Calculus of Variations Problems:

- Introduction
- Minimal Surface Area of Revolution Problem
- Brachistochrone Problem
- Isoperimetric Problem

A Typical Calculus of Variations Problem:

Maximize or minimize (subject to side condition(s)):

$$I(y) = \int_a^b F(x, y, y')\,dx$$

Where *y* and *y'* are continuous on $[a,b]$ **, and *F* has continuous first and second partials.**

Example:

$$F\left(x, y, y'\right) = xy^2 + y'$$

and

$$\left[a, b\right] = \left[0, 1\right]$$

For $y = x^2$,

$$I(y) = \int_0^1 \left[x\left(x^2\right)^2 + 2x \right] dx = \int_0^1 \left(x^5 + 2x\right) dx = \frac{7}{6}$$

Here’s a table of values for various functions, *y*:

y	$I(y) = \int_0^1 \left[xy^2 + y' \right] dx$
1	$\frac{1}{2}$
x	$\frac{5}{4}$
x^2	$\frac{7}{6}$
e^x	$\frac{e^2 + 4e - 3}{4}$

Complete the table of values for $I(y) = \int_0^1 xyy'dx$

y	$I(y)$
1	0
x	
x^2	
$\sqrt{x}$	
e^x	

(See handout, page 1!)

Answer:

y	$I(y)$
1	0
x	$\frac{1}{3}$
x^2	$\frac{2}{5}$
$\sqrt{x}$	$\frac{1}{4}$
e^x	$\frac{e^2+1}{4}$

What's the equivalent of setting $I'(y)$ equal to zero, and solving for the critical points(functions)?

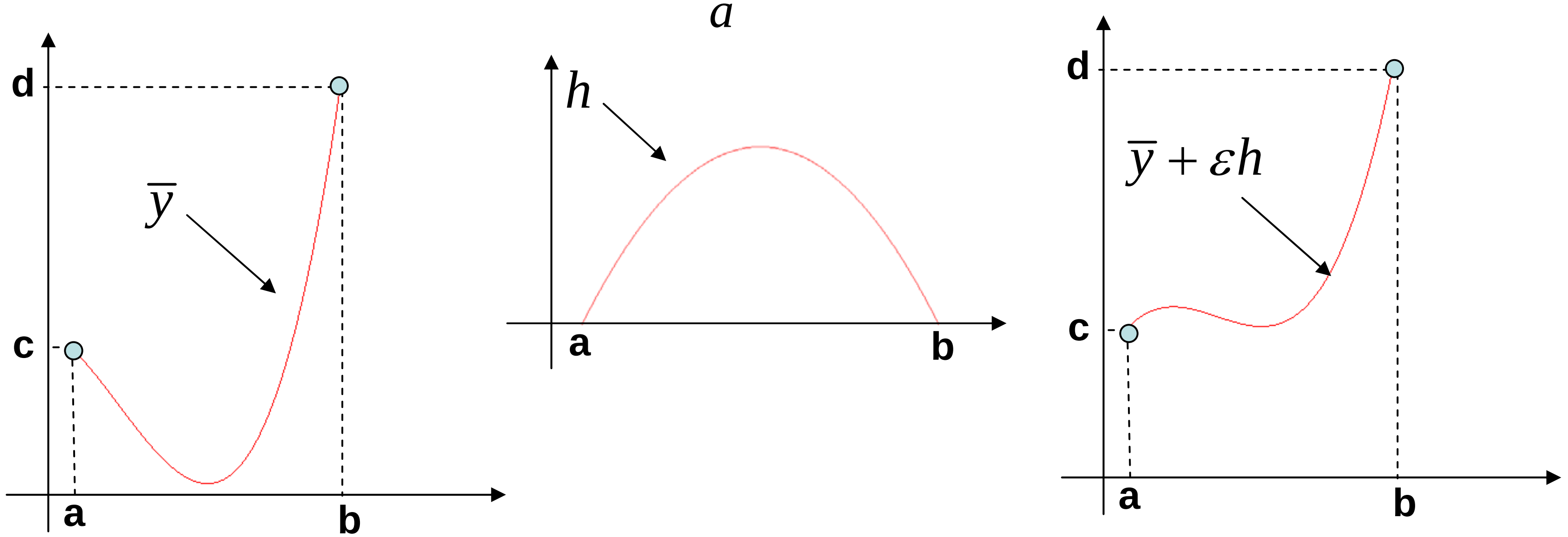
Consider $$I(y) = \int_a^b F(x, y, y')\,dx$$

With the side conditions $y(a) = c$ and $y(b) = d$.

This is called a fixed endpoint problem.

If $\bar{y}$ is a minimizing or maximizing function, and h is a continuously differentiable function with $h(a)=0$ and $h(b)=0$, then we can let ε be a real variable and consider the *ordinary* function

$$f(\varepsilon)=I(\bar{y}+\varepsilon h)=\int_a^b F(x,\bar{y}+\varepsilon h,\bar{y}'+\varepsilon h')\,dx$$



With $\overline{y}$ and h held fixed, f is a real valued function of ε that attains its minimum or maximum at $\varepsilon = 0$, so if $f'(0)$ exists, it must equal zero.

So let's differentiate and see what happens.

Under the assumptions,

$$f'(\varepsilon) = \frac{d}{d\varepsilon}\left[\int_a^b F\left(x, \overline{y} + \varepsilon h, \overline{y}' + \varepsilon h'\right) dx\right]$$

$$= \int_a^b \frac{\partial}{\partial \varepsilon}\left[F\left(x, \overline{y} + \varepsilon h, \overline{y}' + \varepsilon h'\right)\right] dx$$

We can use the Chain Rule to get

$$\frac{\partial}{\partial \varepsilon}\left[F\left(x, \overline{y}+\varepsilon h, \overline{y}'+\varepsilon h'\right)\right]$$

$$=\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial \varepsilon}+\frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial \varepsilon}+\frac{\partial F}{\partial y'}\cdot\frac{\partial y'}{\partial \varepsilon}$$

$$=F_y\left(x, \overline{y}+\varepsilon h, \overline{y}'+\varepsilon h'\right)\cdot h+F_{y'}\left(x, \overline{y}+\varepsilon h, \overline{y}'+\varepsilon h'\right)\cdot h'$$

So now we know that

$$f'(\varepsilon)=\int_a^b\left[F_y\left(x,\overline{y}+\varepsilon h,\overline{y}'+\varepsilon h'\right)\cdot h+F_{y'}\left(x,\overline{y}+\varepsilon h,\overline{y}'+\varepsilon h'\right)h'\right]dx$$

Setting $\varepsilon=0$ yields:

$$f'(0)=\int_a^b\left[F_y\left(x,\overline{y},\overline{y}'\right)\cdot h+F_{y'}\left(x,\overline{y},\overline{y}'\right)h'\right]dx$$

Since this must be zero, we arrive at

$$\int_a^b \left(F_y h + F_{y'} h' \right) dx = 0$$

If we apply integration by parts to the second summand, we get

$$\int_a^b \underbrace{F_{y'}}_{u} \underbrace{h' dx}_{dv} = F_{y'} h \Big|_a^b - \int_a^b \left(\frac{d}{dx} F_{y'} \right) h dx$$

$$= F_{y'}\left(b, \bar{y}, \bar{y}'\right) h(b) - F_{y'}\left(a, \bar{y}, \bar{y}'\right) h(a) - \int_a^b \left(\frac{d}{dx} F_{y'} \right) h dx$$

$$= -\int_a^b \left(\frac{d}{dx} F_{y'} \right) h dx$$

So now we have the equation

$$\int_a^b \left[F_y h - \left(\frac{d}{dx} F_{y'} \right) h \right] dx = 0$$

Or simply

$$\int_a^b \left[F_y - \left(\frac{d}{dx} F_{y'} \right) \right] h \, dx = 0$$

$$\int_a^b \left[F_y - \left(\frac{d}{dx} F_{y'} \right) \right] h \, dx = 0$$

This equation must be true for every

continuously differentiable function *h* with

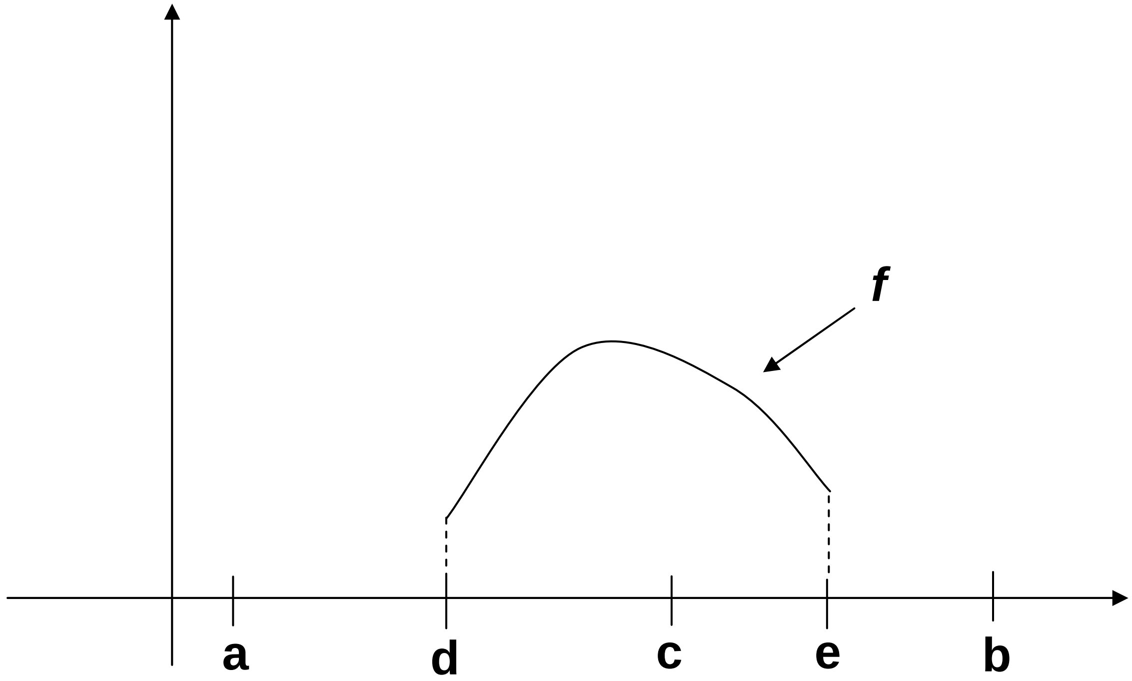
$h(a) = h(b) = 0$.

Suppose that for a continuous function *f*, $\int_a^b f(x)h(x)\,dx = 0$

for all continuously differentiable functions *h*, with

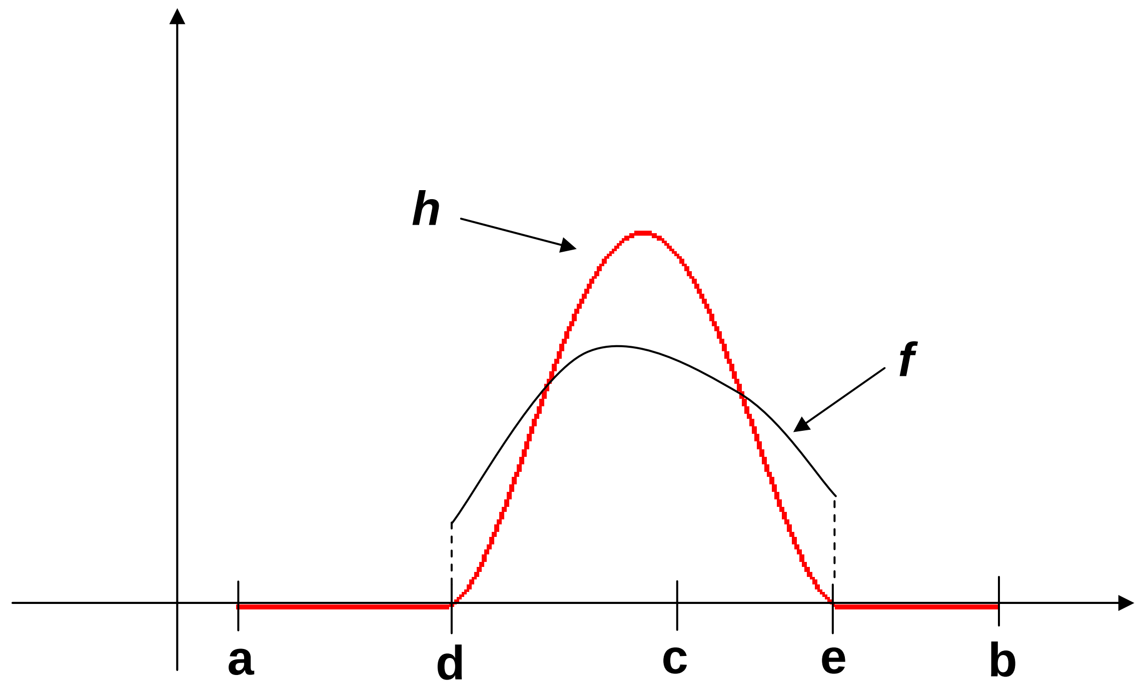
$h(a) = h(b) = 0$. What must *f* equal?

