{-2}$

32. (a) $\left(\left(3^{-1}\right)^{-2}\right)^2$ **(b)** $\left(\left(3^{-2}\right)^{-1}\right)^0$

Simplify the following. No answers should contain negative exponents.

33. (a) $\left(3x^3y^4z^{-2}\right)^3$ **(b)** $\left(3x^3y^4z^{-2}\right)^{-3}$

34. (a) $\left(6x^{-5}y^3z^4\right)^2$ **(b)** $\left(6x^{-5}y^3z^4\right)^{-2}$

35. $\dfrac{\left(x^{-3}x^{-4}x^{-6}\right)^{-1}}{x^{-7}}$

36. $\dfrac{x^2x^{-3}x^{-4}}{\left(x^4x^{-1}\right)^{-1}}$

37. $\dfrac{k^3m^2}{k^{-1}\left(m^{-2}\right)^{-3}}$

38. $\dfrac{a^{-4}\left(b^{-3}\right)^4c^7}{a^3b^{-5}c^9}$

39. $\dfrac{-2a^4b^{-3}}{4^{-1}a^0b^{-9}}$

40. $\dfrac{5d^{-7}e^0}{-3^{-1}d^{-2}e^4}$

41. $\dfrac{a^0+b^0}{(a+b)^0}$

42. $\dfrac{c^0-d^0}{(c-d)^0}$

43. $\left(-\dfrac{3a^3b^6}{2a^{-3}b^2}\right)^{-2}$

44. $\left(\dfrac{5a^{-2}b^2}{-6a^2b}\right)^{-3}$

Write each of the following expressions in simplest radical form or as a rational number (if appropriate). If it is already in simplest radical form, say so.

45. (a) $(36)^{\frac{1}{2}}$ **(b)** $\sqrt{7}$ **(c)** $\sqrt{18}$

46. (a) $\sqrt{20}$ **(b)** $\sqrt{49}$ **(c)** $(32)^{\frac{1}{2}}$

47. (a) $(50)^{\frac{1}{2}}$ **(b)** $\sqrt{14}$ **(c)** $\sqrt{\dfrac{81}{16}}$

48. (a) $(19)^{\frac{1}{2}}$ **(b)** $\sqrt{\dfrac{16}{49}}$ **(c)** $\sqrt{55}$

49. (a) $\sqrt{28}$ **(b)** $\sqrt{72}$ **(c)** $(27)^{\frac{1}{2}}$

50. (a) $(45)^{\frac{1}{2}}$ **(b)** $\sqrt{48}$ **(c)** $\sqrt{500}$

51. (a) $\sqrt{54}$ **(b)** $(80)^{\frac{1}{2}}$ **(c)** $\sqrt{60}$

52. (a) $\sqrt{120}$ **(b)** $\sqrt{180}$ **(c)** $(84)^{\frac{1}{2}}$

53. (a) $\sqrt{\dfrac{1}{5}}$ **(b)** $\left(\dfrac{3}{4}\right)^{\frac{1}{2}}$ **(c)** $\sqrt{\dfrac{2}{7}}$

54. (a) $\sqrt{\dfrac{1}{3}}$ **(b)** $\sqrt{\dfrac{5}{9}}$ **(c)** $\left(\dfrac{2}{5}\right)^{\frac{1}{2}}$

55. (a) $\dfrac{7}{\sqrt{4}}$ **(b)** $\dfrac{1}{\sqrt{10}}$ **(c)** $\dfrac{3}{\sqrt{11}}$

56. (a) $\dfrac{1}{\sqrt{6}}$ **(b)** $\dfrac{11}{\sqrt{9}}$ **(c)** $\dfrac{5}{\sqrt{2}}$

57. (a) $\sqrt{3^5}$ **(b)** $\sqrt{x^4y^5z^7}$

58. (a) $\sqrt{2^7}$ **(b)** $\sqrt{a^2b^9c^5}$

Evaluate the following.

59. (a) $\left(\sqrt{5}\right)^2$ **(b)** $\left(\sqrt{6}\right)^4$ **(c)** $\left(\sqrt{2}\right)^6$

60. (a) $\left(\sqrt{7}\right)^2$ **(b)** $\left(\sqrt{3}\right)^4$ **(c)** $\left(\sqrt{10}\right)^6$

We can evaluate radicals other than square roots. With square roots, we know, for example, that $\sqrt{49}=7$, since $7^2=49$, and $\sqrt{-49}$ is not a real number. (There is no real number that when squared gives a value of -49, since 7^2 and $(-7)^2$ give a value of 49, not -49. The answer is a complex number, which will not be addressed in this course.) In a similar fashion, we can compute the following:

<u>Cube Roots</u>

$\sqrt[3]{125}=5$, since $5^3=125$.

$\sqrt[3]{-125}=-5$, since $(-5)^3=-125$.

<u>Fourth Roots</u>

$\sqrt[4]{10{,}000}=10$, since $10^4=10{,}000$.

$\sqrt[4]{-10{,}000}$ is not a real number.

<u>Fifth Roots</u>

$\sqrt[5]{32}=2$, since $2^5=32$.

$\sqrt[5]{-32}=-2$, since $(-2)^5=-32$.

<u>Sixth Roots</u>

$\sqrt[6]{\frac{1}{64}}=\frac{1}{2}$, since $\left(\frac{1}{2}\right)^6=64$.

$\sqrt[6]{-\frac{1}{64}}$ is not a real number.

Evaluate the following. If the answer is not a real number, state "Not a real number."

61. (a) $\sqrt{64}$ **(b)** $\sqrt{-64}$ **(c)** $-\sqrt{64}$

62. (a) $\sqrt{25}$ **(b)** $\sqrt{-25}$ **(c)** $-\sqrt{25}$

63. (a) $\sqrt[3]{8}$ **(b)** $\sqrt[3]{-8}$ **(c)** $-\sqrt[3]{8}$

64. (a) $\sqrt[4]{81}$ **(b)** $\sqrt[4]{-81}$ **(c)** $-\sqrt[4]{81}$

65. (a) $\sqrt[6]{1{,}000{,}000}$ **(b)** $\sqrt[6]{-1{,}000{,}000}$ **(c)** $-\sqrt[6]{1{,}000{,}000}$

66. (a) $\sqrt[5]{32}$ **(b)** $\sqrt[5]{-32}$ **(c)** $-\sqrt[5]{32}$

67. (a) $\sqrt[4]{\frac{1}{16}}$ **(b)** $\sqrt[4]{-\frac{1}{16}}$ **(c)** $-\sqrt[4]{\frac{1}{16}}$

68. (a) $\sqrt[3]{\frac{1}{27}}$ **(b)** $\sqrt[3]{-\frac{1}{27}}$ **(c)** $-\sqrt[3]{\frac{1}{27}}$

69. (a) $\sqrt[5]{\frac{1}{100{,}000}}$ **(b)** $\sqrt[5]{-\frac{1}{100{,}000}}$ **(c)** $-\sqrt[5]{\frac{1}{100{,}000}}$

70. (a) $\sqrt[6]{1}$ **(b)** $\sqrt[6]{-1}$ **(c)** $-\sqrt[6]{1}$

Section 1.5: Order of Operations

- Evaluating Expressions Using the Order of Operations

Evaluating Expressions Using the Order of Operations

In evaluating expressions involving more than one operation, we need a procedure that clarifies the order in which the operations are performed. To see why this is necessary, consider the expression $9+6\div 3$.

