Parametric Curves, Vectors and Calculus

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Online Masters of Arts in Mathematics at the University of Houston

http://www.math.uh.edu/Matweb/grad_mam.htm

AP Calculus Workshops

(via UH and HACT)

http://www.HoustonACT.org

AP Statistics Listserve

(via UH and HATS)

http://www.HoustonATS.org

Free Online Practice AP Calculus and AP Statistics Exams

(via UH, HACT and HATS)

http://www.HoustonACT.org

http://www.HoustonATS.org

The High School Mathematics Contest at the University of Houston

February 14, 2009

a mathematical love fest :P

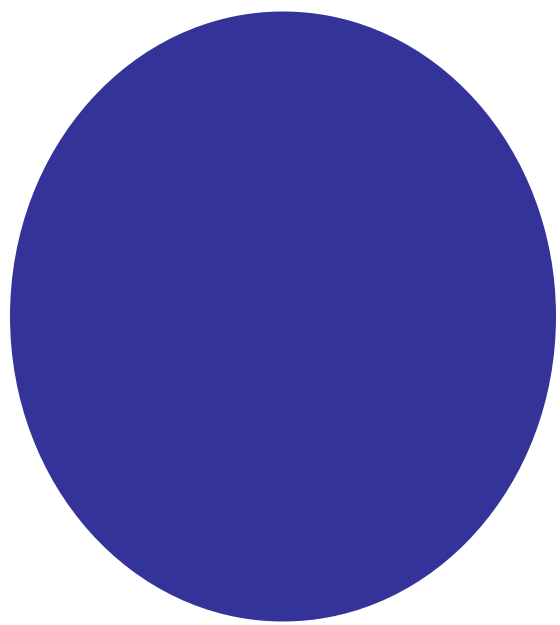
http://mathcontest.uh.edu

Outreach Materials, Including *The Video Calculus Series* (courtesy of UH)

http://online.math.uh.edu

Challenge Question #1

Divide a circular disc into several congruent pieces so that at least one of the pieces does not touch the center of the circle.



Circles and Parametric Equations

- Give a parameterization for the circle of radius 1 centered at the origin in the *xy* plane.
- Give a parameterization for the circle of radius 2 centered at (-3,4) in the *xy* plane.
- Discuss the general process for parameterizing a circle.

Lines and Parametric Equations

- Give a parameterization for the line segment from the point (-3,2) to the point (4,1).
- Give a parameterization for the line passing through the points (3,1) and (3,-7).
- Give a parameterization for the ray initiating at (3,1) and passing through (3,-7).
- Describe the general process for parameterizing line segments, rays and lines.

Parametric Equations for $y = f(x)$

- Give a parameterization for the parabola $y = x^2$.
- Give a parameterization for a portion of the parabola $y = x^2$ from (-2,4) to (3,9).
- Give a parameterization for the graph of $y = \sin(x)$.
- Give a parameterization for a portion of $y = \sin(x)$ from $(-\pi,0)$ to $(\pi/2,1)$.
- Comments…

Important Notes

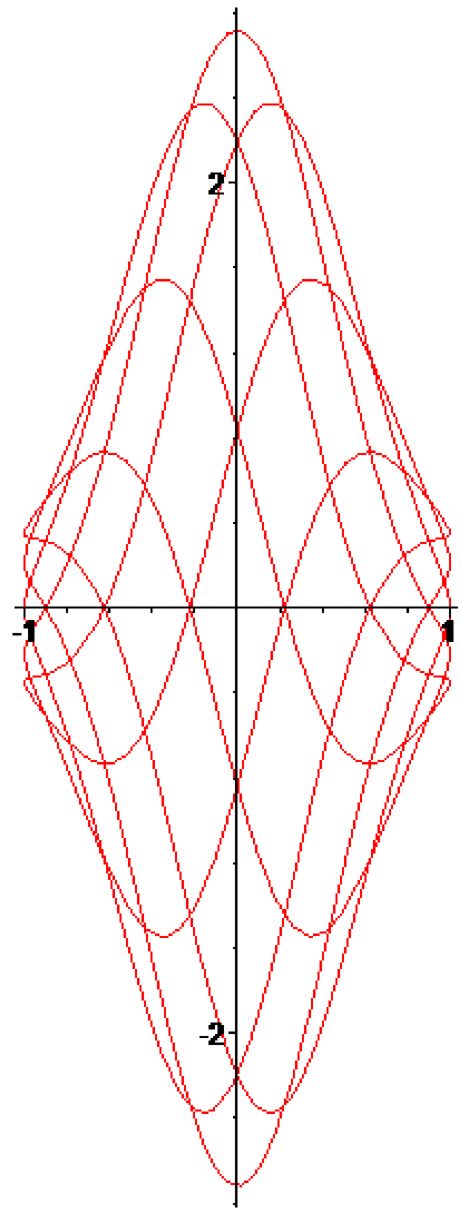
1. A curve can have many different parameterizations.

