Math 1314
Week 6 Session Notes

A few remaining examples from Lesson 7:

Example 17: The model $N(t) = 34.4(1+.32125t)^{0.15}$ gives the number of people in the US who are between the ages of 45 and 55. Note, $N(t)$ is given in millions and t corresponds to the beginning of 1995.

(A) How large is this segment of the population projected to be at the beginning of 2011?
(B) How fast will this segment of the population be growing at the beginning of 2011?

Solving problems that involve velocity is a common application of the derivative. Velocity can be expressed using one of these two formulas, depending on whether units are given in feet or meters:

$$h(t) = -16t^2 + v_o t + h_0 \qquad \text{(feet)}$$
$$h(t) = -4.9t^2 + v_o t + h_0 \qquad \text{(meters)}$$

In each formula, v_o is initial velocity and h_o is initial height.

Example 18: Suppose you are standing on the top of a building that is 28 meters high. You throw a ball up into the air, with initial velocity of 10.2 meters per second. Write the equation that gives the height of the ball at time *t*. Then use the equation to find the velocity of the ball when $t = 2$.

Example 19: Suppose you are standing on the top of a building that is 121 feet high. You throw a ball up into the air, with initial velocity of 36 feet per second. Write the equation that gives the height of the ball at time *t*. Then use the equation to find the velocity of the ball when $t = 3$.

We want to be able to find all values of *x* for which the tangent line to the graph of *f* is horizontal. Since the slope of any horizontal line is 0, we'll want to find the derivative, set it equal to zero and solve the resulting equation for *x*.

Example 20: Find all values of *x* for which $f'(x) = 0$: $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 - 7x + 3$

Example 21: Find all points on the graph of $f(x) = 5 - 3x + 2x^2$ where the tangent line is horizontal.

We can also determine values of *x* for which the derivative is equal to a specified number. Set the derivative equal to the given number and solve for *x* either algebraically or by graphing.

Example 22: Find all values of *x* for which $f'(x) = 3$: $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 - 7x + 3$

Higher Order Derivatives

Sometimes we need to find the derivative of the derivative. Since the derivative is a function, this is something we can readily do. The derivative of the derivative is called the **second derivative**, and is denoted $f''(x)$.

To find the second derivative, we will apply whatever rule is appropriate given the first derivative.

Example 23: Find the second derivative when x = -3: $f(x) = 4x^5 - 3x^4 + 2x^2 - 7x + 5$.

Example 24: Find the value of the second derivative when x = 2: $f(x) = 3x^4 - 5x^3 + 7x + 12$

Example 25: Find the value of the second derivative when x = 2.1 if $f(x) = \dfrac{x^2 + 2x - 5}{x^2 + 1}$

Example 26: Find the value of the second derivative when x = 5 if $f(x) = \dfrac{x^2 \ln x}{(x^2 + 3)^{\frac{1}{3}}}$.

Lesson 8: Business Applications: Break Even Analysis, Equilibrium Quantity/Price

Three functions of importance in business are cost functions, revenue functions and profit functions.

Cost functions model the cost of producing goods or providing services. These are often expressed as linear functions, or functions in the form $C(x) = mx + b$. A linear cost function is made up of two parts, fixed costs such as rent, utilities, insurance, salaries and benefits, etc., and variable costs, or the cost of the materials needed to produce each item. For example, fixed monthly costs might be $125,000 and material costs per unit produced might be $15.88, so the cost function could be expressed as $C(x) = 15.88x + 125{,}000$.

Revenue functions model the income received by a company when it sells its goods or services. These functions are of the form $R(x) = xp$, where x is the number of items sold and p is the price per item. Price however is not always static. For this reason, the unit price is often given in terms of a demand function. This function gives the price of the item in terms of the number demanded over a given period of time (week, month or year, for example). No matter what form you find the demand function, to find the revenue function, you'll multiply the demand function by *x*. So if demand is given by $p = 1000 - 0.01x$, the revenue function is given by $R(x) = x(1000 - 0.01x) = 1000x - 0.01x^2$.

