## Math 1314 <br> Lesson 11: Exponential Functions as Mathematical Models

Exponential models can be written in two forms: $f(x)=a \cdot e^{x}$ or $g(x)=a \cdot b^{x}$ or. In the first model, the base of the exponent is the number $e$, which is approximately 2.71828.... In the second model, the base of the exponent is a positive number other than 1 . In both cases, the variable is located in the exponent, and that's why these are called exponential models.

Exponential functions can be either increasing or decreasing. For a function of the form $f(x)=a \cdot e^{b x}$, the function is increasing if $b>0$ and is decreasing if $b<0$. If $b>0$, the function is an exponential growth function. If $b>0$, the function is an exponential decay function. The value $a$ is the initial value or the initial amount. The value $b$ is called the growth constant or the decay constant, depending on which type of function is given.

For a function of the form $g(x)=a \cdot b^{x}$, the function is increasing if $b>1$ and is decreasing if $0<b<1$.

We can also compute the rate at which an exponential function is increasing or decreasing. We'll do this by finding a numerical derivative.

We can use the regression feature to find an exponential equation for data that's given.
Example 1: Identify each function as a growth function or a decay function. Find the initial value. Calculate $f(10)$ and $f^{\prime}(10)$.
A. $f(x)=12.75 e^{-0.316 x}$
B. $g(x)=150(1.0616)^{x}$

POPPER 8, problems 1 and 2:
For problems 1 and 2, use this function: $Q(t)=1.68(1.047)^{t}$.

1. Is this a growth function or a decay function?
A. 8rowth
B. decay
2. Find $Q(2)$ and round to two decimal places.
A. 5.03
B. 1.79
C. 3.52
D. .84

Example 2: Suppose the points $(0,10000)$ and $(2,10940)$ lie on the graph of a function. Find the equation of the function in the form $f(x)=a \cdot e^{b x}$ using GeoGebra, assuming that the function is exponential.

Example 2, revisited: Suppose the points $(0,10000)$ and $(2,10940)$ lie on the graph of a function. Find the equation of the function in the form $g(x)=a \cdot b^{x}$, assuming that the function is exponential.

## Uninhibited Exponential Growth

Some common exponential applications model uninhibited exponential growth. This means that this is no "upper limit" on the value of the function. It can simply keep growing and growing. Problems of this type include population growth problems and growth of investment assets.

Example 3: A biologist wants to study the growth of a certain strain of bacteria. She starts with a culture containing 25,000 bacteria. After three hours, the number of bacteria has grown to 63,000 . Assume the population grows exponentially and the growth is uninhibited.
A. Find an exponential equation that models this experiment.

B. How many bacteria will be present in the culture 6 hours after she started her study? $\mathcal{E}$

$$
P(6)=158760
$$

C. What will be the rate of growth 6 hours after she started her study? Roc

$$
P^{\prime}(6)=48911.7811
$$



Example 4: The sales from ABC Company for the years 1998 - 2003 are given below. Sales data was collected at the end of each year.

|  | 0 | 1 | 2 | 3 | 4 | 5 | 8 | 9 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Year | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2006 | 2007 |
| Profits in millions of dollars | 51.4 | 53.2 | 55.8 | 56.1 | 58.1 | 59.0 |  |  |

A. Find an exponential regression equation that models this information.

$$
f(t)=51.8595 e^{0.0274 t}
$$

B. Using this model, what were the company's profits 2006 ?

$$
f(8)=44.5719
$$

\# 64.6 million
C. Find the rate at which the company's profits were changing at the end of 2007, using the model. ROC

$$
\begin{aligned}
& f^{\prime}(\overline{9})=1.8188 \\
& \$ 1.8 \mathrm{million} \text { l year }
\end{aligned}
$$