2. Any given parameterization for a curve places an *orientation* on the curve.

Parametric Curves Can Be Complex!!

$$\left(\sin(5t), \cos(7t)e^{\cos(10t)}\right)$$

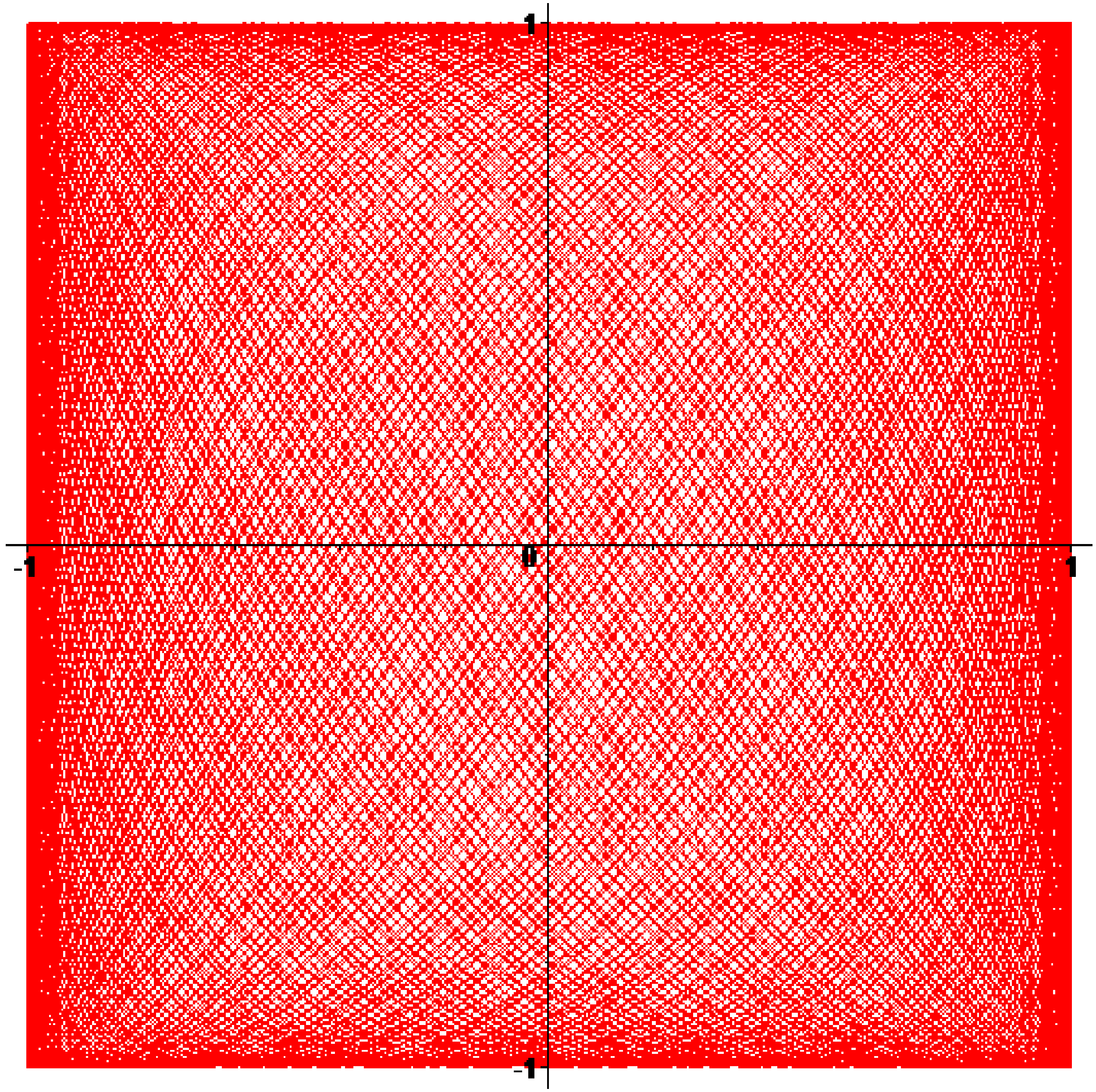
$$\text{for } 0 \le t \le 2\pi$$



$$\left(\sin(t), \cos(\sqrt{2}t)\right)$$

for $0 \le t \le 1000$

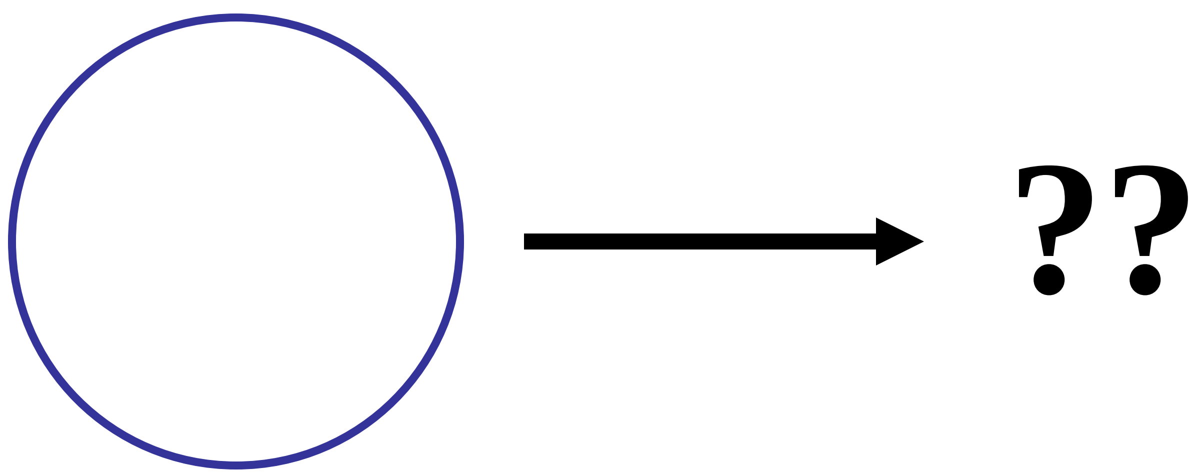
Very Complex!!



Challenge Question #2

If $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$ then describe the curve

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$ and the area it encloses.



Note: You can do much more here than simply give a name for the shape.

What is a Vector in 2 Dimensions?

- Geometric idea…

