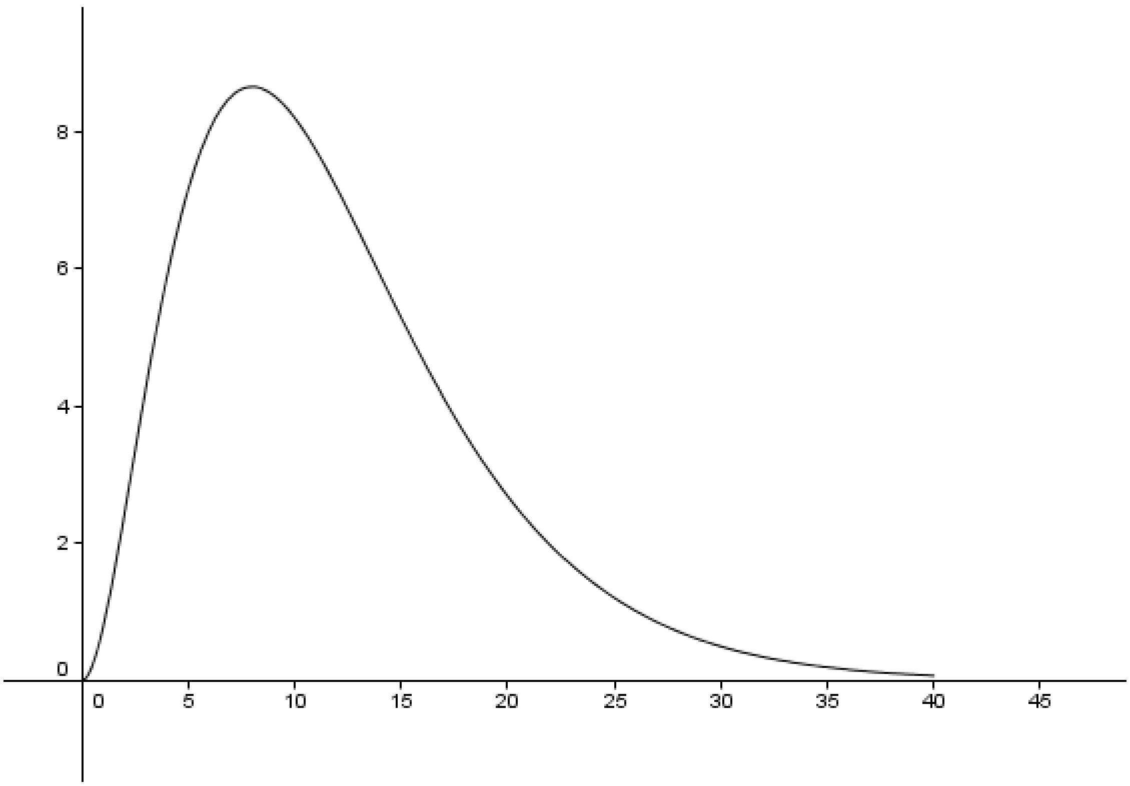
Math 1314 ONLINE
Lesson 12

This lesson will cover analyzing polynomial functions using GeoGebra.

Suppose your company embarked on a new marketing campaign and was able to track sales based on it. The graph below gives the number of sales in thousands shown t days after the campaign began.



Now, suppose you are assigned to analyze this information. For example, you can determine the maximum number of sales and when it occurred. Notice that the number of sales drops off dramatically from that point. Once you know when the maximum occurred and the maximum number of sales, you can try to determine why this happened. Did consumers dislike the product? Was there a problem with the product? Was production not able to keep up with demand? Was there an external factor (e.g., natural disaster) that kept consumers from buying the product?

We can use calculus to answer the following questions:

Where is the function increasing? Where is the function decreasing? (Note that the graph stops at $t = 40$.)

Where does the maximum occur?

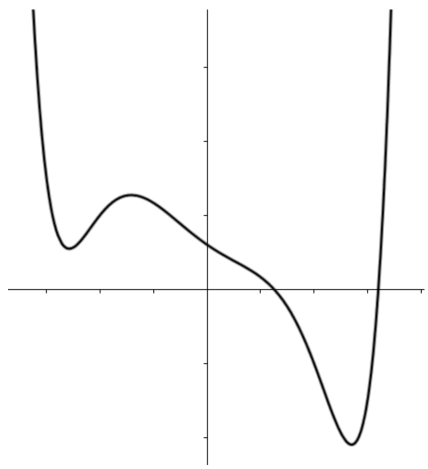
What is the maximum number of sales?
Where does the growth rate change?