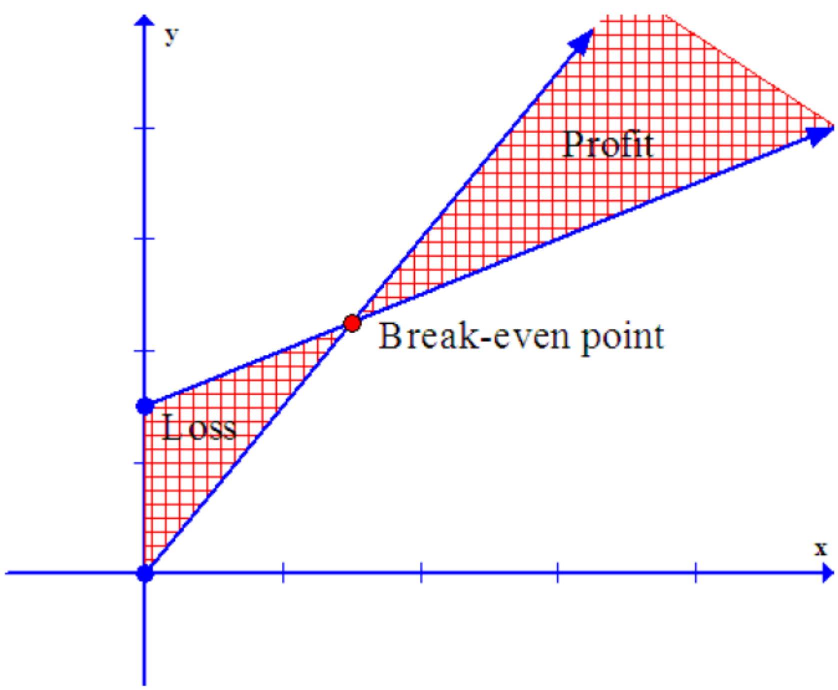
Profit functions model the profits made by manufacturing and selling goods or by providing services. Profits represent the amount of money left over after goods or services are sold and costs are met. We represent the profit as $P(x) = R(x) - C(x)$.

Example 1: Write the cost, revenue and profit functions, given this information:
Suppose a manufacturer has monthly fixed costs of $100,000 and production costs of $12 for each item produced. The item sells for $20.

Example 2: When an item is priced at $100, 500 units are demanded per week. Research shows that for every $20 decrease in price, the quantity demanded will increase by 100 units. Find the demand function.

Break-Even Analysis

The **break-even point** in business is the point at which a company is making neither a profit nor incurring a loss. At the break-even point, the company has met all of its expenses associated with manufacturing the good or providing the service. The *x* coordinate of the break-even point gives the number of units that must be sold to break even. The *y* coordinate gives the revenues at that production and sales level. After the break-even point, the company will make a profit, and that profit will be the difference between its revenues and its costs.



We can find the break-even point algebraically or graphically.

Example 3: Find the break-even point, given this information: Suppose a manufacturer has monthly fixed costs of \$100,000 and production costs of \$12 for each item produced. The item sells for \$20.

Example 4: Suppose a company can model its costs according to the function $C(x) = 0.000003x^3 - 0.04x^2 + 200x + 70,000$ where $C(x)$ is given in dollars and demand can be modeled by $p = -0.02x + 300$. Write the revenue function and find the smallest positive quantity for which all costs are covered.

You may be given raw data concerning costs and revenues. In that case, you'll need to start by finding functions to represent cost and revenue.

Example 5: Suppose you are given the cost data and demand data shown in the tables below. Find a cubic regression equation that models costs, and a quadratic regression equation that models demand. Then find the smallest positive quantity for which all costs are covered.

quantity produced	50	100	200	500	1000	1200	1500	2000
total cost	100400	140100	182000	217400	229300	232600	239300	245500

quantity demanded	50	100	200	500	1000	1200	1500	2000
price in dollars	295	285	280	270	250	248	249	248

Market Equilibrium

The price of goods or services usually settles at a price that is dictated by the condition that the demand for an item will be equal to the supply of the item. If the price is too high, consumers will tend to refrain from buying the item. If the price is too low, manufacturers have no incentive to produce the item, as their profits will be very low.