- Algebraic description…

$$R^2 = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| \, a, b \in R \right\}$$

$$R^2 = \left\{ ai + bj \,\middle|\, a, b \in R \right\}$$

Example

Give 5 different geometric representations of the vector $-2i+3j$.

Addition, Subtraction and Scalar Multiplication of Vectors

- Algebraic

- Geometric

Dot Product and Norm

- Algebraic

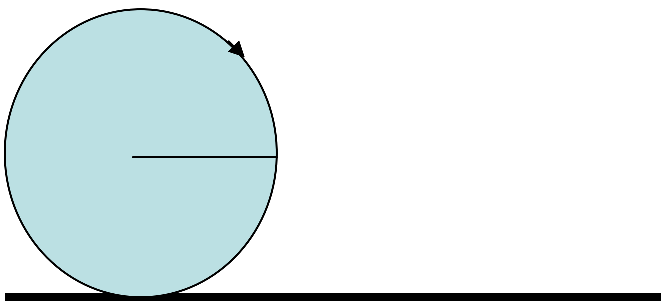
- Geometric

Problem #1

A wheel of radius 1 has a single spoke and sits on a flat road. It rolls forward at the rate of 1 revolution per time unit. Describe the motion both algebraically and geometrically, and use Winplot to create an animation of the motion.