Calculus can't answer the "why" questions, but it can give you some information you need to start that inquiry.

Analyzing Polynomial Functions

There are many features of a polynomial function that we can find using techniques and information from College Algebra.

A polynomial function is a function like $f(x) = x^8 - 7x^4 + 3x^2 - 5x + 3$. The graph of the function is shown below.



The domain of any polynomial function is $(-\infty, \infty)$.
Polynomial functions have no asymptotes.
We can find the *y* intercept by finding $f(0)$.
We can find the *x* intercepts (roots, zeros) by setting the function equal to zero and solving for *x*.

Example 1: Suppose $f(x) = x^3 - 3x^2 - 13x + 15$. Find any *x* intercepts and any *y* intercepts.

Intervals on Which a Function is Increasing/Decreasing

Definition: A function is **increasing** on an interval (*a*, *b*) if, for any two numbers x_1 and x_2 in (*a*, *b*), $f(x_1) < f(x_2)$, whenever $x_1 < x_2$. A function is **decreasing** on an interval (*a*, *b*) if, for any two numbers x_1 and x_2 in (*a*, *b*), $f(x_1) > f(x_2)$, whenever $x_1 < x_2$.

In other words, if the *y* values are getting bigger as we move from left to right across the graph of the function, the function is increasing. If they are getting smaller, then the function is decreasing. We will state intervals of increase/decrease using interval notation.

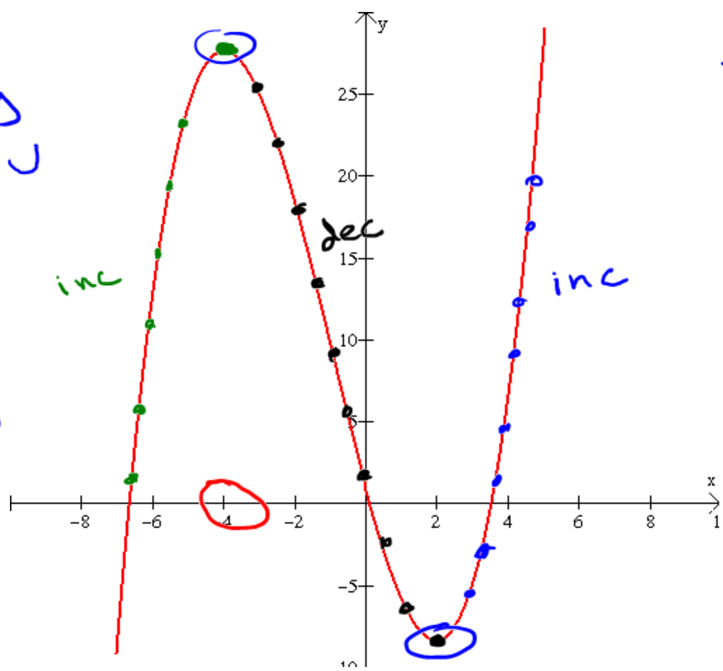
Example 2: State the intervals on which the function graphed below is increasing and intervals on which it is decreasing.

Popper 8, question 5: What is the domain of the function that is graphed below?

A. $(-8, 4)$ B. $(-10, 30)$ C. $(-\infty, \infty)$ D. $(0, \infty)$

f is increasing on $(-\infty, -4) \cup (2, \infty)$

increasing / decreasing

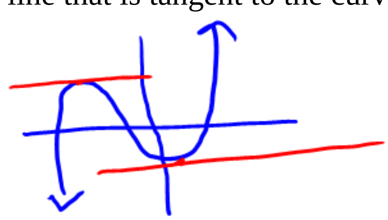


f is decreasing on $(-4, 2)$

We can use calculus to determine intervals of increase and intervals of decrease. A function can change from increasing to decreasing or from decreasing to increasing at its **critical numbers**, so we start with a definition of critical numbers:

The **critical numbers** of a polynomial function are numbers where $f'(x) = 0$.

Critical numbers of a polynomial function occur where the line that is tangent to the curve is horizontal.



to find critical numbers
Find derivative.
Set it equal to zero and solve for x

A function is **increasing** on an interval if the **first derivative** of the function is **positive** for every number in the interval.

A function is **decreasing** on an interval if the **first derivative** of the function is **negative** for every number in the interval.

