Example 2: Find the inverse of a 3 x 3 matrix.(Use Gauss-Jordan)

$$C = \begin{pmatrix} 1 & 4 & -1 \\ 2 & 3 & -2 \\ -1 & 2 & 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 2 & 3 & -2 & 0 & 1 & 0 \\ -1 & 2 & 3 & 0 & 0 & 1 \end{array}\right)$$

$-2R_1 + R_2 \to R_2$

$$\begin{array}{cccccc} -2 & -8 & 2 & -2 & 0 & 0 \\ 2 & 3 & -2 & 0 & 1 & 0 \\ \hline 0 & -5 & 0 & -2 & 1 & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -5 & 0 & -2 & 1 & 0 \\ -1 & 2 & 3 & 0 & 0 & 1 \end{array}\right)$$

$R_1 + R_3 \to R_3$

$$\begin{array}{cccccc} 1 & 4 & -1 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 0 & 1 \\ \hline 0 & 6 & 2 & 1 & 0 & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -5 & 0 & -2 & 1 & 0 \\ 0 & 6 & 2 & 1 & 0 & 1 \end{array}\right) \quad -\tfrac{1}{5}R_2 \to R_2 \quad \left(\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2/5 & -1/5 & 0 \\ 0 & 6 & 2 & 1 & 0 & 1 \end{array}\right)$$

$-6R_2 + R_3 \to R_3$

$$\begin{array}{cccccc} 0 & -6 & 0 & -\frac{12}{5} & \frac{6}{5} & 0 \\ 0 & 6 & 2 & 1 & 0 & 1 \\ \hline 0 & 0 & 2 & -\frac{7}{5} & \frac{6}{5} & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 2 & -\frac{7}{5} & \frac{6}{5} & 1 \end{array}\right)$$

$-4R_2 + R_1 \to R_1$

$$\begin{array}{cccccc} 0 & -4 & 0 & -4/5 & 4/5 & 0 \\ 1 & 4 & -1 & 1 & 0 & 0 \\ \hline 1 & 0 & -1 & -\frac{3}{5} & \frac{4}{5} & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -\frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 2 & -\frac{7}{5} & \frac{6}{5} & 1 \end{array}\right) \quad \tfrac{1}{2}R_3 \to R_3 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -3/5 & 4/5 & 0 \\ 0 & 1 & 0 & 2/5 & -1/5 & 0 \\ 0 & 0 & 1 & -7/10 & 3/5 & 1/2 \end{array}\right)$$

$R_3 + R_1 \to R_1$

$$\begin{array}{cccccc} 0 & 0 & 1 & -\frac{7}{10} & \frac{3}{5} & \frac{1}{2} \\ 1 & 0 & -1 & -\frac{3}{5} & \frac{4}{5} & 0 \\ \hline 1 & 0 & 0 & -\frac{13}{10} & \frac{7}{5} & \frac{1}{2} \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{13}{10} & \frac{7}{5} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & -\frac{7}{10} & \frac{3}{5} & \frac{1}{2} \end{array}\right)$$

$$\boxed{C^{-1} = \begin{pmatrix} -\frac{13}{10} & \frac{7}{5} & \frac{1}{2} \\ \frac{2}{5} & -\frac{1}{5} & 0 \\ -\frac{7}{10} & \frac{3}{5} & \frac{1}{2} \end{pmatrix}}$$

Example 3: Find the inverse.

$$B = \begin{pmatrix} 4 & 2 & 2 \\ -1 & -3 & 4 \\ 3 & -1 & 6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 4 & 2 & 2 & 1 & 0 & 0 \\ -1 & -3 & 4 & 0 & 1 & 0 \\ 3 & -1 & 6 & 0 & 0 & 1 \end{array}\right) \quad R_1 \leftrightarrow R_2 \quad \left(\begin{array}{ccc|ccc} -1 & -3 & 4 & 0 & 1 & 0 \\ 4 & 2 & 2 & 1 & 0 & 0 \\ 3 & -1 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$-R_1 \to R_1 \quad \left(\begin{array}{ccc|ccc} 1 & 3 & -4 & 0 & -1 & 0 \\ 4 & 2 & 2 & 1 & 0 & 0 \\ 3 & -1 & 6 & 0 & 0 & 1 \end{array}\right)$$