Possible Approach:

1. Describe the motion of the center of the wheel.
2. Use #1 to help describe the motion of the spoke.
3. Create the animation.

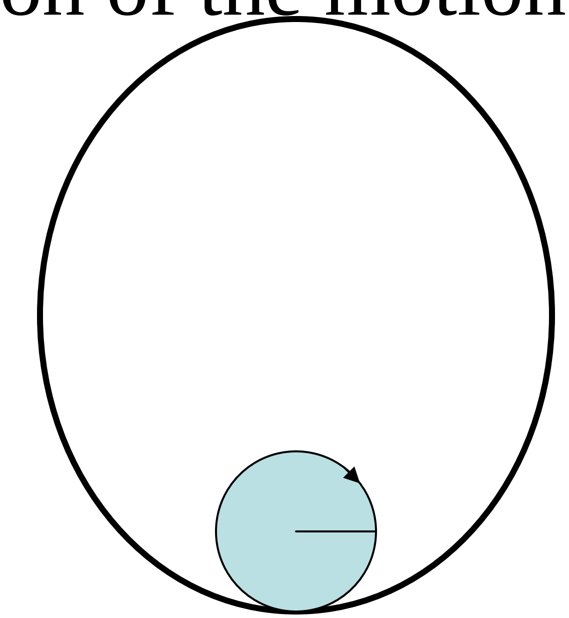


Problem #2

A wheel of radius 1 with a single spoke rotates at one revolution per time unit as it rolls around the inside of a circular track of radius 4 as shown below. Describe the motion both algebraically and geometrically, and use Winplot to create an animation of the motion.

Possible Approach:

1. Describe the motion of the center of the wheel.
2. Use #1 to help describe the motion of the spoke.
3. Create the animation.



Combining Parametric Curves, Vectors and Calculus

Notation: $f : A \to B$

Example:

$$f : [0, 2\pi] \to R^2 \text{ given by } f(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

In General

$$f : I \to R^2 \text{ given by } f(t) = \begin{pmatrix} p(t) \\ q(t) \end{pmatrix}$$

(Here I is an interval in *R*.)

Questions…

$$f : I \to R^2 \text{ given by } f(t) = \begin{pmatrix} p(t) \\ q(t) \end{pmatrix}$$

- How do we take limits of these types of functions?
- How do we differentiate these types of functions?
- How do we integrate these types of functions?

Problem #3

Use Winplot to plot the curve parameterized by

$$f(t) = \begin{pmatrix} \cos(t) \\ \sin(2t) \end{pmatrix},$$ and animate the changes in $f'(\tau)$

at each position $f(\tau)$ along the curve.

In general, what do we expect the derivative to represent in this setting?

Fill in the blanks…

If $f:(a,b)\rightarrow R^2$ is differentiable and $a<\tau<b,$ then

$f'(\tau)$ is ____

to ____

at ____.