Example 3: Find the critical number(s) if $f(x) = 2x^2 - 8x + 5$.

By hand:
1. Find the derivative.
2. Set the derivative equal to zero and solve for x.

① $f'(x) = 4x - 8$

② $4x - 8 = 0$

$4x = 8$

$x = 2$

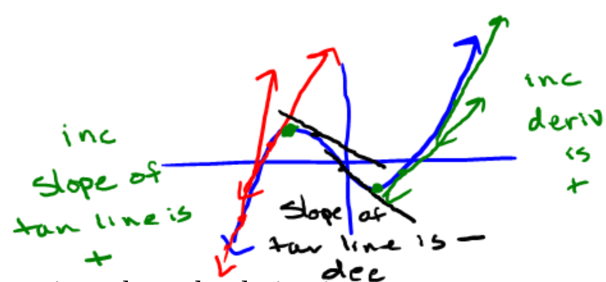
$\boxed{\text{C.N. } x = 2}$

Using GGB:
1. Enter the function in the Input line.
2. Graph the derivative of the function.
3. Find the zero(s) of the derivative.

$f(x) = 2x^2 - 8x + 5$
$f'(x) = 4x - 8$
Point
$A = (2, 0)$

Root[f']

$\boxed{\text{C.N. } x = 2}$



Since we can't test every number in an interval, we determine where the derivative is zero, subdivide the number line into regions based on those numbers and then test each interval as a whole by looking at a graph of the derivative.

if $f'(x)$ is + in an interval f is inc.
if $f'(x)$ is − in an interval f is dec

To find the intervals on which a polynomial function is increasing/decreasing:

1. Graph the derivative of the function. Find the critical number(s) (i.e., find the zero(s) of the derivative).
2. Create a number line, subdividing the line at the zeros of the derivative. i.e. @ critical #s
3. Use the graph of the derivative (or compute the value of a test point) to determine the sign of the y values of the derivative in each interval (positive or negative) and record this on your number line.
4. In each interval in which the derivative is positive, the function is increasing. In each interval in which the derivative is negative, the function is decreasing.

Before we start working these problems using GGB, we'll work one problem by hand.

Example 4: Suppose $f(x) = x^5 - x^4$. Find any critical numbers. Then find the intervals on which the function is increasing/decreasing.

$f(x) = x^5 - x^4$

$f'(x) = 5x^4 - 4x^3$

$5x^4 - 4x^3 = 0$

$x^3(5x-4) = 0$

$x^3 = 0 \qquad 5x-4=0$

① Critical numbers $\boxed{x=0 \qquad x=\frac{4}{5}}$

② inc $+$ | dec $-$ | inc $+$

$-1 \quad 0 \quad \frac{1}{2} \quad \frac{4}{5} \quad 1$

③ $f'(x) = 5x^4 - 4x^3$

$f'(-1) = 5(-1)^4 - 4(-1)^3 = 5+4 = \underline{\underline{9}}$

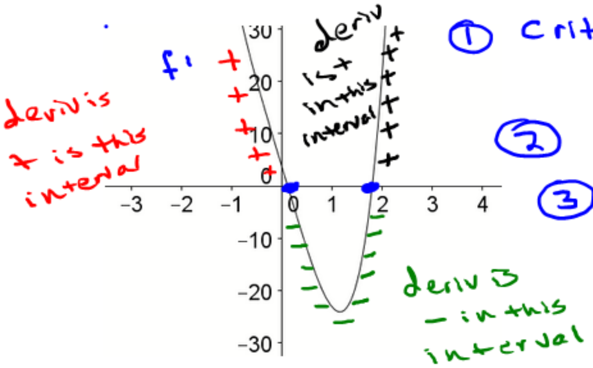
$f'(\frac{1}{2}) = 5(\frac{1}{2})^4 - 4(\frac{1}{2})^3 = 5(\frac{1}{16}) - 4(\frac{1}{8})$
$= \frac{5}{16} - \frac{4}{8} = -\frac{3}{16} = \underline{\underline{-.1875}}$

$f'(1) = 5(1)^4 - 4(1)^3 = 5-4 = 1$

④ f is inc. $(-\infty, 0) \cup (\frac{4}{5}, \infty)$

f is dec on $(0, \frac{4}{5})$

Example 5: Find the interval(s) on which the function is increasing and the interval(s) on which the function is decreasing: $f(x) = x^5 - 16x^2 + 4x$



① critical # Root[f']
↳ x = 0.125 , x = 1.8129

② + − +

0.125 1.8129

③

④ f is inc on $(-\infty, 0.125) \cup (1.8129, \infty)$

f is dec on $(0.125, 1.8129)$

Popper 8, questions 6 - 7: Suppose $f(x) = 2.8x^3 - 7.3x^2 - 9.3x + 5$.