Suppose that *f* is not the zero function. Then there must be a point *c* in $[a,b]$ with $f(c) \neq 0$. If we further suppose that $f(c) > 0$, then since *f* is continuous, there must be an interval $[d,e]$ containing *c* on which *f* is positive.



Choose *h* with $h > 0$ inside $[d, e]$, and $h = 0$ outside $[d, e]$.

For example $h(x) = \begin{cases} (x-d)^2 (x-e)^2 & ; d \le x \le e \\ 0 & ; otherwise \end{cases}$



Clearly, $h(a) = h(b) = 0$. See if you can show that *h* is continuously differentiable on $[a,b]$.

<u>(See handout, page 2!)</u>

For this choice of *h*, what can you conclude about the value

of $\int_a^b f(x)h(x)\,dx$?

So *f* must be the zero function on $[a,b]$.

Since $\int_a^b \left[F_y - \left(\frac{d}{dx} F_{y'} \right) \right] h \, dx = 0$ for all continuously differentiable functions *h* with $h(a) = h(b) = 0$, we can conclude that $F_y - \left(\frac{d}{dx} F_{y'} \right) = 0$ on $[a,b]$.

To clear up one point, when we integrated by parts in the second part of $\int_a^b \left(F_y h + F_{y'} h'\right) dx = 0$, there was no guarantee that $\frac{d}{dx} F_{y'}$ exists or is continuous. But according to the Dubois-Reymond lemma, if $\int_a^b \left(F_y h + F_{y'} h'\right) dx = 0$ for all continuously differentiable functions *h* with $h(a) = h(b) = 0$, then $F_y - \left(\frac{d}{dx} F_{y'}\right) = 0$.

So a maximizing or minimizing function, y, must satisfy:

$$F_y - \left(\frac{d}{dx} F_{y'} \right) = 0$$

Subject to

$$y(a) = c \quad \text{and} \quad y(b) = d$$

The differential equation $F_y - \left(\frac{d}{dx} F_{y'} \right) = 0$ is referred to as the Euler-Lagrange equation.

It is typically converted into a second-order differential equation in *y*, even though the existence of y'' was not assumed or needed in the original problem.

Solutions of the modified Euler-Lagrange equation are solutions of the original Euler-Lagrange equation.

A solution of the original Euler-Lagrange equation which doesn’t have a second derivative is called a weak solution.

It can be shown that if *y* has a continuous first derivative,

satisfies $F_y - \left(\frac{d}{dx} F_{y'} \right) = 0$, *F* has continuous first and second

partials, then y has a continuous second derivative at all points

where $F_{y'y'}\left(x, y, y'\right) \neq 0$.

As an example, let's find the Euler-Lagrange

equation for $$I(y) = \int_a^b y\sqrt{1+(y')^2}\,dx \ .$$

$$F\left(x, y, y'\right) = y\sqrt{1+\left(y'\right)^2}$$

$$F_y = \sqrt{1+\left(y'\right)^2}$$

$$F_{y'} = \frac{yy'}{\sqrt{1+\left(y'\right)^2}}$$

$$\frac{d}{dx}\left(F_{y'}\right) = \frac{\left(y'\right)^4 + yy'' + \left(y'\right)^2}{\left[1+\left(y'\right)^2\right]^{\frac{3}{2}}}$$

So the Euler-Lagrange equation is

$$\underbrace{\sqrt{1+\left(y'\right)^2}}_{F_y} - \underbrace{\frac{\left(y'\right)^4 + yy'' + \left(y'\right)^2}{\left[1+\left(y'\right)^2\right]^{\frac{3}{2}}}}_{\frac{d}{dx}F_{y'}} = 0$$

Or simply

$$yy'' - \left(y'\right)^2 - 1 = 0$$

See if you can find the Euler-Lagrange equation for

$$I(y) = \int_0^{\frac{\pi}{2}} \left[y^2 - (y')^2 \right] dx$$

<u>(See handout, page 3!)</u>

Answer:

$$y'' + y = 0$$

Find all optimal solution candidates of

$$I(y) = \int_0^{\frac{\pi}{2}} \left[y^2 - (y')^2 \right] dx$$

Subject to

$$y(0) = 0 \quad \text{and} \quad y\left(\frac{\pi}{2}\right) = 1$$

(See handout, page 4!)

Answer:

$$y = \sin x$$

Let’s see if it’s a maximum, minimum, or neither?

Suppose that *z* is a function which is continuously differentiable with $z(0)=0$ and $z\left(\frac{\pi}{2}\right)=1$. Then $h=z-\sin x$ would be a continuously differentiable function with $h(0)=0$ and $h\left(\frac{\pi}{2}\right)=0$. So every such function *z* can be represented as $\sin x+h(x)$ for some function *h*. Therefore to investigate the value of $I(z)$ for any such function, *z*, we can consider $I(z)=I\left(\sin(x)+h(x)\right)$, for a continuously differentiable function *h*, with $h(0)=0$ and $h\left(\frac{\pi}{2}\right)=0$.

$$I\left(\sin x+h(x)\right)=\int_0^{\frac{\pi}{2}}\left[\left[\sin x+h(x)\right]^2-\left[\cos x+h'(x)\right]^2\right]dx$$

$$=\int_0^{\frac{\pi}{2}}\left(\sin^2 x-\cos^2 x\right)dx+\underbrace{2\int_0^{\frac{\pi}{2}}\left(h(x)\sin x-h'(x)\cos x\right)dx}_{=0}$$

$$+\int_0^{\frac{\pi}{2}}\left[h(x)^2-\left[h'(x)\right]^2\right]dx$$

$$=\underbrace{I\left(\sin x\right)}_{0}+\int_0^{\frac{\pi}{2}}\left[h(x)^2-\left[h'(x)\right]^2\right]dx$$

So what's going on at the critical function $y = \sin x$, depends

on whether the expression $\int_0^{\frac{\pi}{2}} \left[h(x)^2 - \left[h'(x) \right]^2 \right] dx$ is always

positive, always negative, or can be either positive or

negative for continuously differentiable functions *h* with

$h(0) = 0$ and $h\left(\dfrac{\pi}{2} \right) = 0$. So let's investigate.

We'll use a special case of the Poincare Inequality to get our result: (Assume that $0 \le x \le 1$.)

$$h(x) = \int_0^x h'(t)\,dt$$

so

$$h^2(x) = \left[\int_0^x h'(t)\,dt\right]^2$$

We'll need a version of the Cauchy-Schwarz inequality:

For *f* and *g* continuous functions on $\left[a,b\right]$

$$\left[\int_a^b f\left(x\right)g\left(x\right)dx\right]^2 \leq \left[\int_a^b f^2\left(x\right)dx\right]\left[\int_a^b g^2\left(x\right)dx\right]$$

See if you can prove it.

<u>(See handout, page 5!)</u>

From the Cauchy-Schwarz Inequality, we get

$$h^2(x) \le \int_0^x 1^2\,dt \int_0^x \left[h'(t)\right]^2 dt$$

$$\le x \int_0^x \left[h'(t)\right]^2 dt$$

And so

$$h^2(x) \le \int_0^1 \left[h'(x)\right]^2 dx$$

If we integrate this inequality, we get

$$\int_0^1 h^2(x)\,dx \le \int_0^1 \left[h'(x)\right]^2 dx$$

We can do the same thing for $1 \le x \le \dfrac{\pi}{2}$.

$$h(x) = \int_x^{\frac{\pi}{2}} -h'(t)\,dt$$

so

$$h^2(x) = \left[\int_x^{\frac{\pi}{2}} -h'(t)\,dt \right]^2$$

From the Cauchy-Schwarz Inequality, we get

$$h^2(x) \le \int_x^{\frac{\pi}{2}} 1^2\,dt \int_x^{\frac{\pi}{2}} \left[h'(t) \right]^2 dt$$

$$\le \left(\frac{\pi}{2} - x \right) \int_x^{\frac{\pi}{2}} \left[h'(t) \right]^2 dt$$

And so

$$h^2(x) \le \left(\frac{\pi}{2} - 1\right) \int_1^{\frac{\pi}{2}} \left[h'(x)\right]^2 dx$$

If we integrate this inequality, we get

$$\int_1^{\frac{\pi}{2}} h^2(x)\,dx \le \left(\frac{\pi}{2} - 1\right)^2 \int_1^{\frac{\pi}{2}} \left[h'(x)\right]^2 dx$$

$$\le \int_1^{\frac{\pi}{2}} \left[h'(x)\right]^2 dx$$

So we have

$$\int_0^1 h^2(x)\,dx \le \int_0^1 \left[h'(x)\right]^2 dx \Rightarrow \int_0^1 \left[h^2(x) - \left[h'(x)\right]^2\right] dx \le 0$$

and

$$\int_1^{\frac{\pi}{2}} h^2(x)\,dx \le \int_1^{\frac{\pi}{2}} \left[h'(x)\right]^2 dx \Rightarrow \int_1^{\frac{\pi}{2}} \left[h^2(x) - \left[h'(x)\right]^2\right] dx \le 0$$

If we add them together, we get

$$\int_0^{\frac{\pi}{2}} \left[h^2(x) - \left[h'(x) \right]^2 \right] dx \le 0$$

So

$$I\left(\sin x + h(x)\right) = \underbrace{I\left(\sin x\right)}_{0} + \underbrace{\int_0^{\frac{\pi}{2}} \left[h(x)^2 - \left[h'(x) \right]^2 \right] dx}_{\le 0}$$

Which means that $y = \sin x$ is a maximum.

Let’s just change the previous example a little bit.

$$I(y) = \int_0^{\frac{3\pi}{2}} \left[y^2 - (y')^2 \right] dx$$

The Euler-Lagrange equation is still

$$y'' + y = 0$$

The optimal solution candidate of

$$I(y) = \int_0^{\frac{3\pi}{2}} \left[y^2 - (y')^2 \right] dx$$

Subject to

$$y(0) = 0 \quad \text{and} \quad y\left(\frac{3\pi}{2}\right) = -1$$

is

$$y = \sin x$$

$$I\left(\sin x+h(x)\right)=\int_0^{\frac{3\pi}{2}}\left[\left[\sin x+h(x)\right]^2-\left[\cos x+h'(x)\right]^2\right]dx$$

$$=\int_0^{\frac{3\pi}{2}}\left(\sin^2 x-\cos^2 x\right)dx+\underbrace{2\int_0^{\frac{3\pi}{2}}\left(h(x)\sin x-h'(x)\cos x\right)dx}_{=0}$$

$$+\int_0^{\frac{3\pi}{2}}\left[h(x)^2-\left[h'(x)\right]^2\right]dx$$

$$=\underbrace{I\left(\sin x\right)}_{0}+\int_0^{\frac{3\pi}{2}}\left[h(x)^2-\left[h'(x)\right]^2\right]dx$$

Consider $h_1(x) = \sin 2x$ and $h_2(x) = x\left(\frac{3\pi}{2} - x\right)$

$$h_1(0) = h_2(0) = h_1\left(\frac{3\pi}{2}\right) = h_2\left(\frac{3\pi}{2}\right) = 0$$

Compute $\displaystyle\int_0^{\frac{3\pi}{2}} \left[h_1(x)^2 - \left[h_1'(x) \right]^2 \right] dx$ and

$$\int_0^{\frac{3\pi}{2}} \left[h_2(x)^2 - \left[h_2'(x) \right]^2 \right] dx$$

(See handout, page 6!)

$$\int_0^{\frac{3\pi}{2}} \left[h_1(x)^2 - \left[h_1'(x) \right]^2 \right] dx = -\frac{9\pi}{4} \boxed{<0}$$

$$\int_0^{\frac{3\pi}{2}} \left[h_2(x)^2 - \left[h_2'(x) \right]^2 \right] dx = \frac{9}{8}\pi^3 \left(\frac{9\pi^2}{40} - 1 \right) \boxed{>0}$$

So in this case, $y=\sin x$ is neither a maximum or minimum.

$$I\left(\sin x+\varepsilon\sin 2x\right)=I\left(\sin x\right)+\varepsilon^2\int_0^{\frac{3\pi}{2}}\left[\sin^2 2x-\left[\left(\sin 2x\right)'\right]^2\right]dx$$

$$<I\left(\sin x\right)$$

$$I\left(\sin x+\varepsilon x\left(\frac{3\pi}{2}-x\right)\right)=I\left(\sin x\right)+\varepsilon^2\int_0^{\frac{3\pi}{2}}\left[x^2\left(\frac{3\pi}{2}-x\right)^2-\left[\left(x\left(\frac{3\pi}{2}-x\right)\right)'\right]^2\right]dx$$

$$>I\left(\sin x\right)$$

See if you can find the Euler-Lagrange equation for

$$I(y) = \int_{0}^{\ln 2} \left[y^2 + (y')^2 \right] dx$$

(See handout, page 7!)

Answer:

$$y'' - y = 0$$

Find all optimal solution candidates of

$$I(y) = \int_0^{\ln 2} \left[y^2 + (y')^2 \right] dx$$

Subject to

$$y(0) = 0 \quad \text{and} \quad y(\ln 2) = \frac{3}{4}$$

(See handout, page 8!)

Answer:

$$y = \sinh x$$

See if you can determine if it's a maximum or minimum.