Velocity and Speed

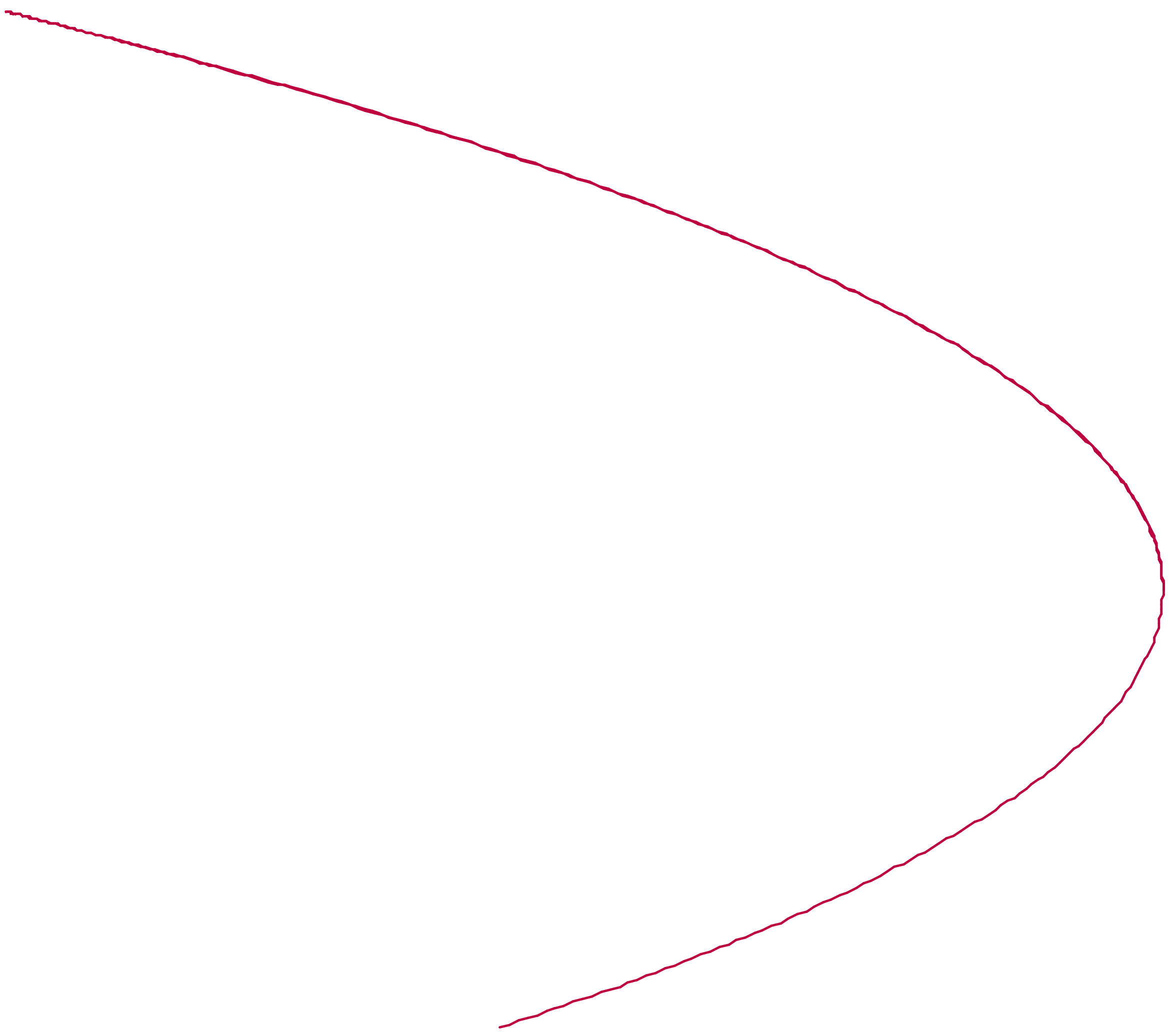
If $r(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ represents the position

of a particle at time t then the velocity of

the particle is $v(t) = r'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$ and the

speed of the partical is $s'(t) = \left| v(t) \right|$.

Principle Unit Tangent



Acceleration - Part I

$$a(t) = v'(t) = r''(t)$$

Acceleration - Part II

$$v(t) = T(t)\left|v(t)\right|$$
$$= T(t)s'(t)$$
$$a(t) = v'(t)$$
$$= T(t)s''(t) + T'(t)s'(t)$$
$$= T(t)s''(t) + \left(\frac{d}{dt}T(t)\right)s'(t)$$
$$= T(t)s''(t) + \left(\frac{d}{ds}T(t)\frac{d}{dt}s(t)\right)s'(t)$$