6. Find any critical numbers.

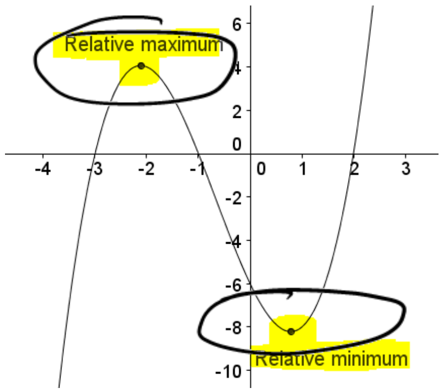
A. -2.2337 and 0.4956
B. -1.2386, 0.421, and 3.4247
C. -3.4247, -0.421 and 1.2386
D. -0.4956 and 2.2337

7. On which of these intervals is the function decreasing?

A. $(-\infty, -2.2337)$
B. $(-\infty, 0.869)$
C. $(-0.4956, 2.2337)$
D. $(0.421, 3.4247)$

Relative Extrema

The relative extrema are the high points and the low points of a function. A relative maximum is higher than all of the points near it; a relative minimum is lower than all of the points near it.



A relative maximum or a relative minimum can only occur at a **critical number**.

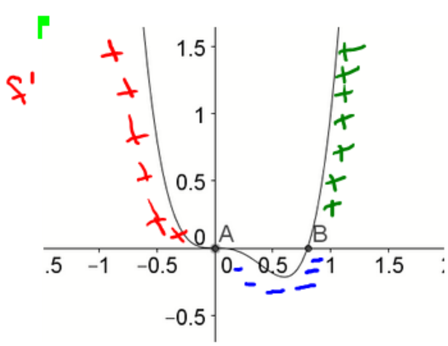
You can use the same number line that you created to determine intervals of increase/decrease to find the *x* coordinate of any relative extrema. Use these three statements to determine if a critical number generates a relative extremum.

1. If the sign of the derivative changes from positive to negative at a critical number, $x = c$, then the function has a relative maximum at the point $(c,\ f(c))$.
2. If the sign of the derivative changes from negative to positive at a critical number, $x = c$, then the function has a relative minimum at the point $(c,\ f(c))$.
3. If the sign of the derivative does not change sign at a critical number, $x = c$, then the function has neither a relative maximum nor a relative minimum at the point $(c,\ f(c))$.

Once you find that $x = c$ generates a relative extremum, you can find the *y* coordinate of the relative extremum by computing $f(c)$.

Example 6: Find any relative extrema. $f(x) = x^5 - x^4$

c.n. $x = 0.8$, $x = 0$



relative maximum at $(0, 0)$

relative minimum at $(0.8, -0.0819)$

relative maximum $(0.125, 0.25)$

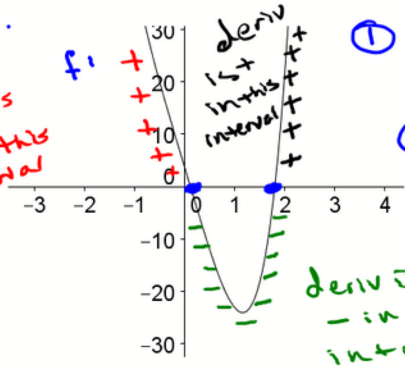
relative minimum $(1.8129, -25.7516)$

to find y's use $f(x)$

these are function values.

from Ex5

Example 7: Find any relative extrema: $f(x)=x^5-16x^2+4x$



f'

deriv is + is this interval

deriv is + in this interval

deriv is − in this interval

① Critical # Root[f']

↳ $x = 0.125$, $x = 1.8129$

② + − +

0.125 1.8129

③

④ f is inc on $(-\infty, 0.125) \cup (1.8129, \infty)$

f is dec on $(0.125, 1.8129)$

Popper 8, problem 8: Suppose $f\left(x\right)=2.8x^{3}-7.3x^{2}-9.3x+5$. Find the x and y coordinates of any relative minima.

f(2.2337)

