Math 1314 ONLINE

More from Lesson 6 – The Limit Definition of the Derivative and Rules for Finding Derivatives.

$$f(x) = 3x^2 - 7x + 4$$

Four steps

$$f(x+h) = 3(x+h)^2 - 7(x+h) + 4$$
$$= 3(x+h)(x+h) - 7x - 7h + 4$$
$$= 3(x^2 + xh + xh + h^2) - 7x - 7h + 4$$
$$= 3(x^2 + 2xh + h^2) - 7x - 7h + 4$$
$$= 3x^2 + 6xh + 3h^2 - 7x - 7h + 4$$

$$f(x+h) - f(x) = 3x^2 + 6xh + 3h^2 - 7x - 7h + 4 - (3x^2 - 7x + 4)$$
$$= \cancel{3x^2} + 6xh + 3h^2 - \cancel{7x} - 7h + \cancel{4} - \cancel{3x^2} + \cancel{7x} - \cancel{4}$$
$$= 6xh + 3h^2 - 7h$$

$$\frac{f(x+h) - f(x)}{h} = \frac{6xh + 3h^2 - 7h}{h}$$
$$= \frac{\cancel{h}(6x + 3h - 7)}{\cancel{h}}$$
$$= 6x + 3h - 7$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (6x + 3h - 7)$$
$$6x - 7$$

$$f'(x) = 6x - 7$$

Example 4: Use the Four-Step Process for finding the derivative of the function $f(x) = \frac{4}{x}$. Then find $f'(1)$.

$$f(x+h) = \frac{4}{x+h}$$

LCD

$$f(x+h) - f(x) = \frac{x}{x}\frac{4}{x+h} - \frac{4}{x}\frac{x+h}{x+h} = \frac{4x}{x(x+h)} - \frac{4(x+h)}{x(x+h)}$$

$$= \frac{4x - 4(x+h)}{x(x+h)} = \frac{\cancel{4x} - \cancel{4x} - 4h}{x(x+h)}$$

$$= \frac{-4h}{x(x+h)}$$

$$f'(x) = \frac{-4}{x^2}$$

$$f'(1) = \frac{-4}{1^2} = -4$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{-4h}{x(x+h)}}{h} = \frac{-4h}{x(x+h)} \div h$$

$$= \frac{-4\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}} = \frac{-4}{x(x+h)}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-4}{x(x+h)} = \frac{-4}{x(x+0)} = \frac{-4}{x^2}$$

POPPER 5, Problem 1:

Using the function shown below, find $\lim_{x \to 1^-} f(x)$

$$f(x) = \begin{cases} x+1, & x<1 \\ x^2+4, & x \geq 1 \end{cases}$$

$1+1=2$

A. dne B. 5 C. 2

We can use the ***average rate of change*** to approximate the slope of the tangent line. The average rate of change is given by the difference quotient, $\frac{f(x+h)-f(x)}{h}$. We can also write this formulas as $\frac{f(b)-f(a)}{b-a}$ where a and b are the endpoints of a closed interval $[a, b]$.

Example 5: Suppose $f(x) = 3x^2 - 4x - 1$. Find the average rate of change of f over the interval [2, 5].

a b

$$\frac{f(5)-f(2)}{5-2}$$

$f(5) = 54$

$f(2) = 3$

$5-2=3$

$$\frac{54-3}{3}$$

$$\frac{51}{3} = 17$$

Example 6: Suppose the distance covered by a car can be measure by the function $s(t) = 4t^2 + 32t$, where $s(t)$ is given in feet and t is measured in seconds.

AROC A. Find the average velocity of the car over the interval [0, 4]. velocity is a rate of change

derivative B. Find the instantaneous velocity of the car when $t = 4$.

A. $\dfrac{s(4) - s(0)}{4 - 0}$

$s(4) = 4 \cdot 4^2 + 32(4)$
$= 64 + 128$
$= 192$

$s(0) = 4(0)^2 + 32(0)$
$= 0$

$4 - 0 = 4$

$\dfrac{192 - 0}{4} = 48$

$\boxed{48 \text{ ft / sec}}$

B. $s(t+h) = 4(t+h)^2 + 32(t+h)$
$= 4(t+h)(t+h) + 32(t+h)$
$= 4(t^2 + th + th + h^2) + 32t + 32h$
$= 4(t^2 + 2th + h^2) + 32t + 32h$
$= 4t^2 + 8th + 4h^2 + 32t + 32h$

$s(t+h) - s(t) = 4t^2 + 8th + 4h^2 + 32t + 32h - (4t^2 + 32t)$

$= \cancel{4t^2} + 8th + 4h^2 + \cancel{32t} + 32h - \cancel{4t^2} - \cancel{32t}$