If the addition is performed before the division, the result is $15\div 3=5$. However, if the division is performed before the addition, the result is $9+2=11$.

We need to establish an order of operations to rule out the possiblity of getting two different results when evaluating expressions involving several operations.

Rules for the Order of Operations:

1) Operations that are within parentheses and other grouping symbols are performed first. These operations are performed in the order established in the following steps. If grouping symbols are nested, evaluate the expression within the innermost grouping symbol first and work outward.

2) Exponential expressions and roots are evaluated first.

3) Multiplication and division are performed next, moving left to right and performing these operations in the order that they occur.

4) Addition and subtraction are performed last, moving left to right and performing these operations in the order that they occur.

Upon removing all of the grouping symbols, repeat the steps 2 through 4 until the final result is obtained.

In evaluating the expression $9+6\div 3$, we follow the order of operations and perform the division first and then the addition.

Thus,

$9+6\div 3=9+2=11.$

Example:

Evaluate: $-13+4\cdot 5-35\div 7$

Solution:

Multiplication and division are performed before addition and subtraction. Working from left to right, peform the multiplication: $4\cdot 5$

$$\begin{aligned} -13+4\cdot 5-35\div 7 &= -13+20-35\div 7 \\ &= -13+20-5 && \text{Perform the division: } 35\div 7 \\ &= 7-5 && \text{Working left to right, perform the addition: } -13+20 \\ &= 2 && \text{Perform the subtraction.} \end{aligned}$$

Example:

Evaluate: $(15-9)(6+1)^2-20$

Solution:

In the order of operations, begin by performing the operations within parentheses (and other grouping symbols): $15-9$ and $6+1$

$$\begin{aligned} (15-9)(6+1)^2-20 &= 6\cdot 7^2-20 \\ &= 6\cdot 49-20 && \text{Evaluate the exponential expression: } 7^2 \\ &= 294-20 && \text{Peform the multiplication: } 6\cdot 49 \\ &= 274 && \text{Perform the subtraction.} \end{aligned}$$

Additional Example 1:

Evaluate: $9+2\cdot 6-8\div 2$

Solution:
In the order of operations, multiplication and division are performed before addition and subtraction.

$$9+2\cdot 6-8\div 2=9+12-8\div 2$$ Working from left to right, perform the multiplication: $2\cdot 6$

$$=9+12-4$$ Peform the division: $8\div 2$

$$=21-4$$ Working left to right, perform the addition: $9+12$

$$=17.$$ Peform the subtraction.

Additional Example 2:
Evaluate: $(10+12)\div 2+5(8-6)$

Solution:
In the order of operations, begin by performing the operations within parentheses (and other grouping symbols): $10+12$ and $8-6$

$$(10+12)\div 2+5(8-6)=22\div 2+5\cdot 2$$

$$=11+5\cdot 2$$ Working from left to right, perform the division: $22\div 2$

$$=11+10$$ Peform the multiplication: $5\cdot 2$

$$=21$$ Perform the addition.

Additional Example 3:
Evaluate: $13+(-8+11)(3+2)^3$

Solution:
In the order of operations, begin by performing the operations within parentheses (and other grouping symbols): $-8+11$ and $3+2$

$$13+(-8+11)(3+2)^3=13+3\cdot 5^3$$

$$=13+3\cdot 125$$ Evaluate the exponential expression: 5^3

$$=13+375$$ Peform the multiplication: $3\cdot 125$

$$=388$$ Perform the addition.

Additional Example 4:

Evaluate: $\dfrac{7\sqrt{81}-5^2}{14+4\sqrt{36}}$

Solution:

Perform the operations in the numerator and the denominator separately. Begin by evaluating the powers and roots: $\sqrt{81}$, 5^2, and $\sqrt{36}$

$$\begin{aligned}
\frac{7\sqrt{81}-5^2}{14+4\sqrt{36}} &= \frac{7\cdot 9-25}{14+4\cdot 6} \\
&= \frac{63-25}{14+24} && \text{Peform the multiplication in the numerator: } 7\cdot 9 \\
& && \text{Perform the multiplication in the denominator: } 4\cdot 6 \\
&= \frac{38}{38} && \text{Peform the subtraction in the numerator: } 63-25 \\
& && \text{Perform the addition in the denominator: } 14+24 \\
&= 1 && \text{Simplify.}
\end{aligned}$$

Additional Example 5:

Evaluate the expression $3|x+y|-20\div z$ for $x=16$, $y=2$, and $z=5$.

Solution:

Substitute 16 for x, 2 for y, and 5 for z in the given expression:

$$3|x+y|-20\div z = 3|16+2|-20\div 5$$

Absolute value symbols serve as grouping symbols. Begin by performing the operation within the absolute value symbol: $16+2$

Then

$$\begin{aligned}
3|x+y|-20\div z &= 3|16+2|-20\div 5 \\
&= 3|18|-20\div 5 \\
&= 3\cdot 18-20\div 5 && \text{Evaluate the absolute value of 18.} \\
&= 54-20\div 5 && \text{Working left to right, perform the multiplication: } 3\cdot 18 \\
&= 54-4 && \text{Peform the division: } 20\div 5 \\
&= 50 && \text{Peform the subtraction.}
\end{aligned}$$

Exercise Set 1.5: Order of Operations

Answer the following.

1. In the abbreviation PEMDAS used for order of operations,

   (a) State what each letter stands for:
   P: ____________________
   E: ____________________
   M: ____________________
   D: ____________________
   A: ____________________
   S: ____________________

   (b) If choosing between multiplication and division, which operation should come first? (Circle the correct answer.)

   Multiplication
   Division
   Whichever appears first

   (c) If choosing between addition and subtraction, which operation should come first? (Circle the correct answer.)

   Addition
   Subtraction
   Whichever appears first

2. When performing order of operations, which of the following are to be viewed as if they were enclosed in parentheses? (Circle all that apply.)