(See handout!, page 9)

$$I\left(\sinh x+h(x)\right)=\int_0^{\ln 2}\left[\left[\sinh x+h(x)\right]^2+\left[\cosh x+h'(x)\right]^2\right]dx$$

$$=\int_0^{\ln 2}\left(\sinh^2 x+\cosh^2 x\right)dx+2\underbrace{\int_0^{\ln 2}\left(h(x)\sinh x+h'(x)\cosh x\right)dx}_{=0}$$

$$+\int_0^{\ln 2}\left[h^2(x)+\left[h'(x)\right]^2\right]dx$$

$$=\underbrace{I\left(\sinh x\right)}_{1}+\underbrace{\int_0^{\ln 2}\left[h^2(x)+\left[h'(x)\right]^2\right]dx}_{\geq 0}$$

So, it’s a minimum!

See if you can find the Euler-Lagrange equation for

$$I(y) = \int_a^b \sqrt{1+(y')^2}\, dx$$

<u>(See handout, page 10!)</u>

Answer:

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right)=0$$

$$\frac{y'}{\sqrt{1+(y')^2}}=C$$

$$(y')^2=C^2\left[1+(y')^2\right] \qquad y'=\frac{\pm C}{\sqrt{1+C^2}}=K$$

Find all optimal solution candidates of

$$I(y) = \underbrace{\int_a^b \sqrt{1+(y')^2}\, dx}_{\text{length of } y}$$

Subject to

$$y(a) = c \quad \text{and} \quad y(b) = d$$

<u>(See handout, page 11!)</u>

Answer:

$$y = \left(\frac{d-c}{b-a} \right) (x-a) + c$$

Do you think that it's a maximum or minimum?

Shortest path between two points in the plane?

Is there a 2nd Derivative Test?

Yes.

If

1. y is a stationary function

2. $F_{y'y'} > 0$ along y

3. There is no number *z* in $(a,b]$ such that

$$-\frac{d}{dx}\left(F_{y'y'}h'\right)+\left(F_{yy}-\frac{d}{dx}F_{yy'}\right)h=0$$

$$h(a)=h(z)=0$$

has a nontrivial solution along y.

Then y is a local minimum.

For $I(y)=\int_a^b \sqrt{1+(y')^2}\,dx$, $F=\sqrt{1+(y')^2}$.

$$F_y=0\,,\ F_{yy'}=0\,,\ F_{y'}=\frac{y'}{\sqrt{1+(y')^2}}\,,\ F_{y'y'}=\frac{1}{\left[1+(y')^2\right]^{\frac{3}{2}}}$$

Along the stationary curve $y=\left(\dfrac{d-c}{b-a}\right)(x-a)+c$,

$$F_{y'y'}=\frac{1}{\left[1+\left(\dfrac{d-c}{b-a}\right)^2\right]^{\frac{3}{2}}}>0$$

Along y, the boundary value problem is

$$-\frac{d}{dx}\left(\frac{1}{\left[1+\left(\frac{d-c}{b-a}\right)^2\right]^{\frac{3}{2}}}h'\right)+\left(0-\frac{d}{dx}0\right)h=0$$

$$h(a)=h(z)=0$$

Or simply

$$h''(x)=0$$

$$h(a)=h(z)=0$$

The boundary value problem has only the trivial solution for

all choices of z in $[a,b]$, and so $y = \left(\dfrac{d-c}{b-a}\right)(x-a)+c$ is

at least a local minimum.

See if you can show that the boundary value problem has only the trivial solution.

(See handout, page 12!)

See if you can find the Euler-Lagrange equation for

$$I(y) = \int_0^1 \left[(y')^2 + 12xy \right] dx$$

(See handout, page 13!)

Answer:

$$y'' = 6x$$

Find all optimal solution candidates of

$$I(y) = \int_0^1 \left[(y')^2 + 12xy \right] dx$$

Subject to

$$y(0) = 0 \text{ and } y(1) = 1$$

(See handout, page 14!)

Answer:

$$y = x^3$$

See if you can determine if it's a maximum or minimum?

(See handout, page 15!)

$$I\left(x^3+h(x)\right)=\int_0^1\left[\left[3x^2+h'(x)\right]^2+12x\left[x^3+h(x)\right]\right]dx$$

$$=\int_0^1 21x^4dx+\int_0^1\left[h'(x)\right]^2dx+\underbrace{\int_0^1\left[12xh(x)+6x^2h'(x)\right]dx}_{=0}$$

$$=\underbrace{I\left(x^3\right)}_{\frac{21}{5}}+\underbrace{\int_0^1\left[h'(x)\right]^2dx}_{\geq 0}$$

So, it’s a minimum!

Find all optimal solution candidates of

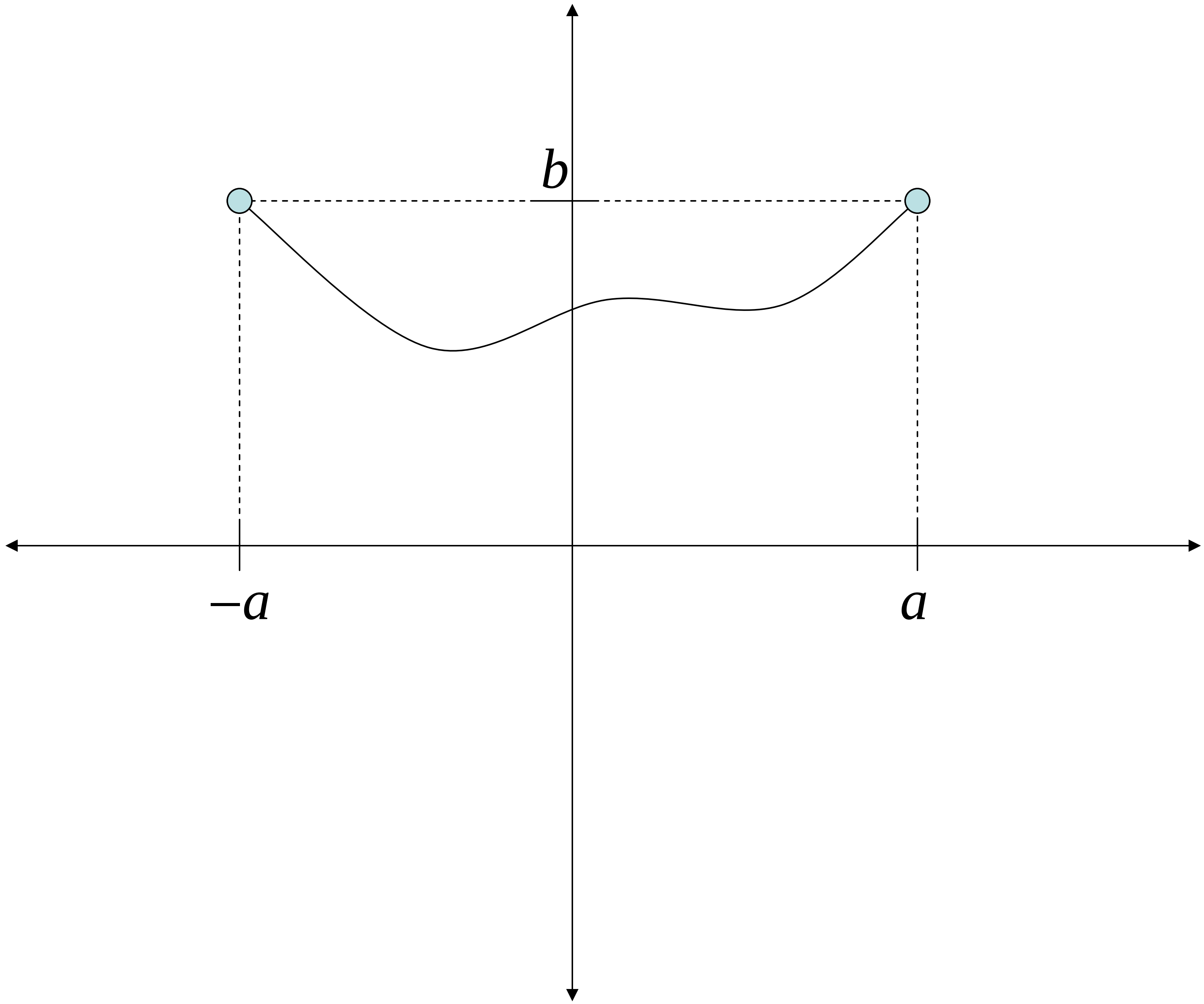
$$I(y) = \int_0^1 xyy'dx$$

Subject to

$$y(0) = 0 \text{ and } y(1) = 1$$

(See handout, page 16!)

A Minimal Surface Area of Revolution Problem



b
−a
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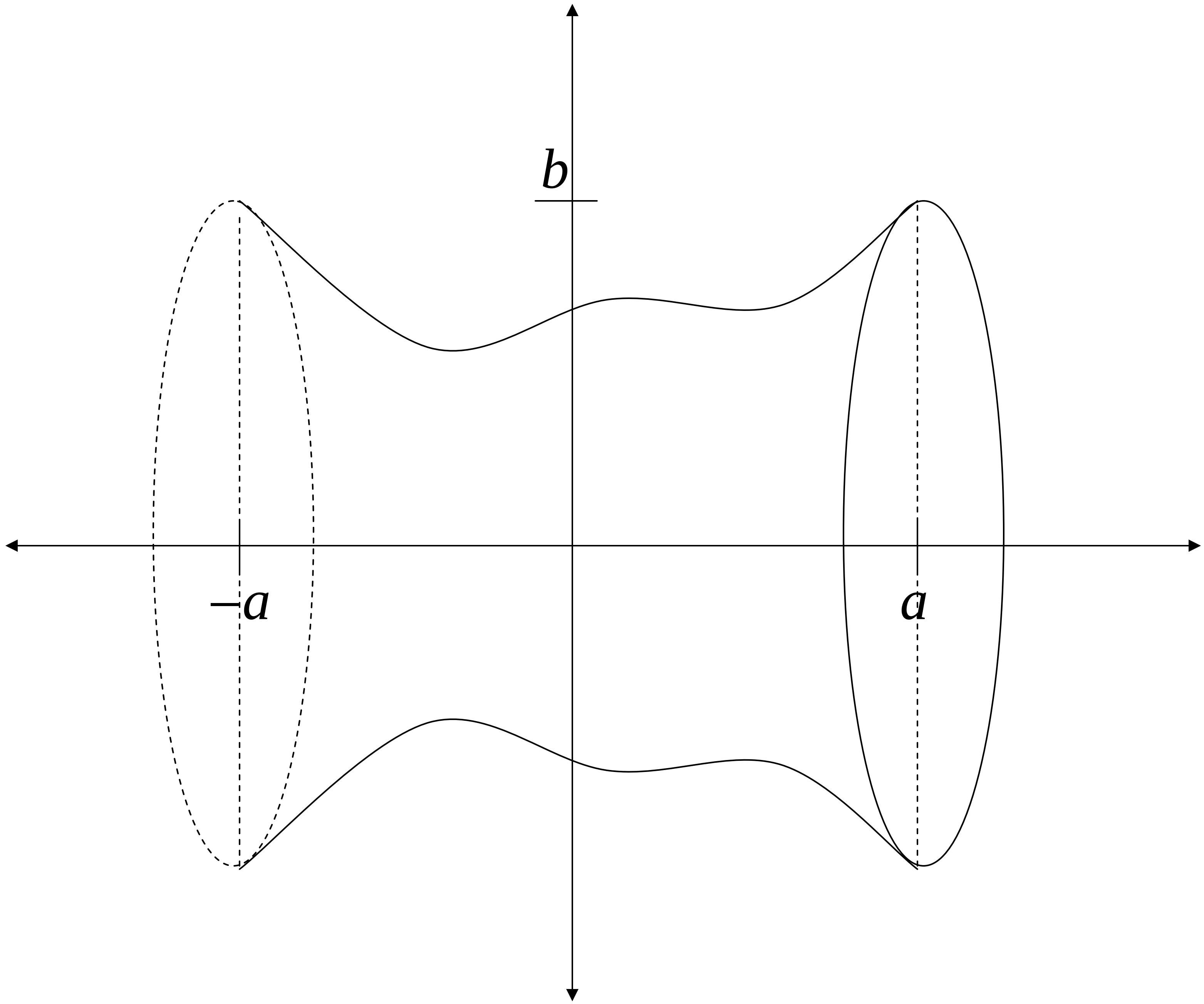


b
−a
a

We want to find a function, $y(x)$, with $y(-a) = y(a) = b$

which makes the surface area

$$A(y) = 2\pi \int_{-a}^{a} y\sqrt{1+(y')^2}\,dx$$

as small as possible.

