Math 1432

Exam 4 Review - KEY

1. Determine whether the given series converges or diverges; state which test you are using to determine convergence/divergence and show all work.

   a. $\sum \frac{k^2 2^k}{(k+1)!}$ converge, ratio test

   b. $\sum \frac{3^{k+1}}{(k+1)^2 e^k}$ diverges; root or ratio test.

   c. $\sum \frac{\ln n}{n}$ diverges; basic comparison test or integral test

   d. $\sum \frac{2n+1}{\sqrt{n^5+3n^3+1}}$ converges; limit comparison to $\sum \frac{1}{n^{3/2}}$

   e. $\sum \frac{4n^2+1}{n^3-n}$ diverges; limit comparison to $\sum \frac{1}{n}$

   f. $\sum \frac{4n^2+1}{n^5-n}$ converges; limit comparison to $\sum \frac{1}{n^3}$

   g. $\sum \left(1+\frac{1}{n}\right)^n$ diverges; BDT

   h. $\sum \frac{n^3}{3^n}$ converges; root or ratio test

   i. $\sum \frac{1}{\sqrt[4]{n^3}}$ diverges; p-series

2. Determine if the following series (A) converge absolutely, (B) converge conditionally or (C) diverge.

   a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+3}$ B

   b. $\sum_{n=1}^{\infty} \frac{\cos \pi n}{n^2}$ A

   c. $\sum_{n=0}^{\infty} \frac{4n(-1)^n}{3n^2+2n+1}$ B

   d. $\sum_{n=0}^{\infty} \frac{3(-1)^n}{\sqrt{3n^2+2n+1}}$ B

   e. $\sum_{n=0}^{\infty} \frac{3n(-1)^n}{\sqrt{3n^2+2n+1}}$ C

3. Find the radius of convergence and interval of convergence for the following Power series:

   a. $\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{(n+1)3^{n+1}}$ $R=3, [-1,5)$

   b. $\sum_{n=0}^{\infty} \frac{1}{3^n}(x-1)^n$ $R=3, (-2,4)$

   c. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{4^n}$ $R=4, (-4,4)$

4. Give the derivative of each power series below:

   a. $\sum_{n=0}^{\infty} \frac{(n+1)x^n}{n^2+2}$ $deriv = \sum_{n=0}^{\infty} \frac{n(n+1)x^{n-1}}{n^2+2}$

   b. $\sum_{n=0}^{\infty} \frac{x^n}{2n+1}$ $deriv = \sum_{n=0}^{\infty} \frac{nx^{n-1}}{2n+1}$

5. For each of the problems in number 4, give the antiderivative F of the power series so that F(0)=0.

   a. $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n^2+2}$

   b. $\sum_{n=0}^{\infty} \frac{x^{n+1}}{(2n+1)(n+1)}$

6. Use the Taylor series expansion (in powers of x) for $f(x)=e^x$ to find the Taylor series expansion $f(x)=\cosh x$. $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots=\sum_{k=0}^{n}\frac{x^{2k}}{(2k)!}$

7. Determine the Taylor polynomial in powers of x of degree 8 for the function $f(x)=x-\cos(x^2)$. $-1+x+\frac{x^4}{2!}-\frac{x^8}{4!}$

8. Determine the Taylor polynomial in powers of x of degree 5 for the function $f(x)=\frac{1-e^x}{x}$ $-1-\frac{x}{2!}-\frac{x^2}{3!}-\frac{x^3}{4!}-\frac{x^4}{5!}-\frac{x^5}{6!}$

9. Determine the Taylor polynomial in powers of $x-\pi$ of degree 4 for the function $f(x)=\sin(2x)$. $2(x-\pi)-\frac{4}{3}(x-\pi)^3$

10. Assume that f is a function such that $|f^{(n)}(x)|\le 2$ for all n and x.

    a. Estimate the maximum possible error if $P_4(0.5)$ is used to approximate $f(0.5)$ $\frac{2}{5!}(.5)^5 \approx 0.00052$

b. Find the least integer n for which $P_n(0.5)$ approximates $f(0.5)$ with an error less than 10^{-3}. $n=4$

11. Use the values in the table below and the formula for Taylor polynomials to give the 5th degree Taylor polynomial for f centered at $x = 0$.

$f(0)$	$f'(0)$	$f''(0)$	$f'''(0)$	$f^{(4)}(0)$	$f^{(5)}(0)$
1	0	-2	3	-4	1

$$P_5(x)=1-\frac{2}{2!}x^2+\frac{3}{3!}x^3-\frac{4}{4!}x^4+\frac{1}{5!}x^5=1-x^2+\frac{1}{2}x^3-\frac{1}{6}x^4+\frac{1}{120}x^5$$

12) Write an expression for the nth term of the sequence:

a) 1, 4, 7, 10, … $a_n = 3n - 2$

b) 2, -1, 1/2, -1/4, 1/8,… $a_n = (-1)^{n+1} 2^{2-n}$

13) Determine if the following sequences are (i) monotonic, (ii) bounded. If bounded, give the least upper bound and/or greatest lower bound.

a) $a_n = \dfrac{2n}{1+n}$; monotonic (increasing), bounded, LUB: 2, GLB: 1

b) $a_n = \dfrac{\cos n}{n}$; not monotonic, bounded, LUB: cos(1), GLB: cos(3)/3

c) $a_n = \dfrac{(-1)^{n+1}}{n}$; not monotonic, bounded, LUB: 1, GLB: -1/2

d) $a_n = \dfrac{n^2}{n+1}$; monotonic (increasing), not bounded above, LUB: none, GLB: 1/2.