Example 5: In 1975, the national debt was approximately $\$ 500$ billion. In 1990, the national debt was about $\$ 2.9$ trillion. In 2003, the national debt was approximately $\$ 6.2$ trillion. Currently, the national debt is about $\$ 16.5$ trillion. Find an exponential function (assume that growth of the national debt is uninhibited) that models the national debt, and use it to predict the national debt in 2016.

$$
\begin{aligned}
& \begin{array}{r|c|c|c|c|c}
\text { years } & 1975 & 1990 & 2003 & 38 & 41 \\
\text { debt } & .5 & 2.9 & 6.2 & 16.5 &
\end{array} \\
& f(t)=0.5754 e^{0.0894} \\
& f(41)=22.1296 \\
& \approx \text { \#22.1 trillion } \\
& 1,000,000,000,000
\end{aligned}
$$

## Limited Growth Models

Some exponential growth is limited. Here are some examples:

## Learning Curves

A worker on an assembly line performs the same task repeatedly throughout the workday. With experience, the worker will perform at or near an optimal level. However, when first learning to do the task, the worker's productivity will be much lower. During these early experiences, the worker's productivity will increase dramatically. Then, once the worker is thoroughly familiar with the task, there will be little change to his/her productivity.

The function that models this situation will have the form
$Q(t)=C-A e^{-k t}$

This model is called a learning curve and the graph of the function will look something like this:


The graph will have a y intercept at $\mathrm{C}-\mathrm{A}$ and a horizontal asymptote at $\mathrm{y}=\mathrm{C}$, Because of the horizontal asymptote, we know that this function does not model uninhibited growth.

$$
\begin{aligned}
& \text { limited growth bloc we have } \\
& \text { an ensymptote }
\end{aligned}
$$

asymptote dy $=50$
Example 6: Suppose your company's HR department determines that an employee will be able to assemble $Q(t)=50-30 e^{-0.5 t}$ products per ry, $t$ months after the employee starts working on the assembly line.

$$
\text { y int }=50-30=20
$$

(A) How many units can a new employee assemble as she starts the first day at work? FV

$$
Q(0)=20
$$

20 units
(B) How many units should an employee be able to assemble after one month at work? FV After two months at work? After six months at work?

$$
\begin{aligned}
& Q(D=31.8041 \\
& Q(2)=38.9636 \\
& Q(4)=48.5064
\end{aligned}
$$


(C) How many units should an experienced worker be able to assemble?

$$
\lim _{t \rightarrow \infty} Q(t)=50^{t \rightarrow \infty}
$$

50 units
(D) At what rate is an employee's productivity changing 4 months after starting to work?

$$
\begin{aligned}
& \text { Q. }(\vec{L})=2.03 \\
& \text { Roc is approx } 2.03 \text { units l month after } \\
& 4 \text { months }
\end{aligned}
$$

## POPPER 8, question 3:

## $Q^{\prime}(\omega)$

Suppose your company's HR department determines that an employee will be able to assemble $Q(t)=50-30 e^{-0.5 t}$ products per day, $t$ months after the employee starts working on the assembly line. At what rate will the employee's productivity be changing 6 months after s/he starts working for the company?

What kind of problem is this?
A. Limit at infinity
B. PROC
C. AV
D. R OC

## Logistic Functions

The last growth model that we will consider involves the logistic function. The general form of the equation is

$$
\left.Q(t)=\frac{A}{\left(1+B e^{-k t}\right.}\right)
$$

and the graph looks something like this:


If we looked at this graph up to around $x=2$ and didn't consider the rest of it, we might think that the data modeled was exponential. Logistic functions typically reach a saturation point - a point at which the growth slows down and then eventually levels off. The part of this graph to the right of $x=2$ looks more like our learning curve graph from the last example. Logistic functions have some of the features of both types of models.

Note that the graph has a limiting value at $y=5$. In the context of a logistic function, this asymptote is called the carrying capacity.