   Absolute value bars
   Radical symbols
   Fraction bars

Evaluate the following.

3. **(a)** $3+4\cdot 5$ **(b)** $(3+4)\cdot 5$
   (c) $3-4\cdot 5$ **(d)** $(3-4)\cdot 5$
   (e) $3-4+5$ **(f)** $3-(4+5)$

4. **(a)** $10-6\cdot 7$ **(b)** $(10-6)\cdot 7$
   (c) $10+6(7)$ **(d)** $10(6+7)$
   (e) $7-10+6$ **(f)** $7-(10+6)$

5. **(a)** $|-3-7|$ **(b)** $|-7+3|$
   (c) $|-3|-7$ **(d)** $|-7|+3$

6. **(a)** $|-2+5|$ **(b)** $|-2|+5$
   (c) $|-2-5|$ **(d)** $|-2|-|5|$

7. **(a)** $-2-7+5$ **(b)** $-2-(7+5)$
   (c) $-2-(-7)+5$ **(d)** $-2-7(-5)$
   (e) $-2(7-(-5))$ **(f)** $2(7)-5+7$

8. **(a)** $-6-2+(-4)$ **(b)** $-6-\left(2+(-4)\right)$
   (c) $-6-2(-4)$ **(d)** $(-6-2)(-4)$
   (e) $2-(-6)+4$ **(f)** $2-4(-6+2)$

9. **(a)** $\frac{2}{5}-\frac{1}{3}+\frac{1}{4}$ **(b)** $\frac{2}{5}-\left(\frac{1}{3}+\frac{1}{4}\right)$
   (c) $\left(\frac{2}{5}-\frac{1}{3}\right)\frac{1}{4}-\frac{1}{4}$ **(d)** $\frac{2}{5}\left(-\frac{1}{3}+\frac{1}{4}\right)$

10. **(a)** $\frac{3}{2}\left(\frac{5}{6}-1\right)$ **(b)** $\frac{3}{2}\left(\frac{5}{6}\right)-1$
    (c) $\left(1-\frac{3}{2}\right)\cdot\frac{5}{6}$ **(d)** $1-\frac{3}{2}+\frac{5}{6}$

11. **(a)** $5(4-7)^2$ **(b)** $-1(-7)^2$
    (c) $5-1(4-7)$ **(d)** $-7+4(1-5)^2$
    (e) -5^2-1^2 **(f)** $(-5-1)^2$

12. **(a)** $-2-3^2$ **(b)** $-2^3(-3)^2$
    (c) $-2+3(1+4)$ **(d)** $(-2+3)(1+4)^3$
    (e) 2^2+3^2 **(f)** $(2+3)^2$

13. **(a)** $20\div 2(10)$ **(b)** $20\div(2\cdot 10)$
    (c) $-20\cdot 10\div(-2)\cdot 10\cdot 5$

14. **(a)** $24\div 4(-2)$ **(b)** $(24\div 4)-2$
    (c) $24(-2)\div 4\div 2(-2)$

15. **(a)** $10^2\div 5\cdot 2$ **(b)** $(10\div 5\cdot 2)^2$
    (c) $2\left(10+(2-5)^2\right)-5$

16. **(a)** $(3+9)\div 3\cdot 4$ **(b)** $3+(9\div 3)\cdot 4$
    (c) $3-\left(9-(3-4)^3\right)$

17. **(a)** $\left(3+\frac{1}{6}\right)^{-1}$ **(b)** $\left(3\cdot\frac{1}{6}\right)^{-1}$ **(c)** $3\left(\frac{1}{6}\right)^{-1}$

18. **(a)** $\left(5+\frac{2}{3}\right)^{-1}$ **(b)** $\left(5\cdot\frac{2}{3}\right)^{-1}$ **(c)** $5\left(\frac{2}{3}\right)^{-1}$

Exercise Set 1.5: Order of Operations

19. $7+\left(-4^{-1}+5^{-1}\right)$

20. $8\left(3^{-1}-7^{-1}\right)$

21. $7^{2}-5+2\left(3^{-4}\right)$

22. $3\left(2^{-3}\right)+3^{-2}+4$

23. $\frac{1}{2}\div\frac{1}{3}\cdot\left(-\frac{3}{4}\right)$

24. $-\frac{3}{5}\div\frac{3}{10}\cdot\frac{10}{3}$

25. $\frac{\sqrt{25}}{5+3\cdot 3}$

26. $\frac{\sqrt{16}}{3-2\sqrt{16}}$

27. $2-3\left(\sqrt{4}+1\right)$

28. $2-3\sqrt{4}+1$

29. $2-3\left(\sqrt{4+1}\right)$

30. $(2-3)\left(\sqrt{4}+1\right)$

31. $2-3\left(\sqrt{4}+1\right)^{2}$

32. $2-3\left(\sqrt{4+1}\right)^{2}$

33. $\frac{(3-7)-(7-3)}{12\div 2\cdot 3-3}$

34. $\frac{(2-4)^{3}(-1)^{5}-1}{5-12\div 6+3}$

35. $-\sqrt{81}-2\left(4+3^{2}(-2)\right)$

36. $\sqrt{64}-\left(-5^{2}+4\left(2^{3}\right)\right)$

37. $-4^{2}+\sqrt{121}-\left(5^{2}-4\cdot 3\right)$

38. $-\sqrt{144}+5^{2}-2\left(6^{2}\div 12\cdot 3\right)$

39. $\frac{\sqrt{49}\left(3-2^{2}\right)}{3\sqrt{49}}$

40. $\frac{3\sqrt{49}-2^{2}}{3\sqrt{49}}$

41. $\frac{\sqrt{9+16}\left(-1^{2}\right)}{\sqrt{9}+\sqrt{16}}$

42. $\frac{\sqrt{9}+\sqrt{16}\left(-1^{2}\right)}{\sqrt{9+16}}$

43. $\frac{\left(\left|-2-3^{2}\right|-5\right)^{2}}{2+8\div 2\cdot 4}$

44. $\frac{-2-\left|3^{2}-5\right|}{2+8\div(2\cdot 4)}$

45. $\frac{\left|5-3^{2}\right|+\sqrt{3^{2}+7}}{\sqrt{4}+2(2+1)^{2}}-\frac{\left|2^{3}-4^{2}\right|-1^{4}}{\sqrt{81}+\left(2\sqrt{3}\right)^{2}}$

46. $\frac{\left|5-\left(2-\sqrt{25}\right)^{2}\right|}{\sqrt{81}-\sqrt{16}\div 2^{2}}+\frac{\sqrt{2^{3}+2\div 2}-\left|3-3^{2}\right|}{1+3\sqrt{1+\sqrt{1+4\cdot 2}}}$

Exercise Set 1.5: Order of Operations

Evaluate the following expressions for the given values of the variables.