e) $a_n = \dfrac{n+1}{n!}$; monotonic (decreasig), bounded, LUB: 2, GLB: 0

14) Determine if the following sequences converge or diverge. If they converge, find the limit.

a) $\left\{ \dfrac{(-1)^n n}{n+1} \right\}$; diverges (since oscillates around 1 and -1)

b) $\left\{ \dfrac{6n^2 + n - 1}{4n^2 + 1} \right\}$; converges to 6/4 = 3/2.

c) $\left\{\frac{(n+2)!}{n!}\right\}$; diverges

d) $\left\{\frac{3}{e^n}\right\}$; converges to 0.

e) $\left\{\frac{4n-1}{n^2+1}\right\}$; converges to 0.

f) $\left\{\frac{e^n}{n^3}\right\}$; diverges

g) $\left\{\frac{5^n}{n!}\right\}$; converges to 0.

h) $\left\{\frac{5^n}{\pi^n}\right\}$;diverges (since 5/pi >1)

i) $\left\{\left(\frac{2}{3}\right)^{n+1}\right\}$; converges (since 2/3 <1)

j) $\left\{\frac{n!}{4^{n+1}}\right\}$; diverges

k) $\left\{n^{2/n}\right\}$ converges to 1.

l) $\left\{5n^{-1/n}\right\}$ converges to 5.

m) $\left\{\left(1-\frac{2}{n}\right)^{5n}\right\}$ converges to e^(-10).

n) $\left\{\frac{\ln(n+2)}{5n}\right\}$ converges to 0.

o) $\left\{\ln\left(5n^2+1\right)-\ln\left(n^2+4\right)\right\}$ converges to ln(5).

p) $a_n = \int_{-n}^{n} \frac{1}{4+x^2}\,dx$; converges to pi/2.

15) . The series $4-3+\frac{9}{4}-\frac{27}{16}+\ldots$ is a geometric series. Find the general term, a_n, and write the sum in sigma notation. Does this series converge? If so, what is the sum?

$$\sum_{k=0}^{\infty}\frac{(-1)^k 3^k}{4^{k-1}}$$

Converges (since geometric, with radius = ¾ <1)

Converges to 16/7.

12. Express the following using Sigma notation.

$$4+\frac{4}{3}+\frac{4}{9}+\frac{4}{27}+\ldots=\sum_{k=0}^{\infty}\frac{4}{3^k}$$

$$-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\ldots=\sum_{k=1}^{\infty}(-1)^k\frac{1}{k}$$

$$\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5}+\ldots=\sum_{k=2}^{\infty}\frac{k+1}{k}$$

16) Determine whether the series converge or diverge; justify your answer. If convergent, find the sum (if possible).

a. $\sum_{k=0}^{\infty}\left(-\frac{3}{4}\right)^k$ Converges to 4/7.

b. $\sum_{k=2}^{\infty}5\left(\frac{2}{3}\right)^k$ Converges to 20/3.

c. $\sum_{k=0}^{\infty}\left(\frac{5}{4}\right)^{k-1}$ Divergent

d. $\sum_{n=2}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$ Converges to 5/6.

e. $\sum_{n=3}^{\infty}\left(\frac{1}{n(n+1)}\right)$ Converges to 1/3.

f. $\sum_{k=0}^{\infty}\frac{6^{k+1}}{7^{k-2}}$ Converges to 2058.

g. $\sum\left(1+\frac{1}{n}\right)^n$ Diverges by BDT (since $\lim\left(1+\frac{1}{n}\right)^n=e$, not 0.)

h. $\sum\frac{n^2+1}{4n^2-n}$ Diverges by BDT (since $\lim\left(\frac{n^2+1}{4n^2-n}\right)=\frac{1}{4}$, not 0.)

Determine whether the series converge or diverge; justify your answer.

a. $\sum \left(1+\frac{1}{n}\right)^n$ div. by BDT ($a_n \to e \neq 0$)

b. $\sum \frac{n^2+1}{4n^2-n}$ div. by BDT ($a_n \to 1/4 \neq 0$)

c. $\sum \frac{k^2 2^k}{(k+1)!}$ conv. by ratio test

d. $\sum \frac{3^{k+1}}{(k+1)^2 e^k}$ divergent by ratio test

e. $\sum \frac{\ln n}{n}$ $\sim \sum \frac{1}{n}$ divergent by basic comp. test ($\frac{\ln n}{n} > \frac{1}{n}$ for $n = 3, 4, \ldots$)

f. $\sum \frac{2n+1}{\sqrt{n^5+3n^3+1}}$ $\sim \sum \frac{1}{n^{3/2}}$ conv. by L.C.T

g. $\sum \frac{4n^2+1}{n^3-n}$ $\sim \sum \frac{1}{n}$ div. by L.C.T

h. $\sum \frac{4n^2+1}{n^5-n}$ $\sim \sum \frac{1}{n^3}$ conv. by LCT

i. $\sum \frac{n^3}{3^n}$ conv. by root test

j. $\sum \frac{1}{\sqrt[4]{n^3}}$ p-series, $p = 3/4 < 1$ divergent

k. $\sum \frac{1}{n \ln^2(n)}$ integral test; convergent (done in class)