Market equilibrium occurs when the quantity produced equals the quantity demanded. The quantity produced at market equilibrium is called the **equilibrium quantity** and the corresponding price is called the **equilibrium price.**

Mathematically speaking, market equilibrium occurs at the point where the graph of the supply function and the graph of the demand function intersect. We can solve problems of this type either algebraically or graphically.

Example 6: Suppose that a company has determined that the demand equation for its product is $5x+3p-30=0$ where *p* is the price of the product in dollars when *x* of the product are demanded (*x* is given in thousands). The supply equation is given by $52x-30p+45=0$, where *x* is the number of units that the company will make available in the marketplace at *p* dollars per unit. Find the equilibrium quantity and price.

Example 7: The quantity demanded of a certain electronic device is 8000 units when the price is $260. At a unit price of $200, demand increases to 10,000 units. The manufacturer will not market any of the device at a price of $100 or less. However for each $50 increase in price above $100, the manufacturer will market an additional 1000 units. Assume that both the supply equation and the demand equation are linear. Find the supply equation, the demand equation and the equilibrium quantity and price.

Lesson 9: Marginal Analysis

Sometimes, we are interested in seeing what happens when very small changes are made to the quantities we are studying. This topic is called **marginal analysis**.

We will be looking to see how one quantity – such as cost, revenue or profit – changes when there is a very small change to another quantity – such as the number of items produced and sold.

Earlier in the semester, we saw that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ gives the exact value for the derivative of a function. Now suppose our value for h is a very small number (such as 1); then we can state that $f'(x) \approx \frac{f(x+h) - f(x)}{h}$. So, if x = 100 and h = 1, $f'(100) \approx \frac{f(100+1) - f(100)}{1}$. Simplifying, we have $f'(100) \approx f(101) - f(100)$. To find the value of the function for the 101st item, we can do a lot of arithmetic, or we can just use the derivative. Let's take a look at an example:

Marginal Cost

Sometimes the business owner will want to know how much it costs to produce one more unit of this product. The cost of producing this additional item is called the **marginal cost.**

Example 1: Suppose the total cost in dollars per week by ABC Corporation for producing its best-selling product is given by $C(x) = 10{,}000 + 3000x - 0.4x^2$. Find the *actual cost* of producing the 101st item.

Note that $\frac{C(101) - C(100)}{101 - 100} = \frac{C(x+h) - C(x)}{(x+h) - x}$ where x = 100 and h = 1.

Example 2: Suppose the total cost in dollars per week by ABC Corporation for producing its best-selling product is given by $C(x) = 10{,}000 + 3000x - 0.4x^2$. Find $C'(100)$ and interpret the results.

Note that our results in example 1 and our results in example 2 are extremely close. But the computation using the derivative is usually much simpler and more efficient. For this reason, we will use the derivative to approximate the actual cost of producing the next item. We call the derivative of the cost function the *marginal cost function*, and we will use $C'(x)$ to approximate the actual cost of producing the $(x+1)^{st}$ item.

Example 3: A company produces noise-canceling headphones. Management of the company has determined that the total daily cost of producing *x* headsets can be modeled by the function $C(x) = 0.0001x^3 - 0.03x^2 + 135x + 15{,}000$.

(A) Find the marginal cost function.

(B) Use the marginal cost function to approximate the actual cost of producing the 25^{th} headset.

Marginal Revenue

Similarly, we can study the effect of a small change in the number of sales (i.e., 1) on the revenues of a company. We will use the derivative of the revenue function to as an approximation of the actual revenues realized on the sale of the $(x+1)^{st}$ item. We'll call this derivative the *marginal revenue function.*

Example 5: A company estimates that the demand for its product can be expressed as $p=-.04x+800$, where *p* denotes the unit price and *x* denotes the quantity demanded.