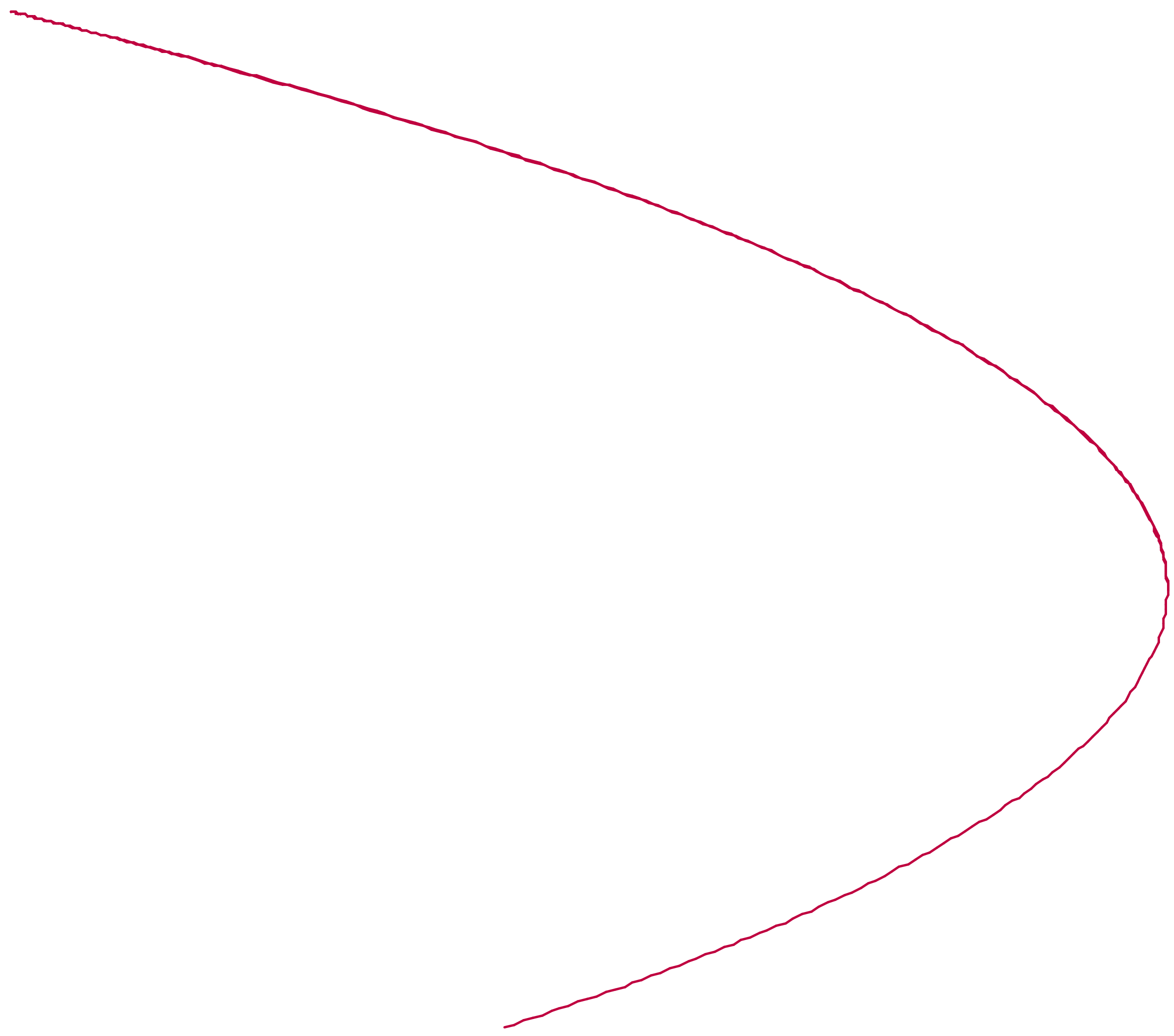
So,

$$a(t) = T(t)s''(t) + \left(\frac{d}{ds}T(t)\right)\left(s'(t)\right)^2$$
$$= T(t)s''(t) + N(t)\left|\frac{d}{ds}T(t)\right|\left(s'(t)\right)^2$$
$$= T(t)s''(t) + N(t)\kappa(t)\left(s'(t)\right)^2$$

Where

$$\kappa(t) = \left|\frac{d}{ds}T(t)\right| \quad \text{and} \quad N(t) = \left(\frac{d}{ds}T(t)\right)\Big/\kappa(t)$$