A. $\left(0.869,\ -15.644\right)$

B. $\left(2.2337,\ -20.9905\right)$

C. $\left(2.2337,\ 0\right)$

D. $\left(-0.4956,\ 7.4752\right)$

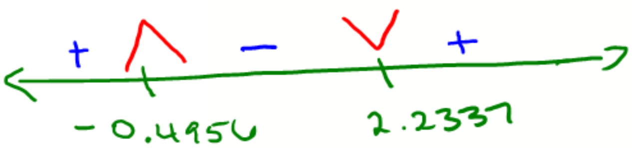
$f(x) = 2.8\,x^3 - 7.3\,x^2 - 9.3\,x + 5$

$f'(x) = 8.4\,x^2 - 14.6\,x - 9.3$

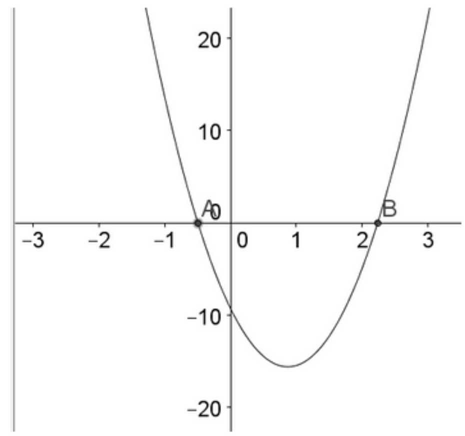
Point

A = (-0.4956, 0)   c.n.

B = (2.2337, 0)

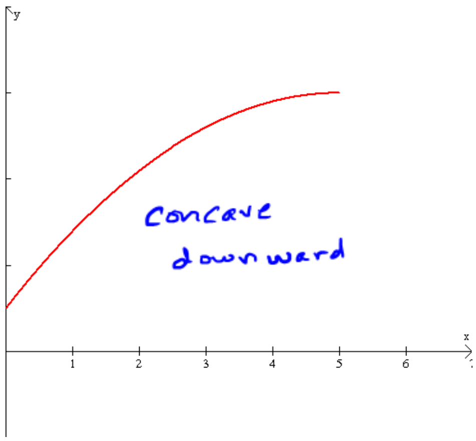


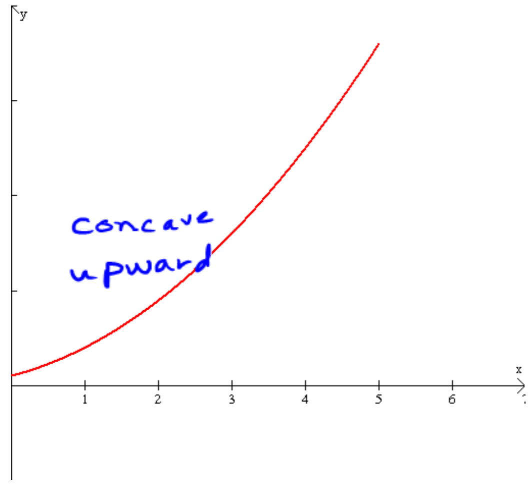
rel min @ x=2.2337



Concavity

Look at the two graphs that are given below. Both of the functions are increasing, but the shapes of the graphs are different. This difference in shape is described by the **concavity** of the function. The graph on the left is concave downward. The graph on the right is concave upward.

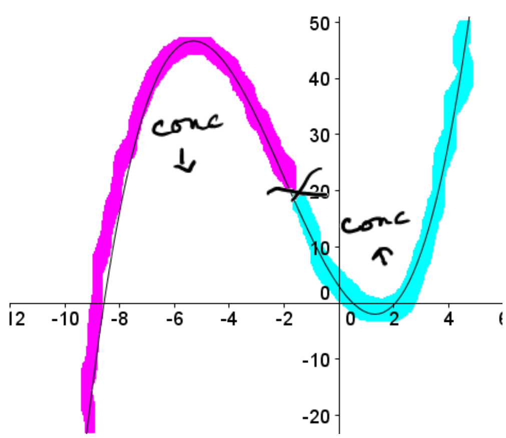




Technically speaking, the shape of the curve differs depending on whether the slopes of tangent lines are increasing or decreasing. This is the idea of concavity.

You are familiar with the graphs of $f(x) = x^2$ and $g(x) = -x^2$. The graph of *f* is a parabola that opens upwards. It is concave upward. The graph of *g* is a parabola that opens downwards. It is concave downward. These two functions may help you visualize concavity.