$= 8th + 4h^2 + 32h$

$\dfrac{s(t+h) - s(t)}{h} = \dfrac{8th + 4h^2 + 32h}{h} = \dfrac{\cancel{h}(8t + 4h + 32)}{\cancel{h}}$

$\lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h} = \lim\limits_{h \to 0} (8t + 4h + 32) = 8t + 32$

$s'(t) = 8t + 32$

$s'(4) = 8(4) + 32 = 64$

$\boxed{64 \text{ ft per sec}}$

$$\frac{f(x+h)-f(x)}{h}$$

POPPER 5, problem 2:

Suppose $f(x)=x^2-3x+2$.

Find the difference quotient.

A. $2xh+h^2-3h$ B. $2x+h-3$ C. $2x-3$

POPPER 5, problem 3:

Suppose $f'(x)=2x-3$. Find $f'(-1)$.

$f'(-1)=2(-1)-3=-2-3=-5$

A. Not enough information is given to answer. B. -1 C. -5 D. -2

$$f(x+h)=(x+h)^2-3(x+h)+2$$
$$=(x+h)(x+h)-3x-3h+2$$
$$=x^2+2xh+h^2-3x-3h+2$$
$$f(x+h)-f(x)=x^2+2xh+h^2-3x-3h+2-(x^2-3x+2)$$
$$=\cancel{x^2}+2xh+h^2-\cancel{3x}-3h+\cancel{2}-\cancel{x^2}+\cancel{3x}-\cancel{2}$$
$$=2xh+h^2-3h$$
$$\frac{f(x+h)-f(x)}{h}=\frac{2xh+h^2-3h}{h}=\frac{\cancel{h}(2x+h-3)}{\cancel{h}}$$
$$=2x+h-3$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}(2x+h-3)=2x-3$$

Rules for Finding Derivatives

We can use the limit definition of the derivative to find the derivative of every function, but it isn't always convenient. Fortunately, there are some rules for finding derivatives which will make this easier.

First, a bit of notation:

$\frac{d}{dx}[f(x)]$ is a notation that means "the derivative of *f* with respect to *x*, evaluated at *x*."

Rule 1: The Derivative of a Constant

$\frac{d}{dx}[c] = 0,$ where *c* is a constant.

Example 1: If $f(x) = 12,$ find $f'(x)$.

$f'(x) = 0$ b/c derivative of any constant is zero

Rule 2: The Power Rule

$f(x) = x^2$
$f'(x) = 2x^1$
$= 2x$

$\frac{d}{dx}[x^n] = nx^{n-1}$ for any real number *n*

Example 2: If $f(x) = x^4,$ find $f'(x)$.

$f'(x) = 4x^{4-1} = \boxed{4x^3}$

Example 3: If $f(x) = \sqrt[3]{x}$, find $f'(x)$.

$\frac{d}{dx}[x^n] = nx^{n-1}$

Rewrite $f(x) = x^{1/3}$

Use rule $f'(x) = \frac{1}{3}x^{1/3-1}$

$f'(x) = \frac{1}{3}x^{-2/3}$

$= \frac{1}{3x^{2/3}}$

$\boxed{f'(x) = \frac{1}{3x^{2/3}}}$

Rewrite final answer with no negative exponents

$\frac{d}{dx}[x^n] = nx^{n-1}$

Example 4: If $f(x) = \frac{1}{x^5}$, find $f'(x)$.

Rewrite $f(x) = x^{-5}$

Use rule $f(x) = -5x^{-6} = \frac{-5}{x^6}$

Rewrite w/ no negative exponent

POPPER 5, problem 4:

$\frac{d}{dx}[x^n] = nx^{n-1}$

Suppose $f(x) = x^7$. Find $f'(x)$

A. $f'(x) = x^6$ B. $f'(x) = 7x^6$ C. $f'(x) = 7x^8$

$\boxed{f'(x) = 7x^6}$

Rule 3: Derivative of a Constant Multiple of a Function

$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$ where c is any real number

$\frac{d}{dx}[x^3] = 3x^2$

Example 5: If $f(x) = 6x^3$, find $f'(x)$.

$\frac{d}{dx}[6x^3] = 6\frac{d}{dx}[x^3]$

$= 6 \cdot 3x^2$

$= 18x^2$

$f(x) = 6x^3$

$\boxed{f'(x) = 18x^2}$

Example 6: If $f(x) = -3\sqrt{x}$, find $f'(x)$.