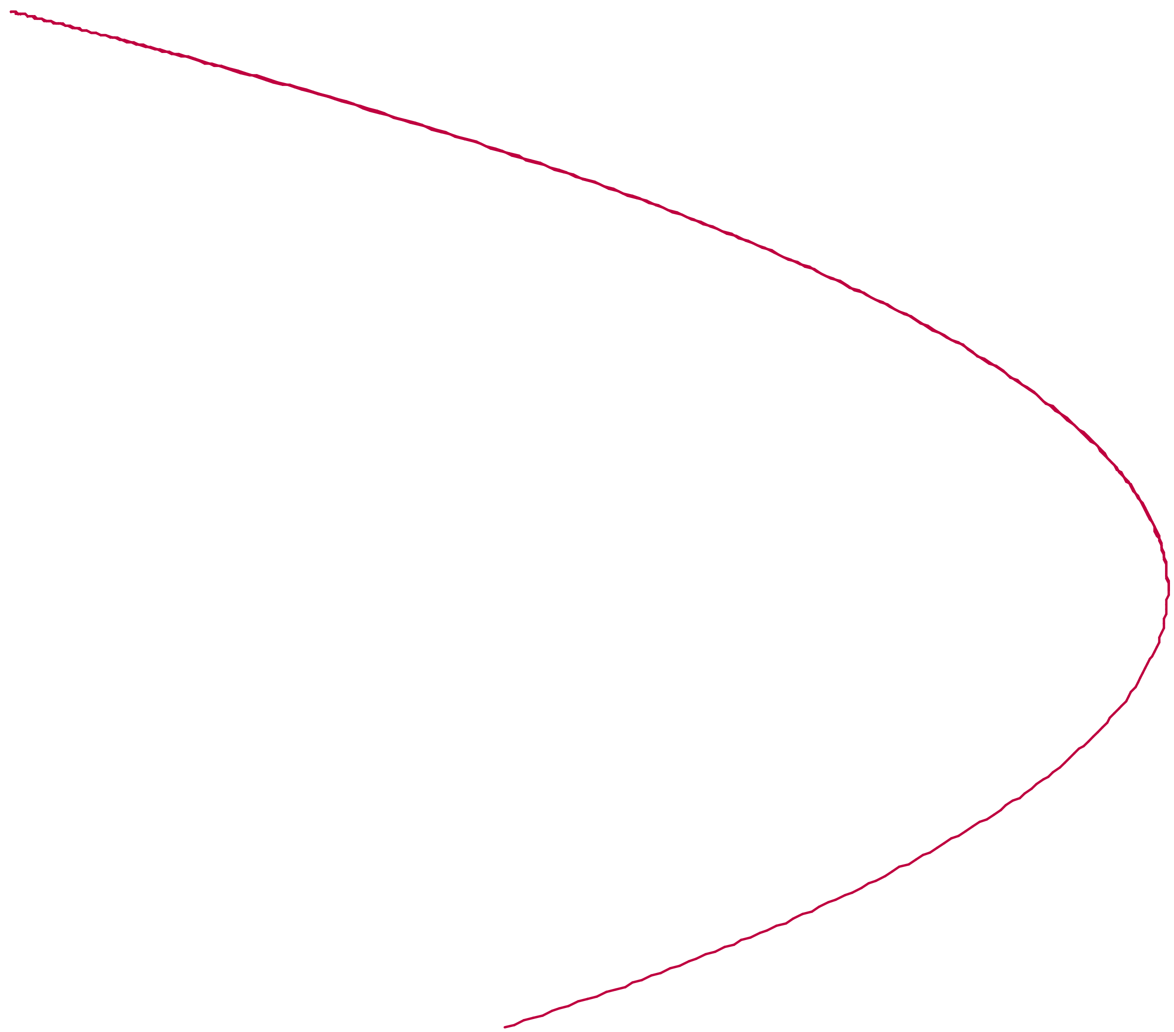
Principle Unit Normal



Speed and Change in Speed
versus
Velocity and Acceleration

Curvature and Circle of Curvature
(algebraic relationship)

Curvature and Circle of Curvature (geometrically)



Acceleration In Terms Of…

- Principle unit tangent
- Principle unit normal
- Speed
- Change in speed
- Curvature
- Circle of curvature

Problem #4

Use Winplot to plot the curve parameterized by

$$f(t) = \begin{pmatrix} 2\cos(2t) \\ 3\sin(t) \end{pmatrix}.$$

Suppose this parameterization gives the position of a particle at time t. Animate the changes in the velocity, acceleration and circle of curvature as the particle moves along the curve.