Here’s the Euler-Lagrange equation(we essentially found it earlier):

$$2\pi\sqrt{1+\left(y'\right)^2} - 2\pi\frac{\left(y'\right)^4 + yy'' + \left(y'\right)^2}{\left[1+\left(y'\right)^2\right]^{\frac{3}{2}}} = 0$$

Or simply

$$yy'' - \left(y'\right)^2 - 1 = 0$$

So we have to solve

$$yy' - (y')^2 - 1 = 0$$

Subject to

$$y(-a) = y(a) = b$$

If we let $p = \frac{dy}{dx}$, then $y'' = \frac{dp}{dx} = \frac{dy}{dx} \cdot \frac{dp}{dy} = p \cdot \frac{dp}{dy}$, and the differential equation becomes

$$yp\frac{dp}{dy} - p^2 - 1 = 0$$

Separation of variables leads to

$$\frac{p}{p^2+1}dp = \frac{1}{y}dy$$

$$\int \frac{p}{p^2+1}dp = \int \frac{1}{y}dy$$

$$\ln\left(p^2+1\right) = \ln y^2 + C_1$$

$$p^2 + 1 = C_1 y^2$$

$$p^2 = C_1 y^2 - 1$$

$$y' = \pm\sqrt{C_1 y^2 - 1}$$

Another separation of variables leads to

$$\int \frac{dy}{\sqrt{C_1 y^2 - 1}} = \pm \int dx$$

$$\int \frac{dy}{\sqrt{C_1 y^2 - 1}} = \pm x + C_2$$

$$\frac{1}{\sqrt{C_1}} \cosh^{-1}\left(\sqrt{C_1}\, y\right) = \pm x + C_2$$

$$\cosh^{-1}\left(\sqrt{C_1}\,y\right) = \sqrt{C_1}\left(\pm x + C_2\right)$$

$$y = \frac{\cosh\left[\sqrt{C_1}\left(\pm x + C_2\right)\right]}{\sqrt{C_1}}$$

$$y = \frac{\cosh\left(\pm C_1 x + C_2\right)}{C_1}$$

$$y(-a) = y(a) = b$$

From the symmetry and boundary conditions, we get:

$$\cosh\left(C_1 a\right) = bC_1$$

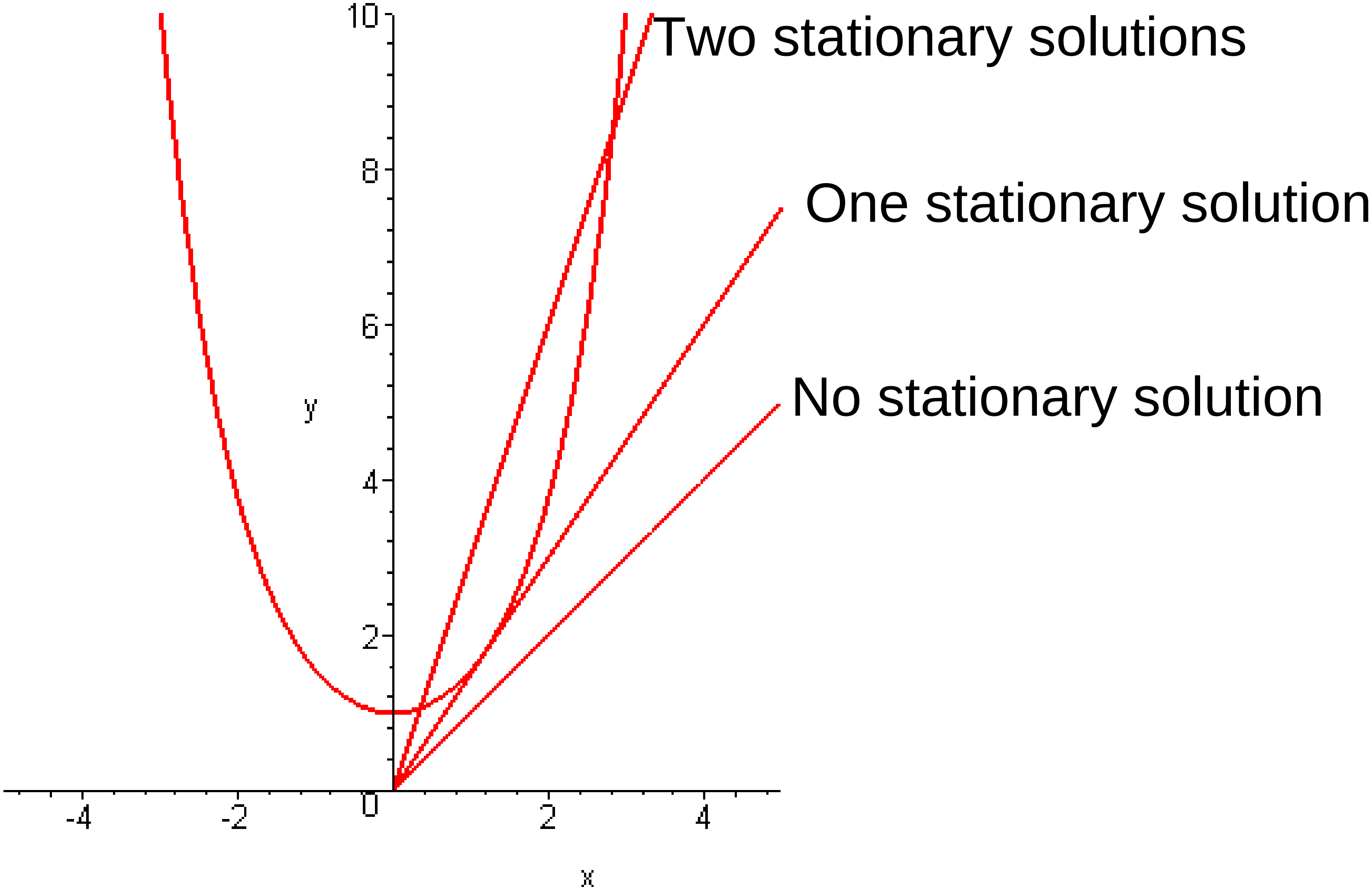
Let $r = \dfrac{b}{a}$

$$\cosh\left(\frac{bC_1}{r}\right) = \frac{bC_1}{r} \cdot r$$

So if we can solve the equation

$$\cosh\left(x\right) = rx$$

For $r > 0$, then we'll have a stationary function for each solution of the equation $\cosh(x) = rx$



If $r < 1.508879563$ then no stationary solution.

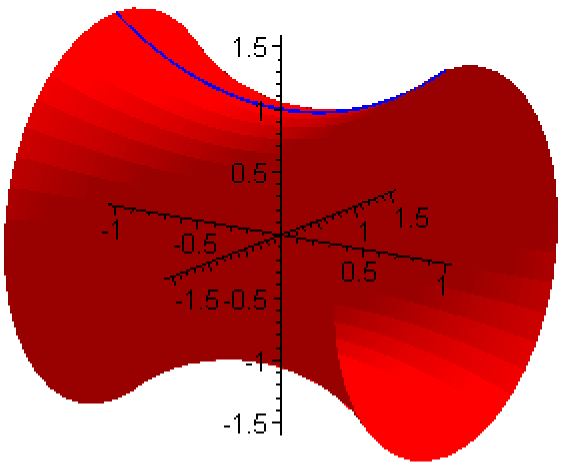
If $r = 1.508879563$ then one stationary solution.

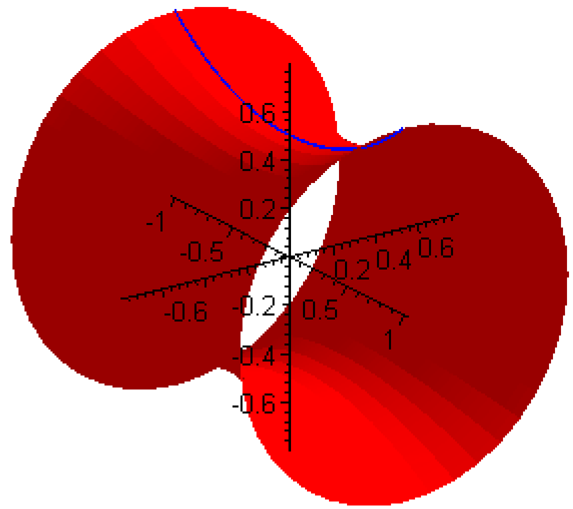
If $r > 1.508879563$ then two stationary solutions.

In other words, if the distance from the point (a,b) to the y-axis, a, is larger than $.6627434187b$, then there is no stationary solution. If a is equal to $.6627434187b$, then there is one stationary solution. If a is less than $.6627434187b$, then there are two stationary solutions.

It can be shown that for $a > .6627434187b$, the two disks give the absolute minimum area, and there are no other local minima; for $.5276973968b < a < .6627434187b$, the two disks give the absolute minimum area and the two stationary solutions are local minima; for $a < .5276973968b$, then one stationary solution is the absolute minimum, the other is a local minimum, and the two disks are a local minimum.

By the way, surfaces of revolution generated by catenaries(graphs of hyperbolic cosines) are called catenoids.





Let’s check the mathematical prediction with reality.

The large rings have a radius of approximately 12.75 cm.

The medium rings have a radius of approximately 10 cm.

The small rings have a radius of approximately 5 cm.

So for the large rings:

You should see a catenoid for separation distances between 0 and 13.5 cm.(The other catenoid and two discs are local minima.)

You should see the two discs as the absolute minimum for separation distances between 13.5 cm and 16.9 cm.(The two catenoids are local minima.)

You should have the two discs as the absolute minimum for separation distances larger than 16.9 cm.(There are no other local minima.)

So for the medium rings:

You should see a catenoid for separation distances between 0 and 10.6 cm.(The other catenoid and two discs are local minima.)

You should see the two discs as the absolute minimum for separation distances between 10.6 cm and 13.3 cm.(The two catenoids are local minima.)

You should have the two discs as the absolute minimum for separation distances larger than 13.3 cm.(There are no other local minima.)

So for the small rings:

You should see a catenoid for separation distances between 0 and 5.3 cm.(The other catenoid and two discs are local minima.)

You should see the two discs as the absolute minimum for separation distances between 5.3 cm and 6.6 cm.(The two catenoids are local minima.)

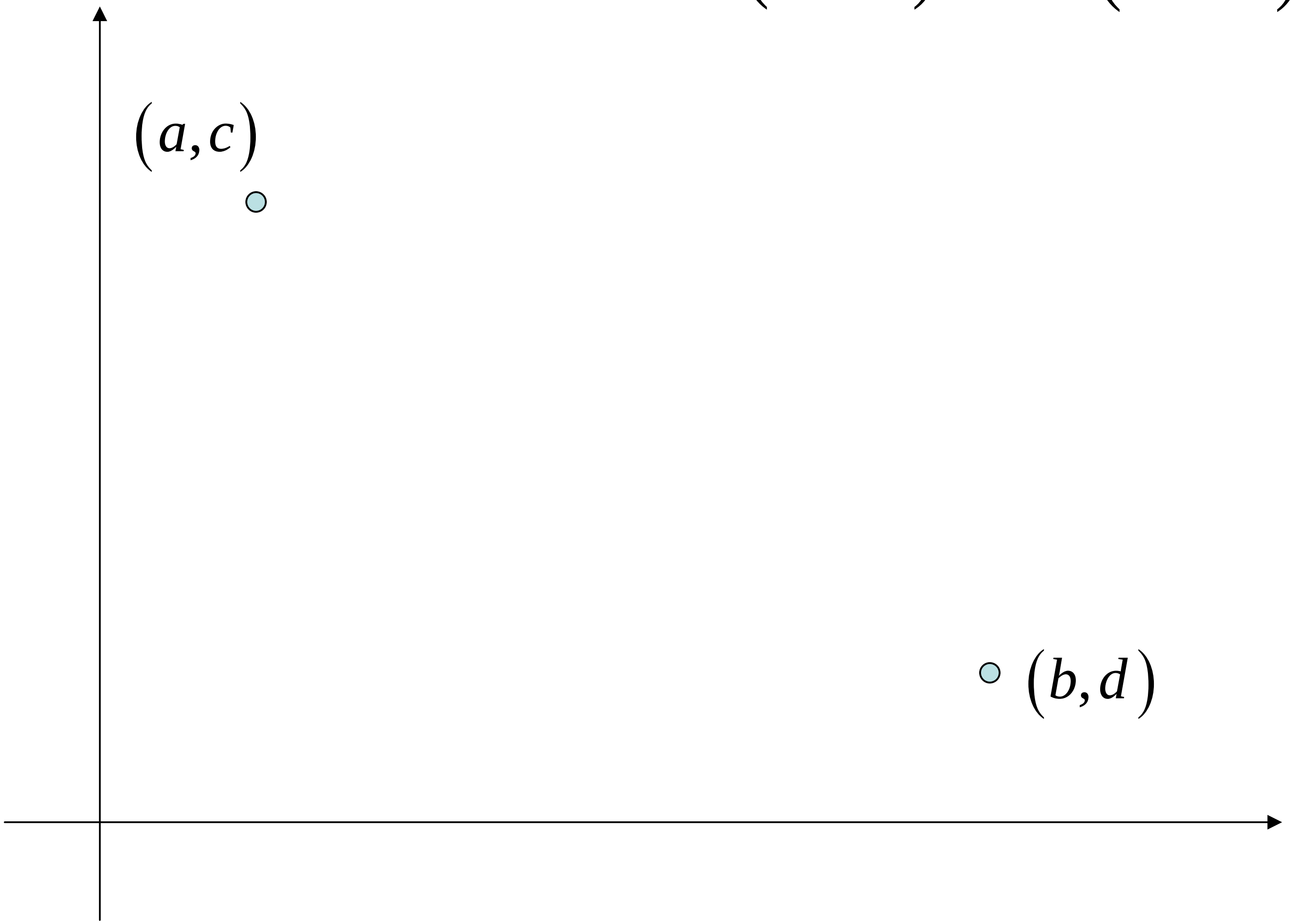
You should have the two discs as the absolute minimum for separation distances larger than 6.6 cm.(There are no other local minima.)

