Math 1313 Section 3.2

Example 4: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$3x + y = 1$$
$$-7x - 2y = -1$$

$$\left(\begin{array}{cc|c} 3 & 1 & 1 \\ -7 & -2 & -1 \end{array}\right) \xrightarrow{\frac{1}{3}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{1}{3} \\ -7 & -2 & -1 \end{array}\right)$$

$\hookrightarrow 0$

$7R_1 + R_2 \to R_2$

$$\begin{array}{ccc} 7 & \frac{7}{3} & \frac{7}{3} \\ -7 & -2 & -1 \\ \hline 0 & \frac{1}{3} & \frac{4}{3} \end{array}$$

$$\left(\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{4}{3} \end{array}\right) \xrightarrow{3R_2 \to R_2} \left(\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 4 \end{array}\right)$$

$\to 0$

$-\frac{1}{3}R_2 + R_1 \to R_1$

$$\begin{array}{ccc} 0 & -\frac{1}{3} & -\frac{4}{3} \\ 1 & \frac{1}{3} & \frac{1}{3} \\ \hline 1 & 0 & -\frac{3}{3} = -1 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \end{array}\right)$$

$$\boxed{\begin{array}{l} x = -1 \\ y = 4 \end{array}}$$

Question 5: Is the following matrix row reduced?

$$\begin{pmatrix} 1 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

a. Yes
b. No

Example 5: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$\begin{aligned} y - 8z &= 9 \\ x - 2y + 3z &= -3 \\ 7y - 5z &= 12 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & -8 & 9 \\ 1 & -2 & 3 & -3 \\ 0 & 7 & -5 & 12 \end{array}\right) \quad R_1 \leftrightarrow R_2 \quad \left(\begin{array}{ccc|c} 1 & -2 & 3 & -3 \\ 0 & 1 & -8 & 9 \\ 0 & 7 & -5 & 12 \end{array}\right)$$

$\to 0$

$-7R_2 + R_3 \to R_3$

$$\begin{array}{cccc} 0 & -7 & 56 & -63 \\ 0 & 7 & -5 & 12 \\ \hline 0 & 0 & 51 & -51 \end{array}$$

$\to 0$

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & -3 \\ 0 & 1 & -8 & 9 \\ 0 & 0 & 51 & -51 \end{array}\right)$$

$2R_2 + R_1 \to R_1$

$$\begin{array}{cccc} 0 & 2 & -16 & 18 \\ 1 & -2 & 3 & -3 \\ \hline 1 & 0 & -13 & 15 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -13 & 15 \\ 0 & 1 & -8 & 9 \\ 0 & 0 & 51 & -51 \end{array}\right)$$

$\to 1$

$\frac{1}{51} R_3 \to R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & -13 & 15 \\ 0 & 1 & -8 & 9 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$13R_3 + R_1 \to R_1$

$$\begin{array}{cccc} 0 & 0 & 13 & -13 \\ 1 & 0 & -13 & 15 \\ \hline 1 & 0 & 0 & 2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -8 & 9 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$\to 0$

$8R_3 + R_2 \to R_2$

$$\begin{array}{cccc} 0 & 0 & 8 & -8 \\ 0 & 1 & -8 & 9 \\ \hline 0 & 1 & 0 & 1 \end{array}$$

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \end{array}$$

$$\boxed{\begin{aligned} x &= 2 \\ y &= 1 \\ z &= -1 \end{aligned}}$$

Math 1313 Section 3.2

Example 6: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$2x + 4y - 6z = 38$$
$$x + 2y + 3z = 7$$
$$3x - 4y + 4z = -19$$

$$\left(\begin{array}{ccc|c} 2 & 4 & -6 & 38 \\ 1 & 2 & 3 & 7 \\ 3 & -4 & 4 & -19 \end{array}\right) \quad R_1 \leftrightarrow R_2 \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 2 & 4 & -6 & 38 \\ 3 & -4 & 4 & -19 \end{array}\right)$$

$-2R_1 + R_2 \to R_2$

-2	-4	-6	-14
2	4	-6	38
0	0	-12	24

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 0 & -12 & 24 \\ 3 & -4 & 4 & -19 \end{array}\right)$$