Rewrite $f(x) = -3x^{1/2}$

Use rule $f'(x) = -\frac{3}{2}x^{-1/2}$

$x^{-1/2} = \frac{1}{x^{1/2}} = \frac{1}{\sqrt{x}}$

$-3 \cdot \frac{1}{2} = -\frac{3}{2}$

$\boxed{f'(x) = \frac{-3}{2x^{1/2}} = \frac{-3}{2\sqrt{x}}}$

Rule 4: The Sum/Difference Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

Example 7: Find the derivative: $f(x) = 5x^4 + 3x^3 - 4 - \frac{6}{x^1}$.

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}[5x^4] + \frac{d}{dx}[3x^3] - \frac{d}{dx}[4] - \frac{d}{dx}[6x^{-1}]$$

$$= 20x^3 + 9x^2 - 0 - (-6x^{-2})$$

$$= \boxed{20x^3 + 9x^2 + \frac{6}{x^2}}$$

Example 8: Find the derivative: $f(x) = 4x^3 - 7x^2 + 8x - 5$

$$\boxed{f'(x) = 12x^2 - 14x + 8}$$

$$\frac{d}{dx}[8x^1] =$$

$$8x^0 = 8 \cdot 1 = 8$$

$$\frac{d}{dx}[ax] = a$$

Note, there are many other rules for finding derivatives "by hand." We will not be using those in this course. Instead, we will use GeoGebra for finding more complicated derivatives.

$\frac{1}{3} \cdot 3 = ?$

POPPER 5, problem 5

Suppose $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x - 5$. Find $f'(x)$.

A. $f'(x) = x^2 + x + 6$ B. $f'(x) = x^2 + x + 1$ C. $f'(x) = \frac{1}{9}x^2 + \frac{1}{4}x + 6$

D. $f'(x) = x^2 + x - 6$ E. $f'(x) = \frac{1}{3}x^2 + \frac{1}{2}x + 1$

$$\boxed{f'(x) = x^2 + x + 6}$$

Lesson 7: Derivatives at a Point and Numerical Derivatives

Many of our problems will ask for the rate at which something is changing at a specific number. The number may express time, or quantity produced and sold, or many other quantities. To find this rate, find the derivative and substitute the desired number into the derivative and evaluate. State your answer using the correct units.

Example 1: Find the derivative when $x = 3$: $f(x) = 3x^3 - 8x + 9$

$f'(x) = 9x^2 - 8$

$f'(3) = 9 \cdot 3^2 - 8 = 9 \cdot 9 - 8 = 81 - 8 = \boxed{73}$

Example 2: The height of a rocket can be modeled by the function $h(t) = -16t^2 + 48t + 6$ where $h(t)$ gives the height in feet at time t given in seconds. At what rate is the height changing when $t = 1$?

ROC $t = 1$

$h'(1)$

$h'(t) = -32t + 48$

$h'(1) = -32(1) + 48 = 16$

$\boxed{16 \text{ ft} + 1 \text{ sec}}$

We can find numerical derivatives using GeoGebra.

What is the command?

Example 3: Find the derivative when $x = 3$: $f(x) = 3x^3 - 8x + 9$

$f'(3)$

$f'(\#)$

Derivative [function]

$f'(\#)$

Example 4: Find the numerical derivative of $f(x) = 5x\sqrt{2x^2 + 3x + 2}$ at the point (2, 40).

$f'(2) = 33.75$

Example 5: Find the numerical derivative of $f(x) = x^{\frac{3}{2}} - x$ when x = 4.

Example 6: Find the numerical derivative of $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}$ at x = 2.

Example 7: Find the numerical derivative of $f(x) = \dfrac{x^2 - 6x + 5}{x^2 - 3x + 2}$ at x = 4.

Example 8: Find the numerical derivative of $C(x) = 0.000003x^3 - 0.04x^2 + 200x + 70000$ when x = 2500.

Example 9: Find the numerical derivative of $C(x) = \dfrac{2x - 3e^x}{x^2 - 4x - 8}$ when $x = 2.5$

Some Applications of the Derivative

Now we'll start looking at the kinds of problems we can work using derivatives.

Example 10: Find the slope of the tangent line when x = -1 if $f(x) = \dfrac{x\sqrt{x+2}}{(2x+3)^2}$.

Example 11: Write an equation of the line that is tangent to $f(x) = \dfrac{1}{3}x^3 - 2x^2 + 7x$ when $x = 3$.

Example 12: Write an equation of the line that is tangent to $f(x) = 1.6x^3 - 2.31x^2 + 6.39x - 2.81$ when $x = 3.75$.