(See handout, pages 17-19!)

The Brachistochrone Problem

(Curve of quickest descent)

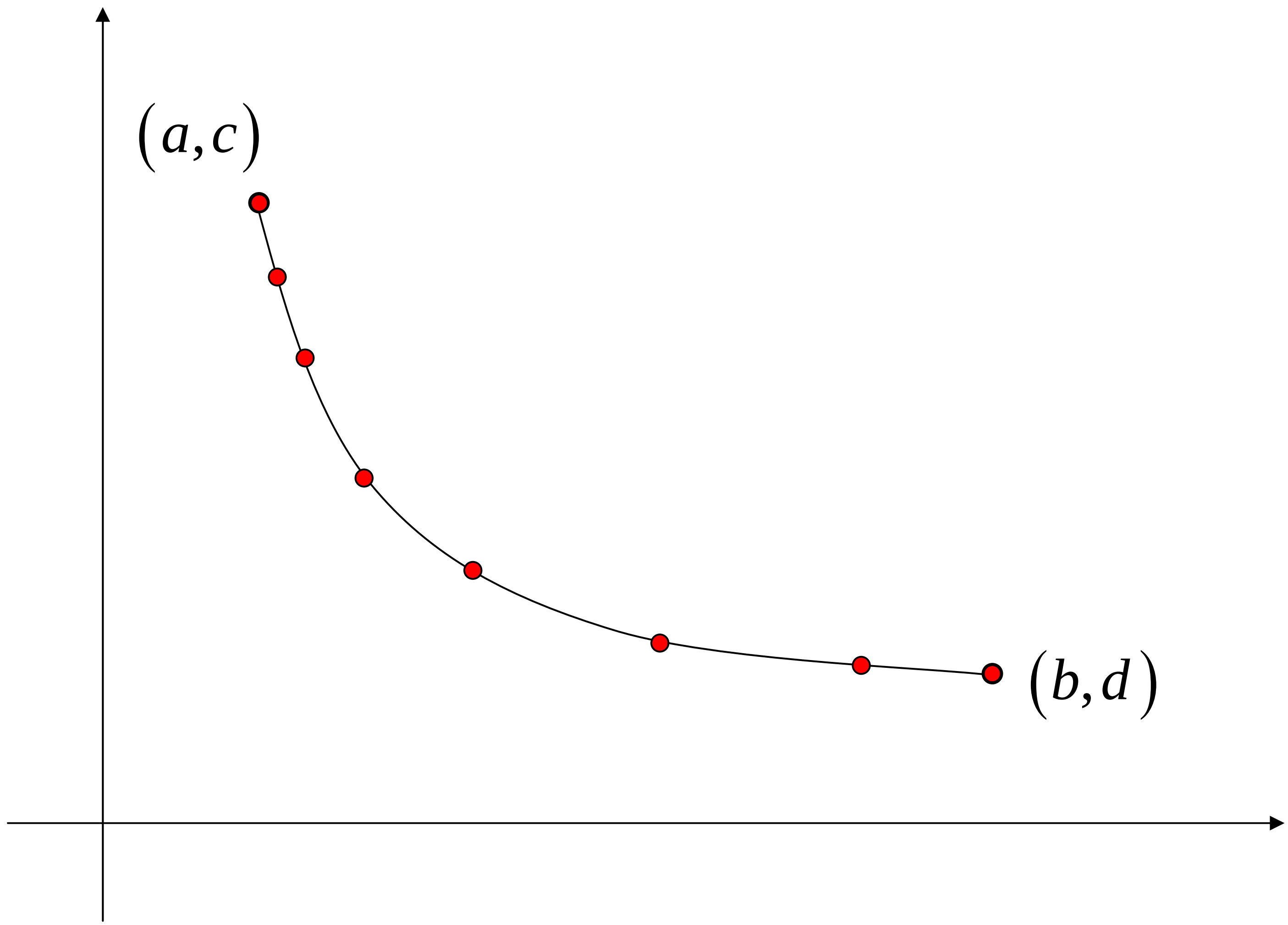
Consider the two points in the plane (a,c) and (b,d).



We want to join the two points by a curve so that a particle

starting from rest at (a,c) and moving along the curve

under the influence of gravity alone will reach the point (b,d)
in minimum time.



To formulate the problem, we'll determine the velocity of the particle in two ways:

Since the velocity, *v*, of the particle is determined by gravity alone, we must have

$$g = \frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = v \cdot \frac{dv}{dy}$$

So we get the initial value problem:

$$v\frac{dv}{dy} = \underbrace{g}_{<0}, \; v(a) = 0$$

The solution of the initial value problem is

$$v = \sqrt{2g\left(y - c\right)}$$

But the velocity must also equal the rate of change of arclength along the curve with respect to time, so we get

$$v = \frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt} = \sqrt{1 + \left(y'\right)^2}\,\frac{dx}{dt}$$

Equating the two formulas for velocity leads to

$$dt = \frac{\sqrt{1+\left(y'\right)^2}}{\sqrt{2g\left(y-c\right)}}dx$$

Which means that the total travel time for the particle is given by

$$\int_a^b \frac{\sqrt{1+\left(y'\right)^2}}{\sqrt{2g\left(y-c\right)}}dx = \frac{1}{\sqrt{2\left(-g\right)}}\int_a^b \frac{\sqrt{1+\left(y'\right)^2}}{\sqrt{c-y}}dx$$

So the mathematical statement of the problem is to find a continuously differentiable function, *y*, that minimizes

$$I(y) = \int_a^b \frac{\sqrt{1+(y')^2}}{\sqrt{c-y}}\,dx$$

Subject to $y(a) = c$ and $y(b) = d$.

(So in other words, it's a fixed endpoint Calculus of Variations problem.)

Equivalent versions of the Euler-Lagrange Equation in special cases:

$$F_y - \left(\frac{d}{dx} F_{y'} \right) = 0$$

<u>Case I:</u> If *F* doesn't involve *y*, then the Euler-Lagrange equation reduces to $\frac{d}{dx} F_{y'} = 0$, which can be integrated into $F_{y'} = C$.

<u>Case II:</u> If *F* doesn't involve *x*, then the Euler-Lagrange

equation reduces to $F - y' F_{y'} = C$.

For the Brachistochrone Problem $F\left(x, y, y'\right) = \dfrac{\sqrt{1+\left(y'\right)^2}}{\sqrt{c-y}}$,

which is independent of *x*.

$$F_{y'} = \frac{y'}{\sqrt{(c-y)\left[1+(y')^2\right]}}$$

So using Case II, we get the Euler-Lagrange Equation as

$$\frac{\sqrt{1+(y')^2}}{\sqrt{c-y}} - \frac{(y')^2}{\sqrt{(c-y)\left[1+(y')^2\right]}} = C$$

Simplifying algebraically, we get $C^2(c-y)\left[1+(y')^2\right]=1$,

and solving for *y'* yields $y' = \pm\sqrt{\dfrac{1-C^2(c-y)}{C^2(c-y)}}$. If we make

the substitution $c-y=\dfrac{1}{C^2}\sin^2\left(\frac{t}{2}\right)$, where *t* is a parameter,

we get $y' = \pm\sqrt{\dfrac{1-\sin^2\left(\frac{t}{2}\right)}{\sin^2\left(\frac{t}{2}\right)}} = -\dfrac{\cos\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}$. The simplification

was chosen to reflect the fact that initially the curve will have a negative slope.

From the Chain Rule, $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx}$, so

$$-\frac{\cos\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)} = \left[-\frac{1}{C^2}\sin\left(\tfrac{t}{2}\right)\cos\left(\tfrac{t}{2}\right)\right]\frac{dt}{dx}$$ and $$dx = \frac{1}{C^2}\sin^2\left(\tfrac{t}{2}\right)dt.$$

This gives us $x = \dfrac{1}{2C^2}\left(t - \sin t\right) + k$. We can solve for *k* if

we insist that $x = a$ when $t = 0$.

See if you can verify the formula for *x* and find the value of *k*.

(See handout, page 20!)

So $k = a$,and we'll let $A = \dfrac{1}{2C^2}$ to get the parametric equations

$$x = A\left(t - \sin t\right) + a$$
$$y = A\left(\cos t - 1\right) + c$$

To satisfy the condition $y\left(b\right) = d$, you'd have to solve

$$b - a = A\left(t - \sin t\right)$$
$$d - c = A\left(\cos t - 1\right)$$

For example: Suppose that $(a,c)=(0,0)$ and $(b,d)=(1,-1)$.

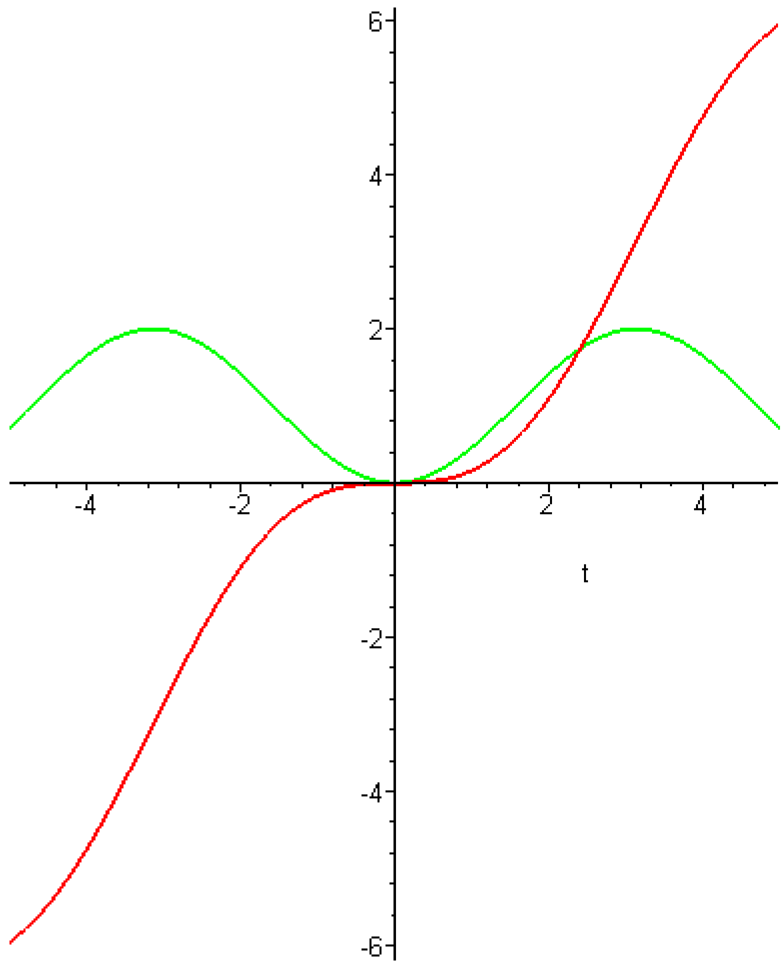
The system would be

$$\begin{aligned} 1 &= A(t-\sin t) \\ -1 &= A(\cos t-1) \end{aligned}$$

If we eliminate *A*, we get

$$t-\sin t=-(\cos t-1)$$

Here's a plot of the left side and right side of the equation:



The t-coordinate of the intersection point between 2 and 3 is the one we want.

See if you can approximate the t-coordinate of this intersection point.

(See handout, page 21!)

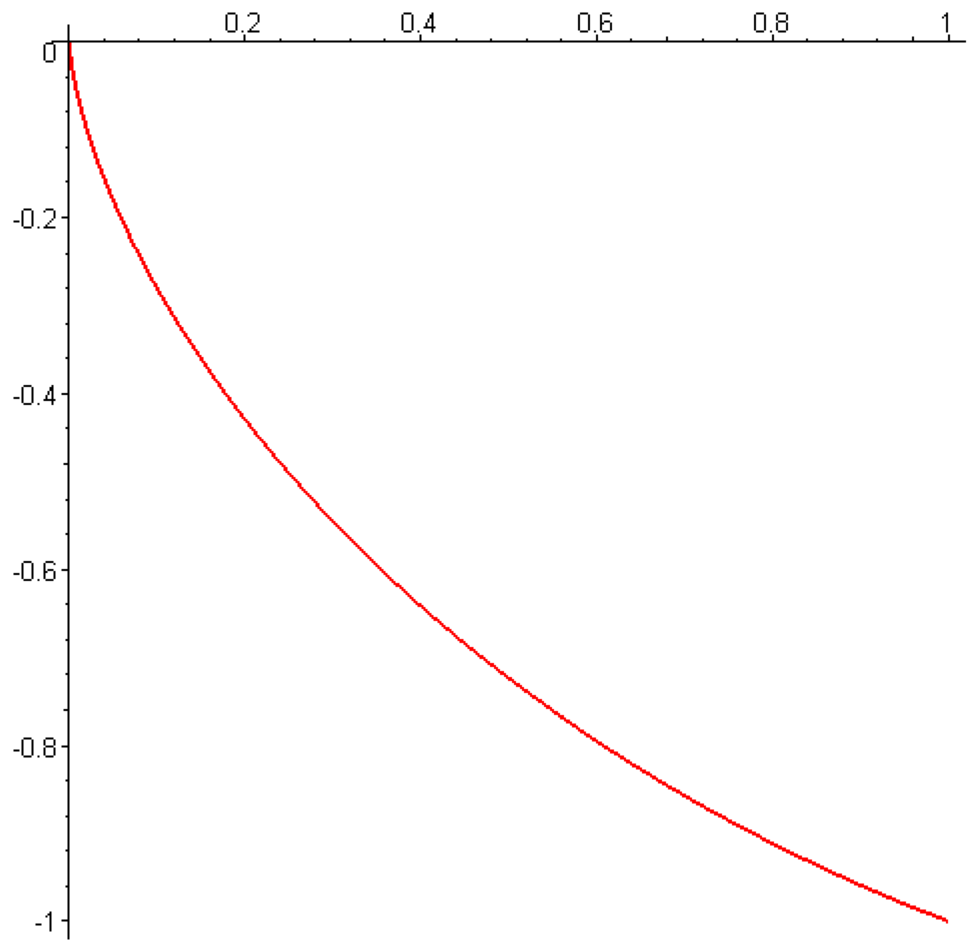
So an approximate parameterization of the solution curve in this case is

$$x = \underbrace{.5729170374}_{A\ value}(t - \sin t)$$

$$;0 \le t \le \underbrace{2.412011144}_{intersection}$$

$$y = \underbrace{.5729170374}_{A\ value}(\cos t - 1)$$

See if you can plot this parametric curve.