$\to 0$

$-3R_1 + R_3 \to R_3$

-3	-6	-9	-21
3	-4	4	-19
0	-10	-5	-40

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 0 & -12 & 24 \\ 0 & -10 & -5 & -40 \end{array}\right) \quad R_2 \leftrightarrow R_3 \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & -10 & -5 & -40 \\ 0 & 0 & -12 & 24 \end{array}\right) \quad -\tfrac{1}{10} R_2 \to R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 1 & \tfrac{1}{2} & 4 \\ 0 & 0 & -12 & 24 \end{array}\right)$$

$-2R_2 + R_1 \to R_1$

0	-2	-1	-8
1	2	3	7
1	0	2	-1

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & \tfrac{1}{2} & 4 \\ 0 & 0 & -12 & 24 \end{array}\right) \quad -\tfrac{1}{12} R_3 \to R_3$$

$\to 1$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & \tfrac{1}{2} & 4 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$-2R_3 + R_1 \to R_1$

0	0	-2	4
1	0	2	-1
1	0	0	3

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & \tfrac{1}{2} & 4 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$-\tfrac{1}{2} R_3 + R_2 \to R_2$

0	0	$-\tfrac{1}{2}$	1
0	1	$\tfrac{1}{2}$	4
0	1	0	5

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$$\boxed{\begin{array}{l} x = 3 \\ y = 5 \\ z = -2 \end{array}}$$

Math 1313 Section 3.2

Infinite Number of Solutions

Example 7: The following augmented matrix in row-reduced form is equivalent to the augmented matrix of a certain system of linear equations. Use this result to solve the system of equations.

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$x - z = 3$

$y + 5z = -2$

$0 = 0$ ✓

$x = z + 3 \leftarrow z = x - 3$

$y = -5z - 2$

$z = ???$

$y = -5(x-3) - 2$

$y = -5x + 15 - 2$

$y = -5x + 13$ (Line)

Infinitely Many Solutions

Example 8: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$x + 2y - 3z = -2$$
$$3x - y - 2z = 1$$
$$2x + 3y - 5z = -3$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 3 & -1 & -2 & 1 \\ 2 & 3 & -5 & -3 \end{array}\right)$$

$-3R_1 + R_2 \to R_2$

$\begin{array}{cccc} -3 & -6 & 9 & 6 \\ 3 & -1 & -2 & 1 \\ \hline 0 & -7 & 7 & 7 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 7 & 7 \\ 2 & 3 & -5 & -3 \end{array}\right)$$

$-2R_1 + R_3 \to R_3$

$\begin{array}{cccc} -2 & -4 & 6 & 4 \\ 2 & 3 & -5 & -3 \\ \hline 0 & -1 & 1 & 1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 7 & 7 \\ 0 & -1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$-R_2 \to R$

Multiples of themselves

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$-2R_2 + R_1 \to R_1$

$\begin{array}{cccc} 0 & -2 & 2 & 2 \\ 1 & 2 & -3 & -2 \\ \hline 1 & 0 & -1 & 0 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x - z = 0$; $x = z$

$y - z = -1$; $y = z - 1$

z = Any Real #

Infinitely Many Solutions

Question 3: State the operation needed for the next appropriate step, in reducing the following matrix

$$\left(\begin{array}{ccc|c} \mathbf{1} & \mathbf{2} & \mathbf{-2} & \mathbf{6} \\ \mathbf{3} & \mathbf{-4} & \mathbf{0} & \mathbf{8} \\ \mathbf{-2} & \mathbf{4} & \mathbf{5} & \mathbf{-6} \end{array}\right)$$

a. $-\frac{1}{4}R_2 \to R_2$

b. $-3 + R_2 \to R_2$

c. $\frac{1}{5}R_3 \to R_3$

d. $2R_1 + R_3 \to R_3$

Math 1313 Section 3.2

A System of Equations That Has No Solution

In using the Gauss-Jordan elimination method the following equivalent matrix was obtained (note this matrix is not in row-reduced form, let's see why):

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

Look at the last row. It reads: 0x + 0y + 0z = -1, in other words, 0 = -1!!! This is never true. So the system is inconsistent and has no solution.

Systems with No Solution

If there is a row in the augmented matrix containing all zeros to the left of the vertical line and a nonzero entry to the right of the line, then the system of equations has no solution.