47. $P+\dfrac{r}{k}$ for $P=5$, $r=-1$, and $k=7$.

48. $\dfrac{x}{y}+\dfrac{y}{z}$ for $x=4$, $y=-3$, and $z=8$.

49. $\dfrac{-b+\sqrt{b^2}-8c}{c^2}$ for $b=4$ and $c=-2$.

50. $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ for $a=-1$, $b=3$, and $c=18$.

Section 1.6: Solving Linear Equations

- Linear Equations

Linear Equations

Rules for Solving Equations:

In solving an equation, we **isolate** the variable on one side of the equation by using the following rules:

(1) If $A = B$, then

$$A + C = B + C$$
and
$$A - C = B - C.$$

Adding or **subtracting** the same quantity to both sides of an equation produces an *equivalent* equation, an equation with the same solutions as the original equation.

(2) If in addtion, if $C \neq 0$, then

$$A \cdot C = B \cdot C$$
and
$$\frac{A}{C} = \frac{B}{C}.$$

Multiplying or **dividing** both sides of an equation by the same nonzero quantity produces an *equivalent* equation.

Linear Equations:

An equation that can be written in the form $ax + b = c$, where a, b, and c are real numbers and $a \neq 0$ is called a linear equation in the variable x.

Example:
Solve the equation $6x + 5 = 17$.

Solution:

$6x+5=17$	
$6x+5-5=17-5$	Subtract 5 from both sides of the equation.
$6x=12$	Simplify.
$\frac{\cancel{6}x}{\cancel{6}}=\frac{12}{6}$	Divide both sides of the equation by 6.
$x=2$	Simplify.

Example:

Solve the equation $2(x+5)=9x-11$.

Solution:

$2(x+5)=9x-11$	
$2x+10=9x-11$	Use the distributive property on the left-hand side (LHS).
$2x+10-9x=9x-11-9x$	Subtract 9x from both sides of the equation.
$-7x+10=-11$	Simplify.
$-7x+10-10=-11-10$	Subtract 10 from both sides of the equation.
$-7x=-21$	Simplify.
$\frac{\cancel{-7}x}{\cancel{-7}}=\frac{-21}{-7}$	Divide both sides of the equation by -7.
$x=3$	Simplify.

Additional Example 1:

Solve the equation $6x+3=4x+33$.

Solution:

$6x+3=4x+33$	
$6x+3-4x=4x+33-4x$	Subtract $4x$ from both sides of the equation.
$2x+3=33$	Simplify.
$2x+3-3=33-3$	Subtract 3 from both sides of the equation.
$2x=30$	Simplify.
$\frac{\cancel{2}x}{\cancel{2}}=\frac{30}{2}$	Divide both sides of the equation by 2.
$x=15$	Simplify.

Additional Example 2:

Solve the equation $2(x-3)+7=-4(x+1)+3$.

Solution:

$$2(x-3)+7=-4(x+1)+3$$

$2x-6+7=-4x-4+3$ Use the distributive property on both sides of the equation.

$2x+1=-4x-1$ Simplify.

$2x+1+4x=-4x-1+4x$ Add $4x$ to both sides of the equation.

$6x+1=-1$ Simplify.

$6x+1-1=-1-1$ Subtract 1 from both sides of the equation.

$6x=-2$ Simplify.

$\frac{\cancel{6}x}{\cancel{6}}=\frac{-2}{6}$ Divide both sides of the equation by 6.

$x=-\frac{1}{3}$ Simplify.

Additional Example 3:

Solve the equation $2x+\frac{x}{12}+\frac{x-3}{6}=x$.

Solution:

$$2x+\frac{x}{12}+\frac{x-3}{6}=x$$

$12(2x)+\frac{12x}{12}+\frac{12(x-3)}{6}=12x$ Multiply both sides of the equation by 12.

$24x+x+2(x-3)=12x$ Simplify.

$24x+x+2x-6=12x$ Use the distributive property on the LHS.

$27x-6=12x$ Combine like terms on the LHS.

$27x-6-12x=12x-12x$ Subtract $12x$ from both sides of the equation.

$15x-6=0$ Simplify.

$15x-6+6=0+6$ Add 6 to both sides of the equation.

$15x=6$ Simplify.

$\frac{\cancel{15}x}{\cancel{15}}=\frac{6}{15}$ Divide both sides by 15.

$x=\frac{6}{15}$ Simplify.

$x=\frac{2}{5}$ Simplify.

Exercise Set 1.6: Solving Linear Equations

Solve the following equations algebraically.

1. $x+5=12$
2. $x-8=9$
3. $x-4=-7$
4. $x+2=-8$
5. $6x=30$
6. $-4x=28$
7. $-6x=-10$
8. $8x=26$
9. $-3x+7=13$
10. $5x-11=6$
11. $2x+3=4x-7$
12. $5x+2=-4x-6$
13. $3(x+2)+9=-5(x-8)-3$
14. $-4(x+3)-5=2(x-4)+3$
15. $3(2-5x)=-4(7x-3)$
16. $7+2(3-8x)=4-6(1+5x)$
17. $\dfrac{x}{5}=-7$
18. $\dfrac{x}{3}=10$
19. $\dfrac{3}{2}x=9$
20. $\dfrac{4}{7}x=12$
21. $-\dfrac{5}{6}x=-3$
22. $-\dfrac{8}{9}x=-4$
23. $\frac{2}{5}x-1=7$
24. $-\frac{3}{4}x-7=2$
25. $\frac{5}{3}(x-7)=\frac{2}{5}x+1$
26. $\frac{4}{9}x-12=-\frac{1}{6}(x-12)-3$
27. $2+\dfrac{2x}{3}-\dfrac{x+5}{7}=3x$
28. $x+\dfrac{x+7}{8}+\dfrac{5x}{6}=\dfrac{-1}{12}$

Section 1.7: Interval Notation and Linear Inequalities

- Linear Inequalities

Linear Inequalities

An inequality in the variable x is **linear** if each term is a constant or a multiple of x.
The inequality will contain an inequality symbol:

$<$	is less than
$\leq$	is less than or equal to
$>$	is greater than
$\geq$	is greater than or equal to

To solve an inequality containing a variable, find all values of the variable that make the inequality true.

Rules for Solving Inequalities:

In solving inequalities, isolate the variable on one side of the inequality symbol by using the following rules.

1) $A \leq B$ is equivalent to $A + C \leq B + C$

Adding the same quantity to both sides of an inequality produces an equivalent inequality.