(A) Find the revenue function.

(B) Then find the marginal revenue function.

(C) Use the marginal revenue function to approximate the actual revenue realized on the sale of the 4007th item.

Example 6: The table below gives pricing information about a company's product.

x	200	400	600	800	1000
Price when *x* items are demanded (in dollars)	525	505	491	484	475

(A) Create a list of points and then find the linear demand function.

(B) Find the revenue function.

(C) Use the marginal revenue function to approximate the actual revenue realized on the sale of the 1101st item.

Marginal Profit

The next function of interest is the profit function. As before, we will find the marginal function by taking the derivative of the function, so the **marginal profit function** is the derivative of $P(x)$. This will give us a good approximation of the profit realized on the sale of the $(x + 1)^{st}$ unit of the product.

Example 6: A company estimates that the cost to produce *x* of its products is given by the function $C(x) = 100x + 200,000$ and the demand function is given by $p = 400 - 0.02x$.

(A) Find the profit function.

(B) Use the marginal profit function to compute the actual profit realized on the sale of the 1001^{st} unit and the 2001^{st} unit.

Example 7:   A company estimates that the cost in dollars to produce x of its products is given by the function $C(x) = 0.000003x^3 - 0.08x^2 + 500x + 250000$. The weekly demand for the product is $p = 600 - 0.8x$, where p gives the wholesale unit price in dollars and x denotes the quantity demanded.

(A) Find the profit function.

(B) Find the marginal profit function.

(C) Use the marginal profit function to approximate the actual profit realized on the sale of the 53^{rd} item.

Example 8: The tables below give the costs associated with the production of x items and the pricing information with x items are demanded. Assume that the cost data is exponential and the demand data is linear.

(A) Find the exponential cost function.

(B) Find the linear demand function.

(C) Find the revenue function.

(D) Find the profit function.

(E) Use the marginal profit function to approximate the actual profit realized on the sale of the 651^{st} item.

Average Cost and Marginal Average Cost

Suppose $C(x)$ is the total cost function for producing *x* units of a certain product. If we divide this function by the number of units produced, *x*, we get the **average cost function**. We denote this function by $\overline{C}(x)$. Then we can express the average cost function as $\overline{C}(x) = \frac{C(x)}{x}$. The derivative of the average cost function is called the **marginal average cost.**

We will use the marginal average cost function solely to determine if the average cost is increasing or if it is decreasing.

If $\overline{C}'(a) > 0$, then the average cost function is increasing when $x = a$.
If $\overline{C}'(a) < 0$, then the average cost function is decreasing when $x = a$.

Example 8: A company produces office furniture. Its management estimates that the total annual cost for producing *x* of its top selling executive desks is given by the function $C(x) = 400x + 500{,}000$.

(A) Find the average cost function.

(B) What is the average cost of producing 3000 desks?

(C) Find the marginal average cost function.

(D) Find the marginal average cost when 3000 desks are produced. Is the function increasing or decreasing at this production/sales level?

(E) What happens to $\overline{C}(x)$ when *x* is very large?

Marginal Propensity to Consume/Save

The marginal propensity to consume is the increase (or decrease) in consumption that an economy experiences when income increases (or decreases). The marginal propensity to consume is a measure of what consumers will do when they have additional income. Suppose the marginal propensity to consume is 0.75. Then for every dollar of increase in income, consumers will spend 75 cents and save 25 cents.

The marginal propensity to consume is the derivative of a consumption function $C(x)$ with respect to income and is denoted $\frac{dC}{dx}$.

The flip side of the marginal propensity to consume is the marginal propensity to save. Consider that there are only two possibilities with respect to the additional income – the consumer can either save it or spend it. We can express a propensity to save function as $S(x)$ where x represents income. Then $S(x) = x - C(x)$.

Take the derivative of both sides of the equation $S(x) = x - C(x)$, so $\frac{dS}{dx} = 1 - \frac{dC}{dx}$. Once you know the propensity to consume, you can find the propensity to save, and vice versa.