Example 9: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$2x + 3y = 2$$
$$x + 3y = -2$$
$$x - y = 3$$

$$\left(\begin{array}{cc|c} 2 & 3 & 2 \\ 1 & 3 & -2 \\ 1 & -1 & 3 \end{array}\right) \quad R_1 \leftrightarrow R_2 \quad \left(\begin{array}{cc|c} 1 & 3 & -2 \\ 2 & 3 & 2 \\ 1 & -1 & 3 \end{array}\right) \quad \begin{array}{l} -2R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 3 & -2 \\ 0 & -3 & 6 \\ 0 & -4 & 5 \end{array}\right) \quad -\tfrac{1}{3}R_2 \to R_2 \quad \left(\begin{array}{cc|c} 1 & 3 & -2 \\ 0 & 1 & -2 \\ 0 & -4 & 5 \end{array}\right) \quad 4R_2 + R_3 \to R_3$$

$$\left(\begin{array}{cc|c} 1 & 3 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -3 \end{array}\right)$$

$0 = -3$

False

$\boxed{\text{No Solution}}$

Example 10: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$-x + 3y - 4z = 12$$
$$4x - 12y + 16z = -36$$

More Variables than Equations

① No Solution
② Infinite

$$\left(\begin{array}{ccc|c} -1 & 3 & -4 & 12 \\ 4 & -12 & 16 & -36 \end{array}\right) \quad 4R_1 + R_2 \to R_2$$

$\times -4$ $\quad \times -4$ $\quad \times -4$ $\quad \times -3$

$0 \quad 0 \quad 0 \mid$ Not 0

$\boxed{\text{No Solution}}$

Example 11: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$3x + y - 4z = 6$$
$$-15x - 5y + 20z = -36$$

Example 12: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$2x - 3y = 13$$
$$x + y = -1$$
$$x - 4y = 14$$

Math 1313 Section 3.2

Question 4: Solve the following system of linear equations using the Gauss-Jordan elimination method for the variable y.

$$-x + y = -1$$
$$3x - 2y = 0$$
$$2x - y = 4$$

a. $y = 1$
b. $y = 3$
c. $y = z$, where z is any real number
d. $y = 2$
e. No Solution

Math 1313 Section 3.3

Section 3.3: Matrix Operations

Addition and Subtraction of Matrices

If A and B are two matrices of the same size,

1. A + B is the matrix obtained by adding the corresponding entries in the two matrices.

2. A – B is the matrix obtained by subtracting the corresponding entries in B from A.

Laws for Matrix Addition

If A, B, and C are matrices of the same dimension, then

1. A + B = B + A

2. (A + B) + C = A + (B + C)

Example 1: Refer to the following matrices: If possible,

$$A = \begin{bmatrix} 8 & -3 & 1 \\ 0 & -9 & -4 \\ 9 & 6 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 4 & -1 \\ 8 & 4 & 8 \\ 10 & 15 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 10 & -8 & 3 \\ 5 & -4 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 1 & 3 \\ 8 & 5 & 1 \end{bmatrix}$$

a. compute $A - B$

b. compute B + C.

c. compute D + C.

Scalar Multiplication

A **scalar** is a real number.

Scalar multiplication is the product of a scalar and a matrix. To perform scalar multiplication, each element in the matrix is multiplied by the scalar; hence, it "scales" the elements in the matrix

Example 2: Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 4 \\ -7 & 9 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 2 & 3 \\ -6 & -9 & 1 \end{pmatrix}$ find, if possible,

a. -3C

b. -2B – A

c. 3B + 2C

Transpose of a Matrix

If A is an m x n matrix with elements a_{ij}, then the **transpose** of A is the n x m matrix A^T with elements a_{ji}.

$$A = \begin{bmatrix} 2 & 5 & 50 \\ 1 & 3 & 27 \\ 16 & 45 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 1 & 16 \\ 5 & 3 & 45 \\ 50 & 27 & 1 \end{bmatrix}$$

Example 3: Given the following matrices, find their transpose.

a. $B = \begin{pmatrix} -3 & 0 & 6 \\ 10 & 100 & 3 \end{pmatrix}$

b. D = $\begin{pmatrix} 0 \\ -4 \\ 11 \\ -3 \end{pmatrix}$

Equality of Matrices

Two matrices are equal if they have the same dimension and their corresponding entries are equal.

Example 4: Solve the following matrix equation for w, x, y, and z.

$$\begin{bmatrix} w + 6 & x \\ y - 2 & z \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 1 & 4 \end{bmatrix}$$

Example 5: Solve for the variables in the matrix equation.

$$-\begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} + 9\begin{bmatrix} u-6 & 2z+5 \\ y & -\frac{1}{3} \end{bmatrix} = -2\begin{bmatrix} 3 & -8 \\ 1 & v \end{bmatrix}$$