2) $A \leq B$ is equivalent to $A - C \leq B - C$

Subtracting the same quantity from both sides of an inequality produces an equivalent inequality.

3) If $C > 0$, then $A \leq B$ is equivalent to $CA \leq CB$

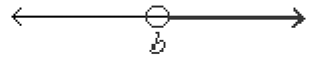
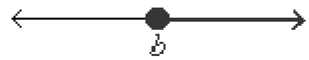
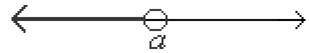
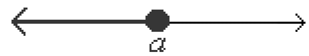
Multiplying both sides of an inequality by a positive number produces an equivalent inequality.

4) If $C < 0$, then $A \leq B$ is equivalent to $CA \geq CB$

Multiplying both sides of an inequality by a negative number reverses the direction of the inequality.

Interval Notation:

After solving a linear inequality, we graph the solution set on the real number line and write the solution in interval notation.

Interval Notation	Description of Interval	Graph
(b, ∞)	includes all real numbers x such that x is greater than b $(x > b)$	
$[b, \infty)$	includes all real numbers x such that x is greater than or equal to b $(x \geq b)$	
$(-\infty, a)$	includes all real numbers x such that x is less than a $(x < a)$	
$(-\infty, a]$	includes all real numbers x such that x is less than or equal to a $(x \leq a)$	
$(-\infty, \infty)$	includes all real numbers x	
(a, b)	includes all real numbers x such that x is between a and b $(a < x < b)$	
$[a, b)$	includes all real numbers x such that x is greater than or equal to a and x is less than b $(a \leq x < b)$	
$(a, b]$	includes all real numbers x such that x is greater than a and x is less than or equal to b $(a < x \leq b)$	
$[a, b]$	includes all real numbers x such that x is between and including a and b $(a \leq x \leq b)$	

Example:

Solve the inequality $2x - 5 > 3$. Give the answer in interval notation and graph the solution set.

Solution:

$$2x - 5 > 3$$

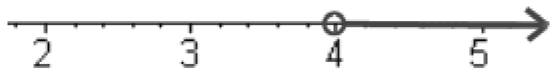
$$2x - 5 + 5 > 3 + 5$$ Add 5 to both sides.

$2x > 8$ Simplify.

$\dfrac{\cancel{2}x}{\cancel{2}} > \dfrac{8}{2}$ Divide both sides by 2.

$x > 4$ Simplify.

The inequality is true for all values of x that are greater than 4. In interval notation the solution is $(4,\infty)$. The graph of the solution is sketched below.



Example:

Solve the inequality $3x+11 \geq 6x+8$. Give the answer in interval notation and graph the solution set.

Solution:

$3x+11 \geq 6x+8$

$3x+11-6x \geq 6x+8-6x$ Subtract $6x$ from both sides.

$-3x+11 \geq 8$ Simplify.

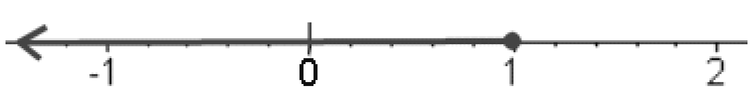
$-3x+11-11 \geq 8-11$ Subtract 11 from both sides.

$-3x \geq -3$ Simplify.

$\dfrac{\cancel{-3}x}{\cancel{-3}} \leq \dfrac{-3}{-3}$ Divide both sides by -3. Reverse the direction of the inequality.

$x \leq 1$ Simplify.

The inequality is true for all values of x that are less than or equal to 1. In interval notation the solution is $(-\infty,1]$. The bracket at 1 indicates that 1 is in the set. The graph of the solution is sketched below.



Example:

Solve the double inequality $-1 < 2x-5 \leq 3$. Give the answer in interval notation and graph the solution set.

Solution:

The solution set consist of all values of x that satisfy both inequalities:

$$-1 < 2x - 5 \text{ and } 2x - 5 \le 3$$

To solve $-1 < 2x - 5 \le 3$, use the rules for inequalities to isolate x in the middle.

$$-1 < 2x - 5 \le 3$$

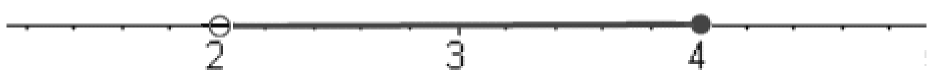
$$-1 + 5 < 2x - 5 + 5 \le 3 + 5 \quad \text{Add 5.}$$

$$4 < 2x \le 8 \quad \text{Simplify.}$$

$$\frac{4}{2} < \frac{\cancel{2}x}{\cancel{2}} \le \frac{8}{2} \quad \text{Divide by 2.}$$

$$2 < x \le 4 \quad \text{Simplify.}$$

The solution in interval notation is $(2, 4]$. The graph of the solution is shown below.



Additional Example 1:

Let $S = \{-4, 0, 3\}$. What elements of S, if any, satisfy the inequality $3x + 5 > x$?

Solution:

Substitute $x = -4$ into the inequality.

$$3x + 5 > x$$

$$3(-4) + 5 \overset{?}{>} -4$$

$$-12 + 5 \overset{?}{>} -4$$

$$-7 \not> -4$$

-4 does not satisfy the inequality.

Substitute $x = 0$ into the inequality.

$$3x + 5 > x$$

$$3(0) + 5 \overset{?}{>} 0$$

$$0+5\overset{?}{>}0$$
$$5\overset{\checkmark}{>}0$$

0 satisfies the inequality.

Substitute $x=3$ into the inequality.

$$3x+5>x$$
$$3(3)+5\overset{?}{>}3$$
$$9+5\overset{?}{>}3$$
$$14\overset{\checkmark}{>}3$$

3 satisfies the inequality.

Additional Example 2:

Solve the inequality $4x-9<11$. Write the solution in interval notation and graph the solution on the real number line.

Solution:

$4x-9<11$	
$4x-9+9<11+9$	Add 9 to both sides.
$4x<20$	Simplify.
$\frac{\cancel{4}x}{\cancel{4}}<\frac{20}{4}$	Divide both sides by 4.
$x<5$	Simplify.

The inequality is true for all values of x that are less than 5. In interval notation the solution is $(-\infty,5)$. The graph of the solution is sketched below.

Additional Example 3:

Solve the inequality $7-3x \ge 9$. Write the solution in interval notation and graph the solution on the real number line.

Solution:

$$7-3x \ge 9$$

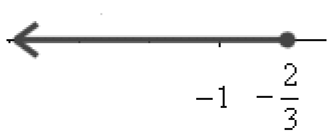
$7-3x-7 \ge 9-7$ Subtract 7 from both sides.

$-3x \ge 2$ Simplify.