0.2
0.4
0.6
0.8
1
0
-0.2
-0.4
-0.6
-0.8
-1

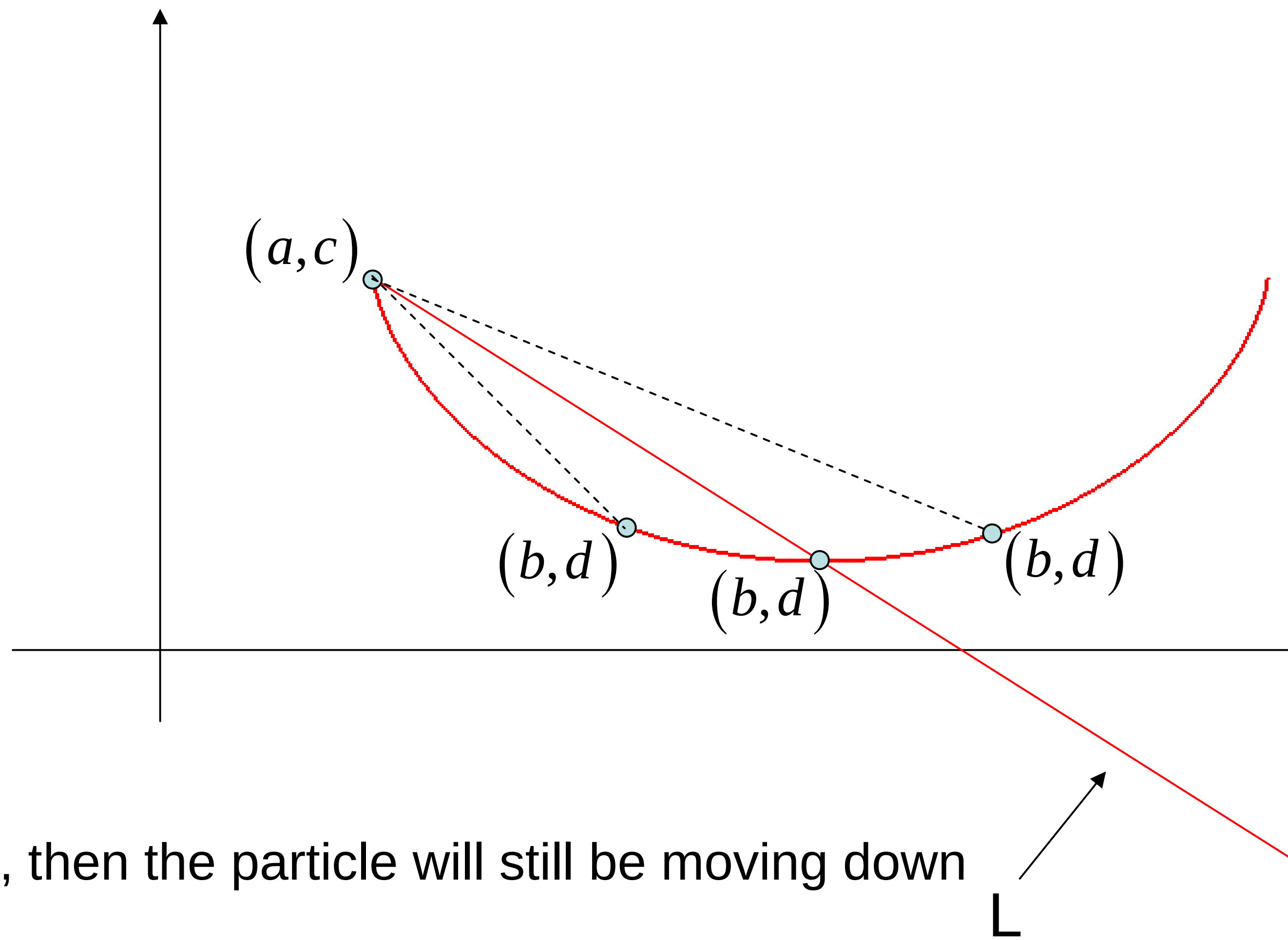
In general, the solution curve is an inverted cycloid, and

furthermore, since $y' = -\dfrac{\cos\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}$ the inverted cycloid will

have its minimum at $t = \pi$. So the coordinates of the

minimum will be $(a + \pi A, c - 2A)$, and they lie on the line

L which passes through the point (a, c) with slope $-\dfrac{2}{\pi}$.



If (b,d) lies below L, then the particle will still be moving down at (b,d).

If (b,d) lies on L, then the particle will just stop moving down at (b,d).

If (b,d) lies above L, then the particle will be moving up at (b,d).

See if you can demonstrate these properties with the Brachistochrone Demo.

Isoperimetric Problem

The Isoperimetric Problem is to find the plane curve of given length that encloses the largest area.

The Generalized Isoperimetric Problem is to find a stationary function for one integral subject to a constraint requiring a second integral to take a prescribed value.

Lagrange Multipliers

Suppose that we want to optimize $z = f(x, y)$ subject to the constraint that $g(x, y) = 0$. One approach is to arbitrarily designate one of the variables *x* and *y* in the equation $g(x, y) = 0$ as independent, say *x*, and the other dependent.

Provided that $\dfrac{\partial g}{\partial y} \neq 0$ on the curve $g(x, y) = 0$, we can use the Chain Rule to calculate $\dfrac{dy}{dx}$.

(See handout, page 22!)

So $\frac{dy}{dx} = -\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}$ on the curve $g(x, y) = 0$. In order for

$z = f(x, y(x))$ to have an optimal value, $\frac{dz}{dx} = 0$. But

$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \cdot \frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}$. So necessary

conditions for $z = f(x, y)$ to be optimized subject to $g(x, y) = 0$ are

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \cdot \frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}} = 0$$

$$g(x, y) = 0$$

An alternative approach to such a problem is to introduce an additional variable, λ, called a Lagrange Multiplier and consider a modified function, $L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$, called the Lagrangian, and investigate its unconstrained extrema. This leads to the system of equations:

$$\frac{\partial L}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0$$

$$\frac{\partial L}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0$$

$$\frac{\partial L}{\partial \lambda} = g(x, y) = 0$$

If $\frac{\partial g}{\partial y} \neq 0$ on the curve $g(x, y) = 0$, then the second equation can be solved as $\lambda = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial g}{\partial y}}$, and if you substitute into the first equation, you get $\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \cdot \frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}} = 0$. So necessary conditions for an unconstrained optimal value of $L(x, y, \lambda)$ are

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \cdot \frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}} = 0$$

$$g(x, y) = 0$$

Example:

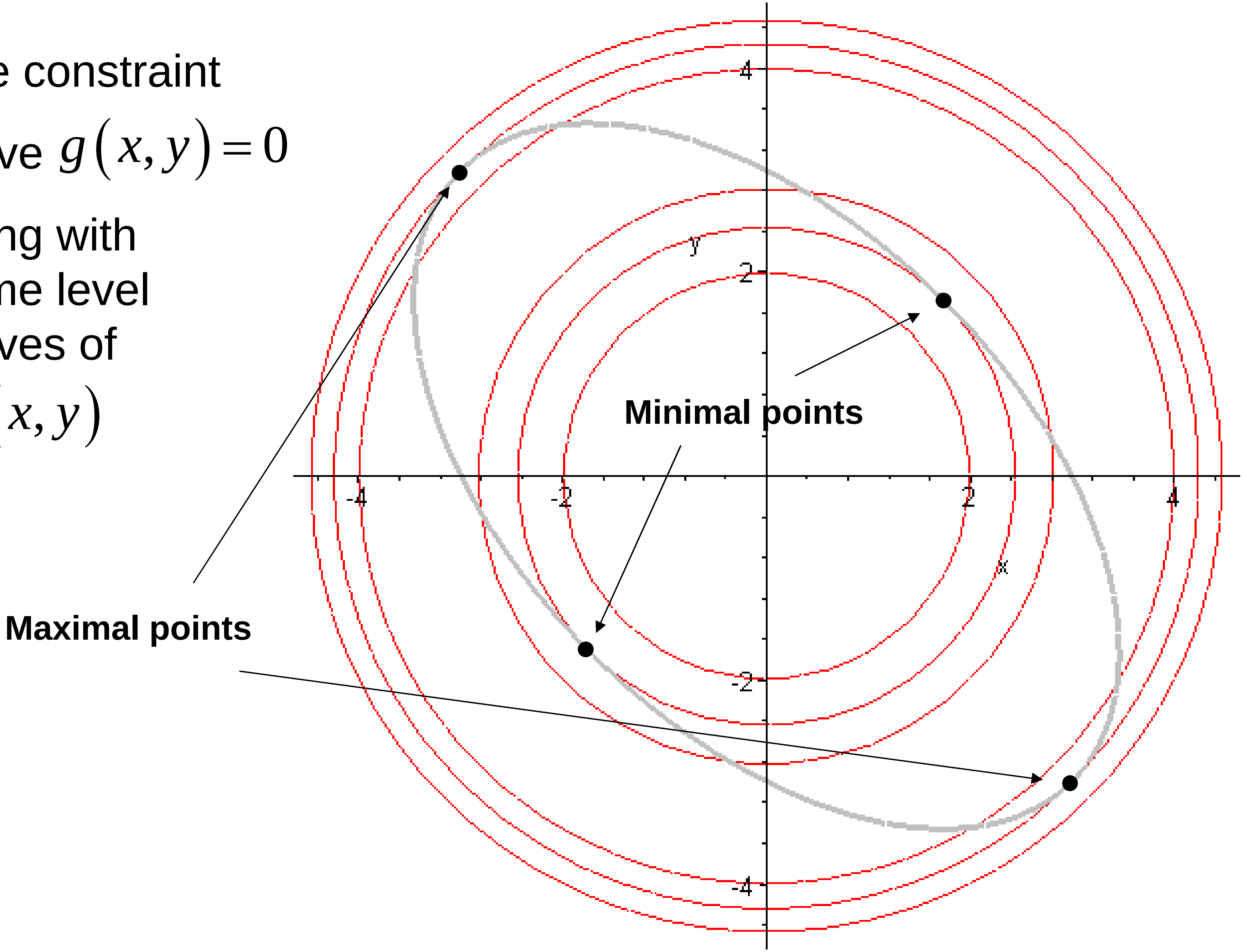
Find the maximum and minimum values of $f(x,y) = x^2 + y^2$

subject to $g(x,y) = x^2 + y^2 + xy - 9 = 0$.

The constraint

curve $g(x, y) = 0$

along with
some level
curves of
$f(x, y)$



The system of equations to solve in order to optimize the Lagrangian is

$$2x+2\lambda x+\lambda y=0$$

$$2y+2\lambda y+\lambda x=0$$

$$x^2+y^2+xy-9=0$$

And it's equivalent to the system

$$2x=-\lambda\left(2x+y\right)$$

$$2y=-\lambda\left(2y+x\right)$$

$$x^2+y^2+xy-9=0$$

If you multiply the first equation by *y,* the second equation by *x*, and subtract, you get

$$
\begin{array}{r}
2xy = -\lambda y\left(2x + y\right) \\
-\left[2xy = -\lambda x\left(2y + x\right)\right] \\
\hline
0 = -\lambda\left(y^2 - x^2\right)
\end{array}
$$

Since $\lambda \neq 0$, we can conclude that $y = x$ or $y = -x$.

Substitute these into $x^2 + y^2 + xy - 9 = 0$ to get the maximum and minimum values.

(See handout, page 23!)

Problem (The Generalized Isoperimetric Problem) :

Find the function y that maximizes or minimizes

$$I(y) = \int_a^b F(x, y, y')\,dx$$

subject to the constraint

$$J(y) = \int_a^b G(x, y, y')\,dx = k\,(constant)$$

with $y(a) = c$ and $y(b) = d$.