$-4R_1 + R_2 \to R_2$

$$\begin{array}{cccccc} -4 & -12 & 16 & 0 & 4 & 0 \\ 4 & 2 & 2 & 1 & 0 & 0 \\ \hline 0 & -10 & 18 & 1 & 4 & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -4 & 0 & -1 & 0 \\ 0 & -10 & 18 & 1 & 4 & 0 \\ 3 & -1 & 6 & 0 & 0 & 1 \end{array}\right)$$

$-3R_1 + R_3 \to R_3$

$$\begin{array}{cccccc} -3 & -9 & 12 & 0 & 3 & 0 \\ 3 & -1 & 6 & 0 & 0 & 1 \\ \hline 0 & -10 & 18 & 0 & 3 & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -4 & 0 & -1 & 0 \\ 0 & -10 & 18 & 1 & 4 & 0 \\ 0 & -10 & 18 & 0 & 3 & 1 \end{array}\right) \quad -R_2 + R_3 \to R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -4 & 0 & -1 & 0 \\ 0 & -10 & 18 & 1 & 4 & 0 \\ 0 & 0 & 0 & \# & \# & \# \end{array}\right)$$

No Inverse

Singular

Matrices That Have No Inverses

If there is a row to the left of the vertical line in the augmented matrix containing all zeros, then the matrix does not have an inverse. Example 3 has this problem and does not have an inverse.

Formula for the Inverse of a 2X2 Matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Suppose $D = ad - bc$ is not equal to zero. Then A^{-1} exists and is given by

$$A^{-1} = \frac{1}{D}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

↖ Determinant

Example 4: Find the inverse of the following matrices.

a. $A = \begin{pmatrix} -5 & 10 \\ 2 & 7 \end{pmatrix}$

$$A^{-1} = \frac{1}{D}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{-55}\begin{pmatrix} 7 & -10 \\ -2 & -5 \end{pmatrix}$$

$$D = ad - bc$$
$$= -5(7) - 10(2)$$
$$= -35 - 20$$
$$= -55$$

$$= \begin{pmatrix} -\frac{7}{55} & \frac{-10}{-55} \\ \frac{-2}{-55} & \frac{-5}{-55} \end{pmatrix} = \boxed{\begin{pmatrix} \frac{-7}{55} & \frac{2}{11} \\ \frac{2}{55} & \frac{1}{11} \end{pmatrix}}$$

b. $B = \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix}$

$$D = ad - bc$$
$$= 8(2) - (-4)(-4)$$
$$= 16 - 16 = 0$$

B^{-1} does not exist

Question 3: Find the determinant of

$$\begin{pmatrix} -3 & 10 \\ 11 & 7 \end{pmatrix}$$

a. 89
b. -22
c. 3
d. -131
e. None of the above

Matrix Representation

A system of linear equations may be written in a compact form with the help of matrices.

Example 5: Given the following system of equations, write it in matrix form.

$$2x - 4y + z = 6$$
$$-3x + 6y - 5z = -1$$
$$x - 3y + 7z = 0$$

$$A = \begin{pmatrix} 2 & -4 & 1 \\ -3 & 6 & -5 \\ 1 & -3 & 7 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad B = \begin{pmatrix} 6 \\ -1 \\ 0 \end{pmatrix}$$

$$AX = B$$

$$(3\times3)(3\times1) = (3\times1)$$

$$3 \times 1$$

$$A \cdot X = B$$

$$\begin{pmatrix} 2 & -4 & 1 \\ -3 & 6 & -5 \\ 1 & -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 0 \end{pmatrix}$$