Example 9: Suppose the consumption function for a country's economy can be modeled by $C(x) = 0.749x^{1.07} + 34.759$, where $C(x)$ and x are both measured in billions of dollars. Find the marginal propensity to consume and the marginal propensity to save when $x = 7$.

Note that $x = 7$ in this problem represents \$7,000,000,000. If we substitute 6 instead of 7 into the derivative, we are actually subtracting \$1 billion. The difference between the final answer using \$7,000,000,000 and using \$6,999,999,999 is negligible, so in the case of marginal propensity to save/consume, we do not reduce the number of interest by 1 befo

Math 1314
Lesson 10: Elasticity of Demand

Suppose you owned a small business and needed to make some decisions about the pricing of your products. It would be helpful to know what effect a small change in price would have on the demand for your product. If a price change will have no real change on demand for the product, it might make good sense to raise the price. However, if a price increase will cause a big drop in demand, then it may not be a good idea to raise prices.

There is a measure of the responsiveness of demand for an object to a change in its price: *elasticity of demand*. This is defined as

$$\frac{\text{percentage change in demand}}{\text{percentage change in price}}.$$

To develop this formula, we'll start by solving our demand function for *x*, so that we have a function $x = f(p)$. Then we have a demand function in terms of price. If we increase the price by *h* dollars, then the price is $p+h$ and the quantity demanded is $f(p+h)$.

The percentage change in demand is $100 \cdot \frac{f(p+h)-f(p)}{f(p)}$ and the percentage change in price is $100 \cdot \frac{h}{p}$.

If we compute the ratio given above, we have $\dfrac{100 \cdot \frac{f(p+h)-f(p)}{f(p)}}{100 \cdot \frac{h}{p}}$.

We can simplify this to

$$\frac{p}{f(p)}\left[\frac{f(p+h)-f(p)}{h}\right].$$

For small values of h, $\frac{f(p+h)-f(p)}{h} \approx f'(p)$, so we have $\frac{p \cdot f'(p)}{f(p)}$.

This quantity is almost always negative, and it is easier to work with a positive value, so we define the negative of this ratio to be the elasticity of demand.

Then

$E(p) = -\frac{p \cdot f'(p)}{f(p)}$, where *p* is price in dollars, $f(p)$ is the demand function and $f(p)$ is differentiable when $x = p$.

Revenue responds to elasticity in the following manner:

If demand is elastic at *p*, then

- An increase in unit price will cause revenue to decrease or
- A decrease in unit price will cause revenue to increase

If demand is unitary at *p*, then

- An increase in unit price will cause the revenue to stay about the same.

If demand is inelastic at *p*, then

- An increase in the unit price will cause revenue to increase
- A decrease in unit price will case revenue to decrease.

We have these generalizations about elasticity of demand:

Demand is said to be elastic if $E(p) > 1$.
Demand is said to be unitary if $E(p) = 1$.
Demand is said to be inelastic if $E(p) < 1$.

So, if demand is elastic, then the change in revenue and the change in price will move in opposite directions.

If demand is inelastic, then the change in revenue and the change in price will move in the same direction.

Example 1: Find $E(p)$ for the demand function $x + 2p - 15 = 0$ and determine if demand is elastic, inelastic, or unitary when $p = 4$.

Example 2: Suppose the demand function for a product is given by $p = -0.02x + 400$. This function gives the unit price in dollars when x units are demanded.

(A) Find the elasticity of demand.

(B) Find $E(100)$ and interpret the results. If the unit price is \$100, will raising the price result in an increase in revenues or a decrease in revenues?

(C) Find $E(300)$ and interpret the results. If the unit price is \$300, will raising the price result in an increase in revenues or a decrease in revenues?

What else can $E(p)$ tell you?

Example 4: If $E(p) = \frac{1}{2}$ when $p = 250$, what effect will a 1% increase in price have on revenue?

Example 5: If $E(p) = \frac{3}{2}$ when $p = 250$, what effect will a 1% increase in price have on revenue?