Suppose that $\bar{y}(x)$ is such a function. Previously, we Considered $y(x) = \bar{y}(x) + \varepsilon h(x)$, where *h* is a continuously differentiable function with $h(a) = h(b) = 0$. This won't be enough in this situation, because there is no guarantee that $y(x)$ will satisfy the integral constraint. So we'll consider $y(x) = \bar{y}(x) + \varepsilon_1 h_1(x) + \varepsilon_2 h_2(x)$, where h_1 and h_2 are continuously differentiable and $h_1(a) = h_2(a) = h_1(b) = h_2(b) = 0$.

So we want

$$I(\varepsilon_1, \varepsilon_2) = \int_a^b F(x, \overline{y} + \varepsilon_1 h_1 + \varepsilon_2 h_2, \overline{y}' + \varepsilon_1 h_1' + \varepsilon_2 h_2')\, dx$$

to have a maximum or minimum at $(0,0)$ subject to

$$J(\varepsilon_1, \varepsilon_2) = \int_a^b G(x, \overline{y} + \varepsilon_1 h_1 + \varepsilon_2 h_2, \overline{y}' + \varepsilon_1 h_1' + \varepsilon_2 h')\, dx = k$$

Let's look at a specific example:

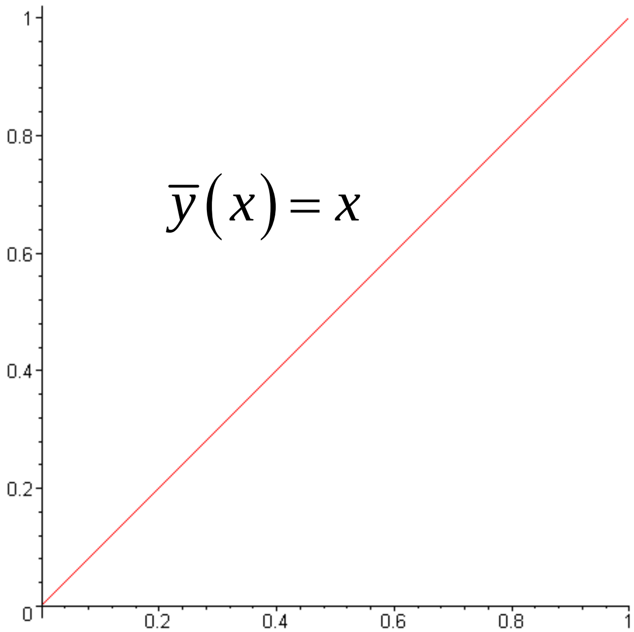
$$F(x,y,y') = \sqrt{1+\left[y'(x)\right]^2} \qquad G(x,y,y') = y \qquad [a,b] = [0,1]$$

$$c = 0\,,\; d = 1 \qquad k = \frac{1}{2} \qquad \bar{y}(x) = x$$

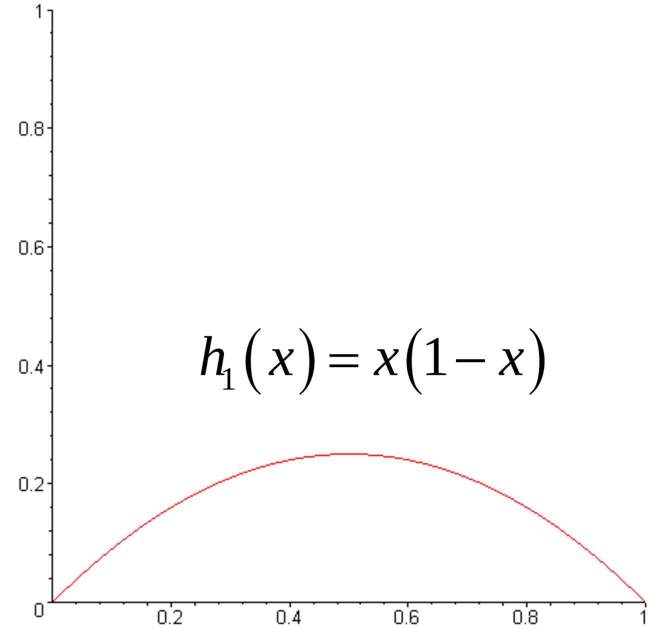
$$h_1(x) = x(1-x) \qquad h_2(x) = \sin(\pi x)$$

We want to minimize $I(y) = \int_0^1 \sqrt{1+\left[y'(x)^2\right]}\,dx$ subject to

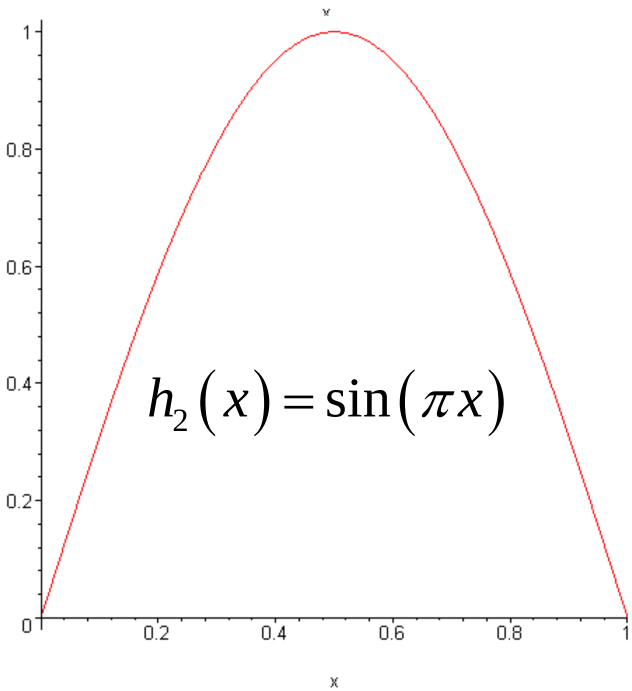
$J(y) = \int_0^1 y(x)\,dx = \frac{1}{2}$ and $y(0) = 0\,,\; y(1) = 1$.



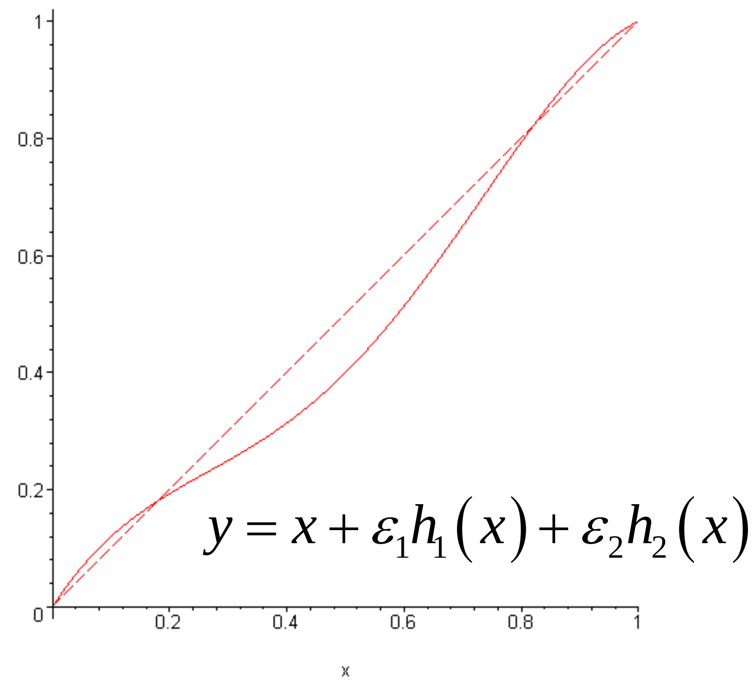
$\bar{y}(x) = x$
0
0.2
0.4
0.6
0.8
1
x



$h_1(x) = x(1-x)$
0
0.2
0.4
0.6
0.8
1



$h_2(x) = \sin(\pi x)$
0
0.2
0.4
0.6
0.8
1
x



$y = x + \varepsilon_1 h_1(x) + \varepsilon_2 h_2(x)$
0
0.2
0.4
0.6
0.8
1
x

$$I(\varepsilon_1,\varepsilon_2)=\int_0^1 \sqrt{1+\left[1+\varepsilon_1(1-2x)+\varepsilon_2\pi\cos(\pi x)\right]^2}\,dx$$

$$J(\varepsilon_1,\varepsilon_2)=\int_0^1 \left[x+\varepsilon_1 x(1-x)+\varepsilon_2\sin(\pi x)\right]dx$$

Let's look at the curve $J(\varepsilon_1,\varepsilon_2)=\frac{1}{2}$.

$$J\left(\varepsilon_1,\varepsilon_2\right)=\frac{12\varepsilon_2+\pi\varepsilon_1+3\pi}{6\pi}$$

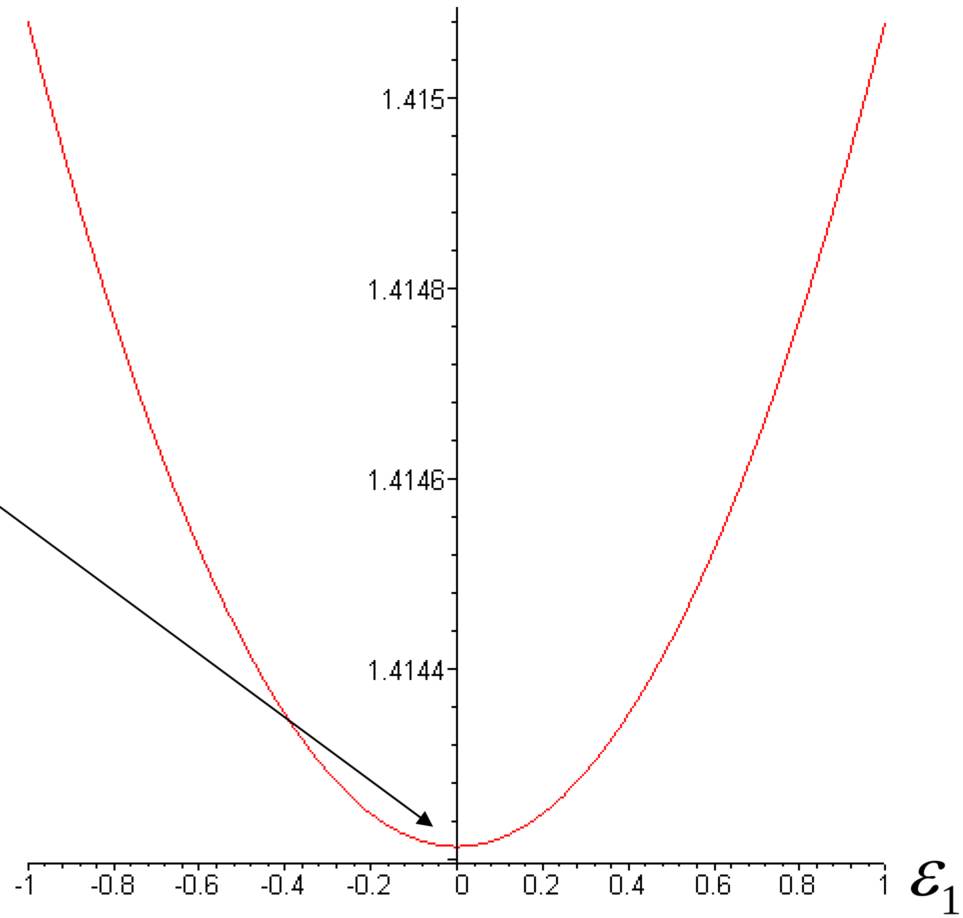
So the constraint curve $J\left(\varepsilon_1,\varepsilon_2\right)=\frac{1}{2}$ is just the line through the origin $\varepsilon_2=-\frac{\pi}{12}\varepsilon_1$. If we substitute this into the formula for I, we get

$$I\left(\varepsilon_1\right)=\int_0^1\sqrt{1+\left[1+\varepsilon_1\left(1-2x\right)-\frac{\pi^2}{12}\varepsilon_1\cos\left(\pi x\right)\right]^2}\,dx$$

Here's a plot of

$$I(\varepsilon_1)=\int_0^1\sqrt{1+\left[1+\varepsilon_1(1-2x)-\frac{\pi^2}{12}\varepsilon_1\cos(\pi x)\right]^2}\,dx$$

The minimum value of $\sqrt{2}$ occurs at $\varepsilon_1=0$. This will be true for all choices of h_1 and h_2.