Question 5: Find the inverse of

$$\begin{pmatrix} -1 & 5 \\ 1 & -4 \end{pmatrix}$$

Find Determinant then use formula

a. $\begin{pmatrix} -4 & 1 \\ 5 & -1 \end{pmatrix}$

b. $\begin{pmatrix} -1 & -3 \\ -5 & -4 \end{pmatrix}$

c. $\begin{pmatrix} -4 & -5 \\ -1 & -1 \end{pmatrix}$

d. $\begin{pmatrix} 4 & 5 \\ 1 & 1 \end{pmatrix}$

e. None of the above

Example 6: Write each system of equations as a matrix equation and then solve the system using the inverse of the coefficient matrix.

$$2x+3y=5$$
$$3x+5y=8$$

$A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \end{pmatrix}$ $B = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$

$AX = B$

$\underbrace{A^{-1} \cdot A}_{I} X = A^{-1} \cdot B$

$X = A^{-1} \cdot B$

$D = 2(5) - 3(3) = 10 - 9 = 1$

$A^{-1} = \frac{1}{1} \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$

$X = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 25 - 24 \\ -15 + 16 \end{pmatrix}$

$X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\boxed{\begin{matrix} x = 1 \\ y = 1 \end{matrix}}$

Question 4: Given the linear system of equations. How would you set up using the coefficient matrix to solve the system?

$$x + 4y = 3$$
$$2x + 3y = 1$$

a. $X = \begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

b. $X = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

c. $X = \begin{pmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{2} & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

d. $X = \begin{pmatrix} \frac{1}{3} & -\frac{1}{4} \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

e. $X = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

The use of inverses to solve systems of equations is advantageous when we are required to solve more than one system of equations **AX = B**, involving the same coefficient matrix, **A**, and different matrices of constants, **B**.

x y

Example 7: A performance theatre has 10,000 seats. The ticket prices are either \$25 or \$35, depending on the location of the seat. Assume every seat can be sold.

a. How many tickets of each type should be sold to bring in a return of \$275,000?
b. How many tickets of each type should be sold to bring in a return of \$300,000?
Let x = number of \$25 tickets and y = number of \$35 tickets

a)

$$x + y = 10{,}000$$
$$25x + 35y = 275{,}000$$

$$A = \begin{pmatrix} 1 & 1 \\ 25 & 35 \end{pmatrix} \qquad B = \begin{pmatrix} 10{,}000 \\ 275{,}000 \end{pmatrix}$$

$$D = ad - bc$$
$$= 1(35) - 1(25)$$
$$= 10$$

$$A^{-1} = \frac{1}{10}\begin{pmatrix} 35 & -1 \\ -25 & 1 \end{pmatrix} = \begin{pmatrix} \frac{35}{10} & \frac{-1}{10} \\ \frac{-25}{10} & \frac{1}{10} \end{pmatrix}$$

$$X = A^{-1} \cdot B = \frac{1}{10}\begin{pmatrix} 35 & -1 \\ -25 & 1 \end{pmatrix}\begin{pmatrix} 10000 \\ 275000 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 350000 - 275000 \\ -250000 + 275000 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} 75000 \\ 25000 \end{pmatrix} = \begin{pmatrix} 7500 \\ 2500 \end{pmatrix}$$

$\boxed{\begin{array}{l} 7500 \text{ of } \$25 \\ 2500 \text{ of } \$35 \end{array}}$

b)

$$x + y = 10000$$
$$25x + 35y = 300000$$

$$B_2 = \begin{pmatrix} 10000 \\ 300000 \end{pmatrix}$$

$$X = A^{-1} \cdot B_2 = \frac{1}{10}\begin{pmatrix} 35 & -1 \\ -25 & 1 \end{pmatrix}\begin{pmatrix} 10000 \\ 300000 \end{pmatrix}$$

$$X = \begin{pmatrix} 5000 \\ 5000 \end{pmatrix}$$

5000 of \$25
5000 of \$35