Our next task will be to analyze a function and to describe its concavity. Often a function will change concavity at some point, so a typical function will be concave downward for a while and then will change and be concave upward. Here's an example:

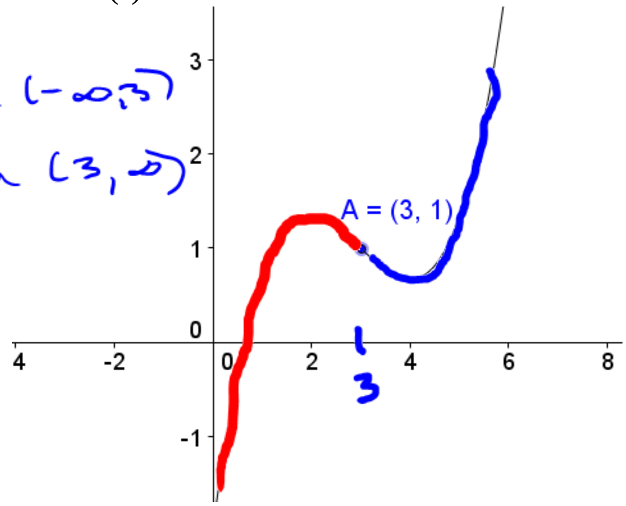


We'll state concavity intervals using interval notation.

Example 8: Using the graph shown below, determine the interval(s) on which the function is concave upward and the interval(s) on which the function is concave downward.

f is concave ↓ on $(-\infty, 3)$

f is concave ↑ on $(3, \infty)$



We find concavity intervals by analyzing the *second derivative* of the function. The analysis is very similar to the method we used to find increasing/decreasing intervals.

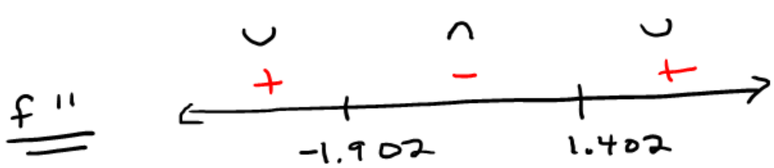
1. Use GeoGebra to graph the second derivative of the function. Then find the zero(s) of the second derivative.
2. Create a number line and subdivide it using the zeros of the second derivative.
3. Use the graph of the second derivative to determine the sign of the *y* values of the second derivative in each interval (positive or negative) and record this on your number line.
5. In each interval in which the second derivative is positive, the function is concave upward. In each interval in which the second derivative is negative, the function is concave downward.

State concavity intervals using interval notation.

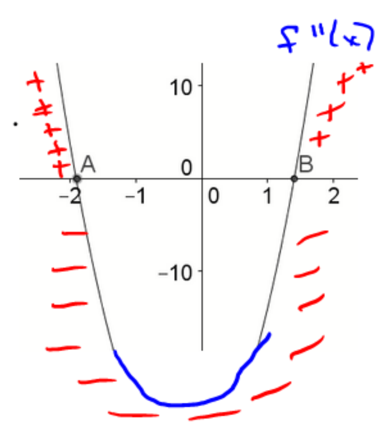
So, on an interval where the second derivative is positive, the function is concave upward and on an interval where the second derivative is negative, the function is concave downward.

Example 9: State intervals on which the function is concave upward and intervals on which the function is concave downward: $f(x) = x^4 + x^3 - 16x^2 + 4x + 16$ where is $f''(x) = 0$

$x = -1.902, 1.402$



$\boxed{\text{f is concave upward on } (-\infty, -1.902) \cup (1.402, \infty) \quad \text{f is concave downward on } (-1.902, 1.402)}$

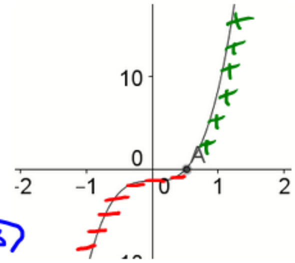


f''

Example 10: State intervals on which the function is concave upward and intervals on which the function is concave downward: $f(x)=\frac{1}{2}x^5-\frac{2}{3}x^2-8x-1$

f''(x) = 0
when x = 0.5109

∩ −   ∪ +
0.5109



f is conc ↓ on (−∞, 0.5109)
f is conc ↑ on (0.5109, ∞)
f has an inflection pt at (0.5109, −5.2438)

You'll also need to be able to identify the point(s) where concavity changes. A point where concavity changes is called a ***point of inflection***, or an ***inflection point***.