$\frac{\cancel{-3}x}{\cancel{-3}} \le \frac{2}{-3}$ Divide both sides by -3. Reverse the direction of the inequality.

$x \le -\frac{2}{3}$ Simplify.

The inequality is true for all values of x that are less than or equal to $-\frac{2}{3}$. In interval notation the solution is $\left(-\infty,-\frac{2}{3}\right]$. The bracket at $-\frac{2}{3}$ indicates that $-\frac{2}{3}$ is in the set. The graph of the solution is sketched below.



Additional Example 4:

Solve the inequality $3(4x-1) \le 15x+12$. Write the solution in interval notation and graph the solution on the real number line.

Solution:

$$3(4x-1) \le 15x+12$$

$12x-3 \le 15x+12$ Use the distributive property on the LHS.

$12x-3-15x \le 15x+12-15x$ Subtract $15x$ from both sides.

$-3x-3 \le 12$ Simplify.

$-3x-3+3 \le 12+3$ Add 3 to both sides.

$-3x \le 15$ Simplify.

$\frac{\cancel{-3}x}{\cancel{-3}} \ge \frac{15}{-3}$ Divide both sides by -3. Reverse the direction of the inequality.

$x \ge -5$ Simplify.

The inequality is true for all values of x that are greater than or equal to -5. In interval notation the solution is $[-5,\infty)$. The bracket at -5 indicates that -5 is in the set. The graph of the solution is sketched below.

Additional Example 5:

Solve the double inequality $1 \le 7 + 2x \le 9$. Write the solution in interval notation and graph the solution on the real number line.

Solution:

$1 \le 7 + 2x \le 9$	Isolate x in the middle.
$1 - 7 \le 7 + 2x - 7 \le 9 - 7$	Subtract 7.
$-6 \le 2x \le 2$	Simplify.
$\frac{-6}{2} \le \frac{\cancel{2}x}{\cancel{2}} \le \frac{2}{2}$	Divide by 2.
$-3 \le x \le 1$	Simplify.

In interval notation the solution is $[-3,1]$. The brackets at -3 and 1 indicate that -3 and 1 are in the set. The graph of the solution is sketched below.

Additional Example 6:

Solve the double inequality $-2 < \frac{4-x}{5} \le \frac{3}{5}$. Write the solution in interval notation and graph the solution on the real number line.

Solution:

$-2 < \frac{4-x}{5} \le \frac{3}{5}$	Isolate x in the middle.
$(5)(-2) < \cancel{5} \cdot \frac{4-x}{\cancel{5}} \le \cancel{5} \cdot \frac{3}{\cancel{5}}$	Multiply by 5.

$-10 < 4 - x \le 3$	Simplify.
$-10 - 4 < 4 - x - 4 \le 3 - 4$	Subtract 4.
$-14 < -x \le -1$	Simplify.
$(-1)(-14) > (-1)(-x) \ge (-1)(-1)$	Multiply by -1. Reverse the direction of the inequalities.
$14 > x \ge 1$	Simplify.

$14 > x \ge 1$ can be written as $1 \le x < 14$.

In interval notation the solution is $[1,14)$. The bracket at 1 indicates that 1 is in the set. The graph of the solution is sketched below.

Additional Example 7:
A rental car company offers two options. Option 1 is $100 per week plus 10 cents for each mile. Option 2 is $125 per week plus 5 cents for each mile. How many miles per week would a person need to drive to make Option 2 more economical than Option 1?

Solution:
Let $x =$ the number of miles per week.
$100 + .10x =$ the weekly cost of Option 1.
$125 + .05x =$ the weekly cost of Option 2.

Write an inequality. We want Option 2 more economical than Option 1.

(The cost of Option 2) is less than (the cost of Option 1).

$125 + .05x < 100 + .10x$	
$125 + .05x - .10x < 100 + .10x - .10x$	Subtract $.10x$ from both sides.
$125 - .05x < 100$	Simplify.
$125 - .05x - 125 < 100 - 125$	Subtract 125 from both sides.
$-.05x < -25$	Simplify.
$\dfrac{\cancel{-.05}x}{\cancel{-.05}} > \dfrac{-25}{-.05}$	Divide both sides by $-.05$. Reverse the direction of the inequality.
$x > 500$	Simplify.

A person must drive more than 500 miles per week for Option 2 to be more economical than Option 1.

Exercise Set 1.7: Interval Notation and Linear Inequalities

For each of the following inequalities:
- **(a) Write the inequality algebraically.**
- **(b) Graph the inequality on the real number line.**
- **(c) Write the inequality in interval notation.**

1. x is greater than 5.

2. x is less than 4.

3. x is less than or equal to 3.

4. x is greater than or equal to 7.

5. x is not equal to 2.

6. x is not equal to -5.

7. x is less than -1.

8. x is greater than -6.

9. x is greater than or equal to -4.

10. x is less than or equal to -2.

11. x is not equal to -8.

12. x is not equal to 3.

13. x is not equal to 2 and x is not equal to 7.

14. x is not equal to -4 and x is not equal to 0.

Write each of the following inequalities in interval notation.

15. $x > 3$

16. $x \geq -5$

17. $x \leq -2$

18. $x < 7$

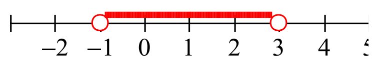
19. $3 < x \leq 5$

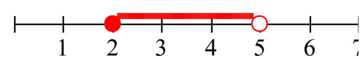
20. $-7 \leq x \leq 2$

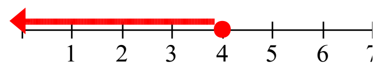
21. $x \neq -7$

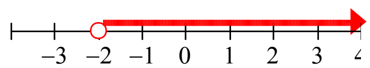
22. $x \neq 9$

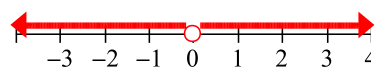
Write each of the following inequalities in interval notation.

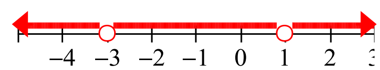
23. 

24. 

25. 

26. 

27. 

28. 