Logistic curves are used to model various types of phenomena. For example, suppose a new gym opens in your neighborhood. At first membership sales are brisk, and the number of members seems to grow exponentially. But eventually, the market is tapped out, and there are fewer and fewer prospective members to attract to the facility. The rate of growth slows down and then may stop all together.

Logistic functions also apply to other physical situations such as population management. Suppose a number of animals are introduced into a protected game reserve, with the expectation that the population will grow. Various factors will work together to keep the population from growing exponentially (in an uninhibited manner). The natural resources (food, water, protection) may not exist to support a population that gets larger without bound. Often such populations grow according to a logistic model.

Example 7: A population study was commissioned to determine the growth rate of the fish population in a certain area of the Pacific Northwest. The function given below models the population where $t$ is measured in years and $N$ is measured in millions of tons.

$$
N(t)=\frac{2.4}{1+2.39 e^{-0.338 t}}
$$

(A) What was the initial number of fish in the population?

$$
N(0)=0.708
$$

.708 million tons
(B) What is the carrying capacity in this population?

$$
\lim _{t \rightarrow \infty} N(t)=2.4
$$

(C) What is the fish population after 3 years?

$$
N(3)=1.2855
$$

(D) How fast is the fish population changing after 2 years? ROC

$$
N^{\prime}(2)=0.2009
$$

0.2009 million tors of fish l year
(E) When will the population reach 2 million tons?
also graph g(x)=2
find $p^{+}$. of intersection
7.3 years

Example 8: Use logistic regression to determine a function that models this data and graph it:

| t | 1 | 3 | 4 | 7 | 8 | 10 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| N | 55 | 108 | 153 | 280 | 291 | 311 |

$$
\text { fitlogistic }[1 i s t]
$$

(A) Use the model to find $N(5)$.

$$
N(5)
$$

(B) When will $N(t)=200$ ?

(C) At what rate is the function changing when $t=6$ ?

Decay

Next, we'll look at some exponential decay problems.


Example 9: At the beginning of a study, there are 50 grams of a substance present. After 17 days, there are 38.7 grams remaining.
A. Find an exponential regression model for this situation. (0,50) (17, 38.7)

$$
f(x)=50 e^{-0.0151 x}
$$

B. How much of the substance will be present after 40 days?

EU

$$
f(40)=27.3643
$$


C. What will be the rate decay $n$ day 40 of the study? Assume the substance decays Roc exponentially.

$$
f^{\prime}(40)=-0.4124
$$

.4124 grams l day

Exponential decay problems frequently involve the half-life of a substance. The half-life of a substance is the time it takes to reduce the amount of the substance by one-half. So, if the halflife of a radioactive substance is 1000 years, and there are 50 mg present at the beginning of an experiment, 1000 years later, there will only be 25 mg of the substance left.

Example 10: A certain drug has a half-life of 4 hours. Suppose you take a dose of 1000 milligrams of the drug. How much of it is left in your bloodstream 28 hours later?

$$
\begin{aligned}
&(0,1000) \text { use half-life to get } \\
& \text { and point } \\
& A(t)=1000 e^{-0.1733 t}
\end{aligned}
$$

Example 11: The half-life of Carbon 145770 vars. Bones found from an archeological dig were found to have $22 \%$ of the amount of Carbon 14 that living bones have. Find the approximate age of the bones.

POPPER 8, question 4:

$$
\begin{aligned}
& (\text { time }, 90) \\
& (0,100)
\end{aligned} \quad(5770,50)
$$

To work this half-life problem, you need two ordered pairs. One ordered pair is $(0,100)$ which represents $100 \%$ present at the time the animal died. Which of these is a second ordered pair that you would use in working this problem?
A. $(100,5770)$
B. $(50,5770)$
C. $(5770,0)$
D. 5770,50 )

$$
A(t)=100 e^{-0.00012 t} \quad\left(f_{i+e x P}\right)
$$

$$
g(t)=22
$$

find pt. of intersection