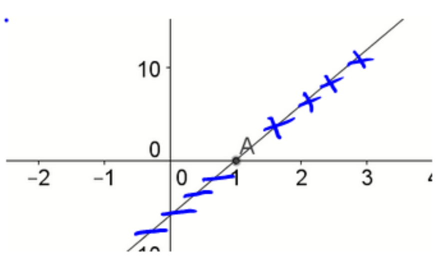
You can use the same number line that you created to determine concavity intervals to find the *x* coordinate of any inflection points. Use these two statements to determine if a zero of the second derivative generates an inflection point.

1. If the sign of the second derivative changes from positive to negative or from negative to positive at a number, $x=c$, then the function has an inflection point at the point $(c,\ f(c))$.
2. If the sign of the second derivative does not change sign at a number, $x=c$, then the function does not have an inflection point at the point $(c,\ f(c))$.

Once you find that $x=c$ generates an inflection point, you can find the *y* coordinate of the inflection point by computing $f(c)$.

Note – you can check your work using the Inflection Point command in GGB.

Example 11: Use the graph of the second derivative of $f(x)=x^3-3x^2-24x+32$ to find any inflection points.



f''(x) = 0 @ x = 1

f'' ∩ −   ∪ +
1

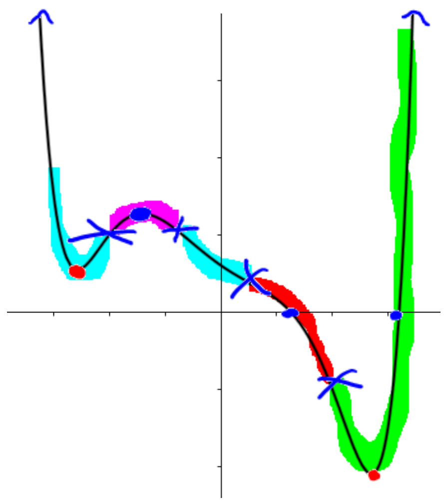
✓ inflection pt at (1, 6)

Example 12: Use the graph of the second derivative of $f(x)=\frac{1}{2}x^5-\frac{2}{3}x^2-8x-1$ to find any inflection points.

OMIT

Now that we can describe all of the features of a polynomial function, we can answer some questions, given a graph of a function.

Example 13: The graph given below is the graph of a polynomial function. Which of the statements shown below is/are true?



2 zeros
1 rel max
2 rel min
2 dec interval
1 inc interval
4 inflection pts

T 1. The function is increasing on two intervals and decreasing on two intervals.
F 2. The function has one relative maximum and one relative minimum.
F 3. The function has a vertical asymptote.
T 4. The function has four inflection points.
T 5. The domain of the function is $(-\infty,\ \infty)$.

Popper 8, problem 9: Suppose $f(x) = 2.8x^3 - 7.3x^2 - 9.3x + 5$. Find all values of x for which $f''(x) = 0$.

A. -0.4956

B. 2.2337

C. 0.869

D. 3.4247

Problem 10: On how many intervals is $f(x) = 2.8x^3 - 7.3x^2 - 9.3x + 5$ concave upward?

A. 0

B. 1

C. 2

D. 3

Example 14: Analyze this function: $f(x)=\frac{3}{2}x^4-2x^3+12x+2$

Domain $(-\infty, \infty)$

X intercept(s) $-1.5695, -0.1675$

Root

Y intercept(s) 2

$f(0)$

Critical numbers

$x = -1$

Increasing interval(s)

f is inc on $(1, \infty)$

Decreasing interval(s)

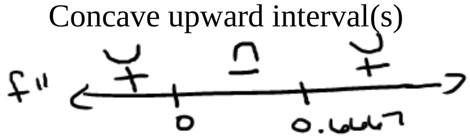
f is dec on $(-\infty, 1)$

Relative extrema

relative min at $(1, 13.5)$

Concave upward interval(s)

$f''(x) = 0$ at $x = 0, 0.6667$



f is conc↑ on $(-\infty, 0) \cup (0.6667, \infty)$

Concave downward interval(s)

f is conc↓ on $(0, 0.6667)$

Inflection points

$(0, 2)$

$(0.6667, 9.7041)$

Asymptotes

none