Given the set $S = \left\{2, 4, -3, \frac{1}{3}\right\}$**, use substitution to determine which of the elements of *S* satisfy each of the following inequalities.**

29. $2x + 5 \leq 10$

30. $4x - 2 > -14$

31. $-2x + 1 > -7$

32. $-3x + 1 \geq 0$

33. $x^2 + 1 < 10$

34. $\frac{1}{x} \leq \frac{2}{5}$

For each of the following inequalities:
- **(a) Solve the inequality.**
- **(b) Graph the solution on the real number line.**
- **(c) Write the solution in interval notation.**

35. $2x < 10$

36. $3x \geq 24$

37. $-5x \geq 30$

38. $-4x < 40$

39. $2x - 5 \geq -11$

40. $3x + 4 \leq -17$

41. $8 - 3x > 20$

42. $10 - x > 0$

43. $4x - 11 < 7x + 4$

44. $5 - 9x \leq 3x - 7$

45. $10x - 7 \geq 2x + 6$

46. $8 - 4x < 6 - 5x$

47. $5 - 8x \geq 4x + 1$

48. $x + 10 \geq 8x - 9$

49. $-3(4 + 5x) < -2(7 - x)$

50. $-4(3 - 2x) \leq -(x + 20)$

51. $\frac{5}{6} - \frac{1}{3}x \leq \frac{1}{2}(x + 5)$

52. $\frac{2}{5}\left(x + \frac{1}{2}\right) > -\frac{1}{3}(10 - x)$

53. $-10 \leq 3x + 2 < 8$

54. $-9 < 2x - 3 < 13$

55. $-4 \leq 3 - 7x \leq 17$

56. $-19 < 5 - 4x \leq -3$

57. $\frac{2}{3} < \frac{3x-10}{15} < \frac{4}{5}$

58. $\frac{3}{4} > \frac{5-2x}{6} > -\frac{5}{3}$

Which of the following inequalities can never be true?

59. **(a)** $5 \leq x \leq 9$
(b) $9 \leq x \leq 5$
(c) $-3 < x \leq 7$
(d) $-5 \geq x > -3$

60. **(a)** $3 > x > 5$
(b) $-8 \leq x < 1$
(c) $-2 < x \leq -8$
(d) $-7 \geq x > -10$

Answer the following.

61. You go on a business trip and rent a car for \$75 per week plus 23 cents per mile. Your employer will pay a maximum of \$100 per week for the rental. (Assume that the car rental company rounds to the nearest mile when computing the mileage cost.)

(a) Write an inequality that models this situation.
(b) What is the maximum number of miles that you can drive and still be reimbursed in full?

62. Joseph rents a catering hall to put on a dinner theatre. He pays \$225 to rent the space, and pays an additional \$7 per plate for each dinner served. He then sells tickets for \$15 each.

(a) Joseph wants to make a profit. Write an inequality that models this situation.
(b) How many tickets must he sell to make a profit?

63. A phone company has two long distance plans as follows:

Plan 1: \$4.95/month plus 5 cents/minute
Plan 2: \$2.75/month plus 7 cents/minute

How many minutes would you need to talk each month in order for Plan 1 to be more cost-effective than Plan 2?

64. Craig's goal in math class is to obtain a "B" for the semester. His semester average is based on four equally weighted tests. So far, he has obtained scores of 84, 89, and 90. What range of scores could he receive on the fourth exam and still obtain a "B" for the semester? (Note: The minimum cutoff for a "B" is 80 percent, and an average of 90 or above will be considered an "A".)

Section 1.8: Absolute Value and Equations

- Absolute Value

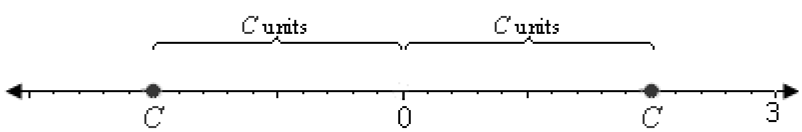
Absolute Value

Equations of the Form $|x| = C$:

To solve an equation involving **absolute value**, use the following property:

If C is positive, then $|x| = C$ is equivalent to $x = \pm C$.

Recall that the absolute value of a real number is its distance from 0 on the number line. Thus, to solve the equation $|x| = C$, we find the two numbers that are exactly C units from 0.



Special Cases for $|x| = C$:

Case 1: If C is negative, then the equation $|x| = C$ has no solution since absolute value cannot be negative.

Case 2: The solution of the equation $|x| = 0$ is $x = 0$.

Example:

Solve the equation $|4x + 7| = 9$.

Solution:

The equation $|4x+7|=9$ is equivalent to two equations: $4x+7=9$ or $4x+7=-9$.

$$\begin{array}{ccc} 4x+7=9 & \text{or} & 4x+7=-9 \\ 4x+7-7=9-7 & & 4x+7-7=-9-7 \\ 4x=2 & & 4x=-16 \\ \dfrac{\cancel{4}x}{\cancel{4}}=\dfrac{2}{4} & & \dfrac{\cancel{4}x}{\cancel{4}}=\dfrac{-16}{4} \\ x=\dfrac{2}{4} & & x=\dfrac{-16}{4} \\ x=\dfrac{1}{2} & & x=-4 \end{array}$$

Check the answers.

$x=\dfrac{1}{2}$ LHS $=\left|4\cdot\dfrac{1}{2}+7\right|=|2+7|=|9|=9=$ RHS ✔

$x=-4$ LHS $=|4(-4)+7|=|-16+7|=|-9|=9=$ RHS ✔

The solutions are $x=\dfrac{1}{2}$ and $x=-4$.

Example:

Solve the equation $3|x+5|+6=15$.

Solution:

$$\begin{aligned} 3|x+5|+6&=15 \\ 3|x+5|+6-6&=15-6 \\ 3|x+5|&=9 \\ \frac{\cancel{3}|x+5|}{\cancel{3}}&=\frac{9}{3} \\ |x+5|&=3 \end{aligned}$$

$|x+5|=3$ is equivalent to two equations: $x+5=3$ or $x+5=-3$.

$$x+5=3 \qquad \text{or} \qquad x+5=-3$$
$$x+5-5=3-5 \qquad\qquad x+5-5=-3-5$$
$$x=-2 \qquad\qquad x=-8$$

Check the answers.

$x=-2$ LHS $=3|-2+5|+6=3|3|+6=3\cdot 3+6=9+6=15=$ RHS ✔

$x=-8$ LHS $=3|-8+5|+6=3|-3|+6=3\cdot 3+6=9+6=15=$ RHS ✔

The solutions are $x=-2$ and $x=-8$.

Example:

Solve the equation $|2x+5|+6=1$.

Solution:

$$|2x+5|+6=1$$
$$|2x+5|+6-6=1-6$$
$$|2x+5|=-5$$

The equation has no solution since absolute value cannot be negative.

Example:

Solve the equation $|4x+1|=0$.

Solution:

$$|4x+1|=0$$
$$4x+1=0$$
$$4x+1-1=0-1$$
$$4x=-1$$
$$\frac{\cancel{4}x}{\cancel{4}}=\frac{-1}{4}$$
$$x=-\frac{1}{4}$$

Check the answer.