In general, we won't be able to conveniently solve for one variable in terms of the other(parametrize the constraint curve), so we'll use the Method of Lagrange. The Lagrangian is

$$L(\varepsilon_1,\varepsilon_2,\lambda)=\int_a^b F\left(x,\bar{y}+\varepsilon_1 h_1+\varepsilon_2 h_2,\bar{y}'+\varepsilon_1 h_1'+\varepsilon_2 h_2'\right)dx$$

$$+\lambda\int_a^b G\left(x,\bar{y}+\varepsilon_1 h_1+\varepsilon_2 h_2,\bar{y}'+\varepsilon_1 h_1'+\varepsilon_2 h'\right)dx$$

First, let's abbreviate:

$$L(\varepsilon_1, \varepsilon_2, \lambda) = \int_a^b H\left(x, \overline{y} + \varepsilon_1 h_1 + \varepsilon_2 h_2, \overline{y}' + \varepsilon_1 h_1' + \varepsilon_2 h_2', \lambda\right) dx$$

where

$$\begin{aligned} H\left(x, \overline{y} + \varepsilon_1 h_1 + \varepsilon_2 h_2, \overline{y}' + \varepsilon_1 h_1' + \varepsilon_2 h_2', \lambda\right) = {} & F\left(x, \overline{y} + \varepsilon_1 h_1 + \varepsilon_2 h_2, \overline{y}' + \varepsilon_1 h_1' + \varepsilon_2 h_2'\right) \\ & + \lambda G\left(x, \overline{y} + \varepsilon_1 h_1 + \varepsilon_2 h_2, \overline{y}' + \varepsilon_1 h_1' + \varepsilon_2 h_2'\right) \end{aligned}$$

It must be that $\dfrac{\partial L}{\partial \varepsilon_1} = \dfrac{\partial L}{\partial \varepsilon_2} = 0$ for $\varepsilon_1 = \varepsilon_2 = 0$.

$$\left.\frac{\partial L}{\partial \varepsilon_1}\right|_{(0,0)} = \frac{\partial}{\partial \varepsilon_1}\left.\left[\int_a^b H\left(x, \bar{y}+\varepsilon_1 h_1+\varepsilon_2 h_2, \bar{y}'+\varepsilon_1 h_1'+\varepsilon_2 h_2', \lambda\right) dx\right]\right|_{(0,0)}$$

$$= \int_a^b \left[H_y\left(x, \bar{y}, \bar{y}', \lambda\right) h_1(x) + H_{y'}\left(x, \bar{y}, \bar{y}', \lambda\right) h_1'(x)\right] dx$$

$$\left.\frac{\partial L}{\partial \varepsilon_2}\right|_{(0,0)} = \frac{\partial}{\partial \varepsilon_2}\left.\left[\int_a^b H\left(x, \bar{y}+\varepsilon_1 h_1+\varepsilon_2 h_2, \bar{y}'+\varepsilon_1 h_1'+\varepsilon_2 h_2', \lambda\right) dx\right]\right|_{(0,0)}$$

$$= \int_a^b \left[H_y\left(x, \bar{y}, \bar{y}', \lambda\right) h_2(x) + H_{y'}\left(x, \bar{y}, \bar{y}', \lambda\right) h_2'(x)\right] dx$$

If we integrate by parts and use the arbitrariness of h_1 and h_2, we can conclude that

$$H_y - \left(\frac{d}{dx} H_{y'} \right) = 0$$

is the Euler-Lagrange equation for the Isoperimetric Problem, and solutions which satisfy $y(a) = c$, $y(b) = d$, and

$\int_a^b G(x, y, y')\,dx = k$ will be candidates for maxima or minima.

Let's reexamine the previous example:

Minimize $I(y)=\int_0^1 \sqrt{1+\left[y'(x)^2\right]}dx$ subject to

$J(y)=\int_0^1 y(x)dx=\frac{1}{2}$ and $y(0)=0$, $y(1)=1$.

$$H(x,y,y',\lambda)=\sqrt{1+\left[y'(x)\right]^2}+\lambda y(x)$$

$$H_y=\lambda \qquad H_{y'}=\frac{y'}{\sqrt{1+(y')^2}} \qquad \frac{d}{dx}H_{y'}=\frac{y''}{\left[1+(y')^2\right]^{\frac{3}{2}}}$$

So the Euler-Lagrange equation is

$$\lambda - \frac{y''}{\left[1+\left(y'\right)^2\right]^{\frac{3}{2}}} = 0$$

or

$$\frac{y''}{\left[1+\left(y'\right)^2\right]^{\frac{3}{2}}} = \lambda$$

A solution curve will have constant curvature, so it will be a circular arc or a line segment. Circular arcs through $(0,0)$

and $(1,1)$ won't satisfy $\int_0^1 y(x)\,dx = \frac{1}{2}$.

So a solution must be a line segment through $(0,0)$ and $(1,1)$.

This leads us to $y(x) = x$.

Let's try another one: Find the curve of fixed length $\frac{5}{4}$ that joins the points $(0,0)$ and $(1,0)$, lies above the *x*-axis, and encloses the maximum area between itself and the *x*-axis.

Maximize $\displaystyle\int_0^1 y(x)\,dx$

Subject to $\displaystyle\int_0^1 \sqrt{1+\left[y'(x)\right]^2}\,dx = \frac{3}{2}$, $y(0)=0$ and $y(1)=0$.

$$H(x, y, y', \lambda) = y + \lambda\sqrt{1 + (y')^2}$$

$$H_y = 1 \qquad H_{y'} = \frac{\lambda y'}{\sqrt{1 + (y')^2}} \qquad \frac{d}{dx} H_{y'} = \frac{\lambda y''}{\left[1 + (y')^2\right]^{\frac{3}{2}}}$$

So the Euler-Lagrange equation is

$$1 - \frac{\lambda y''}{\left[1 + (y')^2\right]^{\frac{3}{2}}} = 0$$

or

$$\frac{y''}{\left[1 + (y')^2\right]^{\frac{3}{2}}} = \frac{1}{\lambda}$$

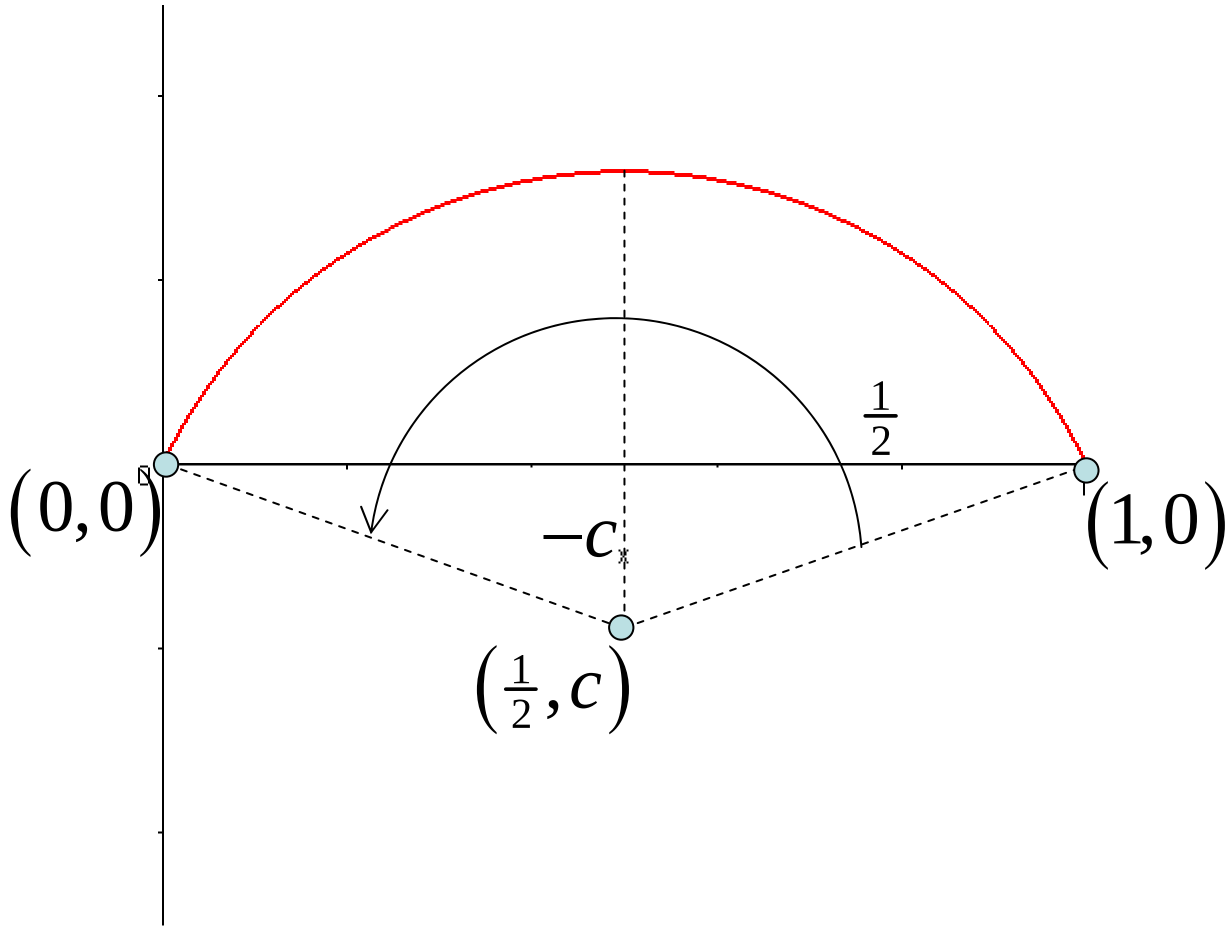
Again, a solution curve must have constant curvature, and so must be a circular arc or a line segment. A line segment through $(0,0)$ and $(1,0)$ would lead to the curve $y=0$, but its length is 1, not $\frac{5}{4}$. So from symmetry a solution curve will

have the form $y=\sqrt{r^2-\left(x-\frac{1}{2}\right)^2}+c$. The conditions

$y(0)=y(1)=0$ imply that $\sqrt{r^2-\frac{1}{4}}+c=0$, or $c=-\sqrt{r^2-\frac{1}{4}}$.

See if you can find the value of *r* so that

$$\int_0^1 \sqrt{1+\left[\frac{-2\left(x-\frac{1}{2}\right)}{\sqrt{r^2-\left(x-\frac{1}{2}\right)^2}}\right]^2}\,dx=\frac{5}{4}$$



$$length=\frac{2\tan^{-1}\left(-\frac{1}{2c}\right)}{2\pi}\cdot 2\pi\sqrt{c^2+\tfrac{1}{4}}=2\tan^{-1}\left(-\frac{1}{2c}\right)\sqrt{c^2+\tfrac{1}{4}}$$

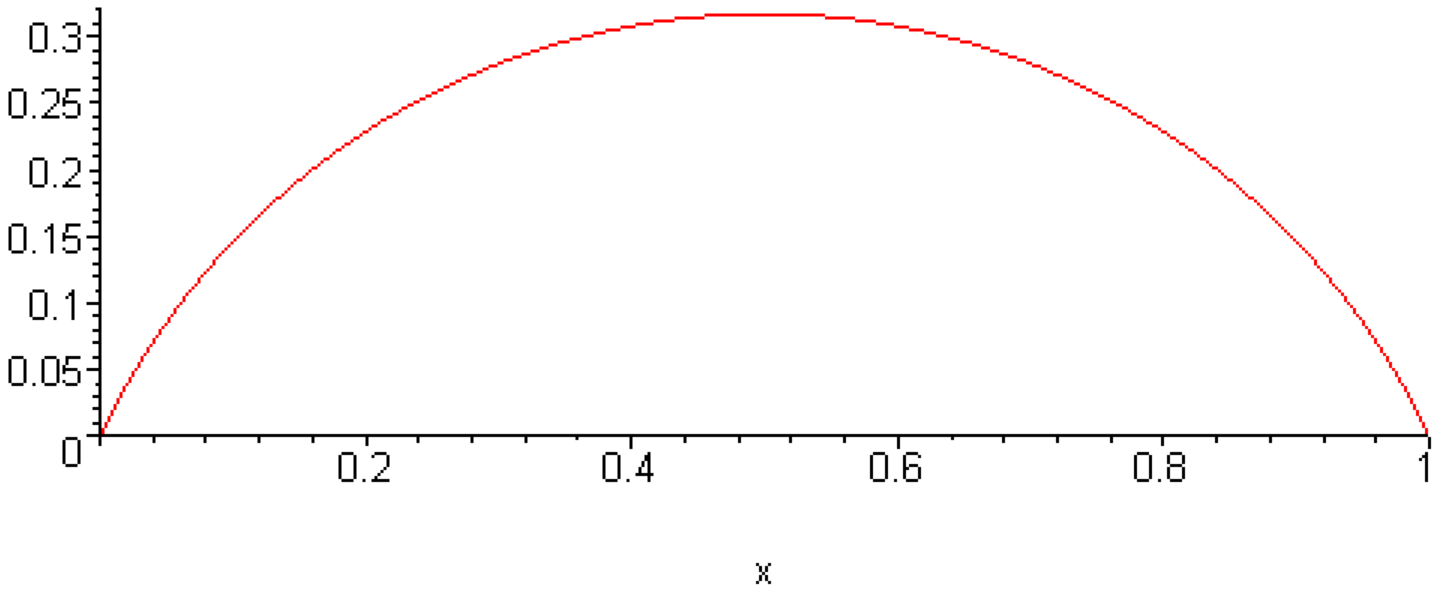
See if you can approximately solve

$$2\tan^{-1}\left(-\frac{1}{2c}\right)\sqrt{c^2+\tfrac{1}{4}}=\frac{5}{4}$$

(See handout, page 24!)

$$c \approx -.2352032224$$

Here's a plot of the solution:



Try to solve the similar problem

Find the curve of fixed length $\frac{\pi}{2}$ that joins the points $(0,0)$ and $(1,0)$, lies above the *x*-axis, and encloses the maximum area between itself and the *x*-axis.

Maximize $\int_0^1 y(x)\,dx$

subject to $\int_0^1 \sqrt{1+[y'(x)]^2}\,dx = \frac{\pi}{2}$, $y(0)=0$ and $y(1)=0$.

(See handout, page 25!)