With this group of word problems, the first step will be to determine what kind of problem we have for each problem. Does it ask for a function value (FV), a rate of change (ROC) or an average rate of change (AROC). From there, we'll apply the appropriate methods.

Example 13: A country's gross domestic product (in millions of dollars) is modeled by the function $G(t) = -2t^3 + 45t^2 + 20t + 6000$ where $0 \le t \le 11$ and $t = 0$ corresponds to the beginning of 1997.

(A) At what rate was GDP changing at the beginning of 2002? At the beginning of 2004? At the beginning of 2009?
(B) What was the average rate of growth of the GDP over the period 1999 – 2004?

Example 14: Data from a homebuilders association shows that the average size of a single family home built after 1970 can be modeled by the function $S(t) = 0.1666t^2 + 29.98t + 1510$ where t is the number of years since 1970. Was the average size of a new single family home growing faster at the end of 1989 or at the end of 1999?

Example 15: According to the US Department of Energy, a typical car's fuel economy depends on the speed it is driven. Fuel economy is approximated by the function $f(x) = 0.00000310315x^4 - 0.000455174x^3 + 0.00287869x^2 + 1.25986x$ where x is measured in miles per hour and $f(x)$ is measured in miles per gallon.

Find the rate of change of f when $x = 20$ and when $x = 50$ and interpret the results.

Example 16: A secretarial school knows that the average student taking an advance typing class will progress according to the rule $N(t) = \dfrac{60t + 180}{t + 6}$, $t \geq 0$, where $N(t)$ measures the number of words per minute that a student can type *t* weeks into the program.

(A) At what rate will the average student's speed in words per minute be changing at the end of the first, third, fourth and seventh weeks of a twelve week course?
(B) What will be the average student's typing speed at the beginning of a 12 week course?
(C) What will be the average student's typing speed at the end of a 12 week course?
(D) What will be the average student's average rate of change during a 12 week course?

Example 17: The model $N(t) = 34.4(1 + .32125t)^{0.15}$ gives the number of people in the US who are between the ages of 45 and 55. Note, $N(t)$ is given in millions and *t* corresponds to the beginning of 1995.

(A) How large is this segment of the population projected to be at the beginning of 2011?
(B) How fast will this segment of the population be growing at the beginning of 2011?

Solving problems that involve velocity is a common application of the derivative. Velocity can be expressed using one of these two formulas, depending on whether units are given in feet or meters:

$$h(t) = -16t^2 + v_o t + h_0 \qquad \text{(feet)}$$
$$h(t) = -4.9t^2 + v_o t + h_0 \qquad \text{(meters)}$$

In each formula, v_o is initial velocity and h_o is initial height.

Example 18: Suppose you are standing on the top of a building that is 28 meters high. You throw a ball up into the air, with initial velocity of 10.2 meters per second. Write the equation that gives the height of the ball at time *t*. Then use the equation to find the velocity of the ball when $t = 2$.

Example 19: Suppose you are standing on the top of a building that is 121 feet high. You throw a ball up into the air, with initial velocity of 36 feet per second. Write the equation that gives the height of the ball at time *t*. Then use the equation to find the velocity of the ball when $t = 3$.

We want to be able to find all values of *x* for which the tangent line to the graph of *f* is horizontal. Since the slope of any horizontal line is 0, we'll want to find the derivative, set it equal to zero and solve the resulting equation for *x*.

Example 20: Find all values of *x* for which $f'(x) = 0$: $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 - 7x + 3$

Example 21: Find all points on the graph of $f(x) = 5 - 3x + 2x^2$ where the tangent line is horizontal.

We can also determine values of *x* for which the derivative is equal to a specified number. Set the derivative equal to the given number and solve for *x* either algebraically or by graphing.

Example 22: Find all values of *x* for which $f'(x) = 3$: $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 - 7x + 3$

Higher Order Derivatives

Sometimes we need to find the derivative of the derivative. Since the derivative is a function, this is something we can readily do. The derivative of the derivative is called the **second derivative**, and is denoted $f''(x)$.

To find the second derivative, we will apply whatever rule is appropriate given the first derivative.

Example 23: Find the second derivative when x = -3: $f(x) = 4x^5 - 3x^4 + 2x^2 - 7x + 5$.

Example 24: Find the value of the second derivative when x = 2: $f(x) = 3x^4 - 5x^3 + 7x + 12$

Example 25: Find the value of the second derivative when x = 2.1 if $f(x) = \dfrac{x^2 + 2x - 5}{x^2 + 1}$

Example 26: Find the value of the second derivative when x = 5 if $f(x) = \dfrac{x^2 \ln x}{(x^2 + 3)^{\frac{1}{3}}}$.