$x=-\frac{1}{4}$ LHS $=\left|4\left(-\frac{1}{4}\right)+1\right|=|-1+1|=|0|=0=$ RHS ✔

The solution is $x=-\frac{1}{4}$.

Example:

Solve the equation $|x-1|=|3x+2|$.

Solution:

The equation $|x-1|=|3x+2|$ is equivalent to two equations: $x-1=3x+2$ or $x-1=-(3x+2)$.

$$\begin{aligned}
x-1&=3x+2\\
x-1-3x&=3x+2-3x\\
-2x-1&=2\\
-2x-1+1&=2+1\\
-2x&=3\\
\frac{\cancel{-2}x}{\cancel{-2}}&=\frac{3}{-2}\\
x&=-\frac{3}{2}
\end{aligned}$$

or

$$\begin{aligned}
x-1&=-(3x+2)\\
x-1&=-3x-2\\
x-1+3x&=-3x-2+3x\\
4x-1&=-2\\
4x-1+1&=-2+1\\
4x&=-1\\
\frac{\cancel{4}x}{\cancel{4}}&=\frac{-1}{4}\\
x&=-\frac{1}{4}
\end{aligned}$$

Check the answers.

$x=-\frac{3}{2}$ LHS $=\left|-\frac{3}{2}-1\right|=\left|-\frac{5}{2}\right|=\frac{5}{2}$ ✔

RHS $=\left|3\left(-\frac{3}{2}\right)+2\right|=\left|-\frac{9}{2}+2\right|=\left|-\frac{9}{2}+\frac{4}{2}\right|=\left|-\frac{5}{2}\right|=\frac{5}{2}$

$x=-\frac{1}{4}$ LHS $=\left|-\frac{1}{4}-1\right|=\left|-\frac{5}{4}\right|=\frac{5}{4}$ ✔

$$\text{RHS} = \left|3\left(-\frac{1}{4}\right)+2\right| = \left|-\frac{3}{4}+2\right| = \left|-\frac{3}{4}+\frac{8}{4}\right| = \left|\frac{5}{4}\right| = \frac{5}{4}$$

The solutions are $x = -\frac{3}{2}$ and $x = -\frac{1}{4}$.

Additional Example 1:

Solve the equation $|3x+2| = 7$.

Solution:

$|3x+2| = 7$ is equivalent to two equations: $3x+2=7$ or $3x+2=-7$.

$$\begin{aligned} 3x+2&=7 \\ 3x+2-2&=7-2 \\ 3x&=5 \\ \frac{\cancel{3}x}{\cancel{3}}&=\frac{5}{3} \\ x&=\frac{5}{3} \end{aligned} \qquad \text{or} \qquad \begin{aligned} 3x+2&=-7 \\ 3x+2-2&=-7-2 \\ 3x&=-9 \\ \frac{\cancel{3}x}{\cancel{3}}&=\frac{-9}{3} \\ x&=-3 \end{aligned}$$

The solutions are $x = \frac{5}{3}$ and $x = -3$.

Additional Example 2:

Solve the equation $3|x+1|-4=5$.

Solution:

To solve the equation for x, begin by isolating the absolute value on one side of the equation.

$$\begin{aligned} 3|x+1|-4&=5 \\ 3|x+1|-4+4&=5+4 \\ 3|x+1|&=9 \\ \frac{\cancel{3}|x+1|}{\cancel{3}}&=\frac{9}{3} \\ |x+1|&=3 \end{aligned}$$

$|x+1|=3$ is equivalent to two equations: $x+1=3$ or $x+1=-3$.

$$\begin{aligned} x+1&=3 \\ x+1-1&=3-1 \\ x&=2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x+1&=-3 \\ x+1-1&=-3-1 \\ x&=-4 \end{aligned}$$

The solutions are $x=2$ and $x=-4$.

Additional Example 3:

Solve the equation $|2x+1|+3=1$.

Solution:

To solve the equation for x, begin by isolating the absolute value on one side of the equation.

$$\begin{aligned} |2x+1|+3&=1 \\ |2x+1|+3-3&=1-3 \\ |2x+1|&=-2 \end{aligned}$$

The equation has no solution since absolute value cannot be negative.

Additional Example 4:

Solve the equation $|3x-5|=0$.

Solution:

$|3x-5|=0$ is equivalent to the equation $3x-5=0$.

$$\begin{aligned} 3x-5&=0 \\ 3x-5+5&=0+5 \\ 3x&=5 \\ \frac{\cancel{3}x}{\cancel{3}}&=\frac{5}{3} \\ x&=\frac{5}{3} \end{aligned}$$

The solution is $x=\frac{5}{3}$.

Additional Example 5:

Solve the equation $|x-4|=|2x+3|$.

Solution:

$|x-4|=|2x+3|$ is equivalent to two equations: $x-4=2x+3$ or $x-4=-(2x+3)$.

$$
\begin{aligned}
x-4 &= 2x+3 \\
x-4-2x &= 2x+3-2x \\
-x-4 &= 3 \\
-x-4+4 &= 3+4 \\
-x &= 7 \\
(-1)(-x) &= (-1)(7) \\
x &= -7
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
x-4 &= -(2x+3) \\
x-4 &= -2x-3 \\
x-4+2x &= -2x-3+2x \\
3x-4 &= -3 \\
3x-4+4 &= -3+4 \\
3x &= 1 \\
\frac{\cancel{3}x}{\cancel{3}} &= \frac{1}{3} \\
x &= \frac{1}{3}
\end{aligned}
$$

The solutions are $x=-7$ and $x=\frac{1}{3}$.

Exercise Set 1.8: Absolute Value and Equations

Solve the following equations.

1. $|x| = 7$
2. $|x| = 5$
3. $|x| = -9$
4. $|x| = -10$
5. $|2x| = 12$
6. $|-3x| = 30$
7. $|x+4| = 5$
8. $|x-7| = 2$
9. $|x| + 4 = 5$
10. $|x| - 7 = 2$
11. $|3x-4| = 8$
12. $|5x+4| = 3$
13. $|3x| - 4 = 8$
14. $|5x| + 4 = 3$
15. $\left|\frac{2}{3}x - 7\right| = 1$
16. $\left|\frac{1}{2}x + \frac{5}{6}\right| = \frac{1}{3}$
17. $|4-3x| + 7 = 10$
18. $|5x-2| + 8 = 2$
19. $3|2x+1| + 5 = 11$
20. $-2|2-9x| + 6 = 4$
21. $-4\left|\frac{1}{2}x + 1\right| + 3 = 11$
22. $5 - |x+7| = -8$
23. $|3x+2| = |5x-1|$
24. $|x+4| = |7x+6|$