1. Riemann sums. Sketch the region associated with the approximation, and then given the requested value.

See next page.

   a. Give the upper sum for $f(x) = 4 - x^2$ on the interval [-1,1] with respect to the partition $P = \left\{-1, -\frac{1}{2}, \frac{1}{2}, 1\right\}$.

   b. Give the lower sum for $f(x) = 4 - x^2$ on the interval [-1,1] with respect to the partition $P = \left\{-1, -\frac{1}{2}, \frac{1}{2}, 1\right\}$.

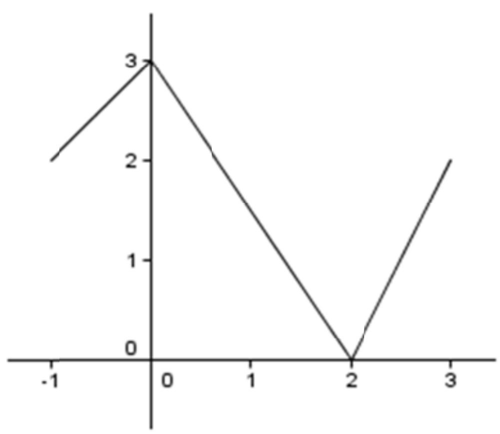
   c. Give the midpoint approximation for $\int_{-1}^{1} \left(4 - x^2\right) dx$ with respect to the partition $P = \left\{-1, -\frac{1}{2}, \frac{1}{2}, 1\right\}$.

   d. Give the midpoint approximation for $\int_{-1}^{1} \left(4 - x^2\right) dx$ using $n = 4$.

   e. Give the left hand endpoint approximation for $\int_{-1}^{1} \left(4 - x^2\right) dx$ using $n = 4$.

   f. Give the right hand endpoint approximation for $\int_{-1}^{1} \left(4 - x^2\right) dx$ using $n = 4$.

   g. Use the graph of $f(x)$ below.



Give the upper sum for $f(x)$ on the interval [-1,3] with respect to the partition $P = \left\{-1, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, 3\right\}$.

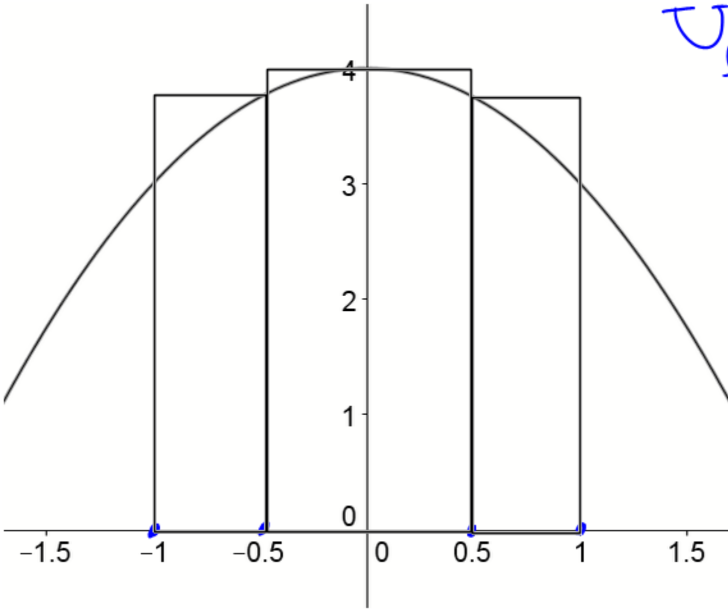
1. Riemann sums. Sketch the region associated with the approximation, and then given the requested value.

a. Give the upper sum for $f(x) = 4 - x^2$ on the interval [-1,1] with respect to the partition $P = \left\{-1, -\frac{1}{2}, \frac{1}{2}, 1\right\}$.

The upper sum for f(x) with respect to P is

$U_f(P) = f(-.5) \cdot \frac{1}{2} + f(0) \cdot 1 + f(.5) \cdot \frac{1}{2}$

$= 7.75$

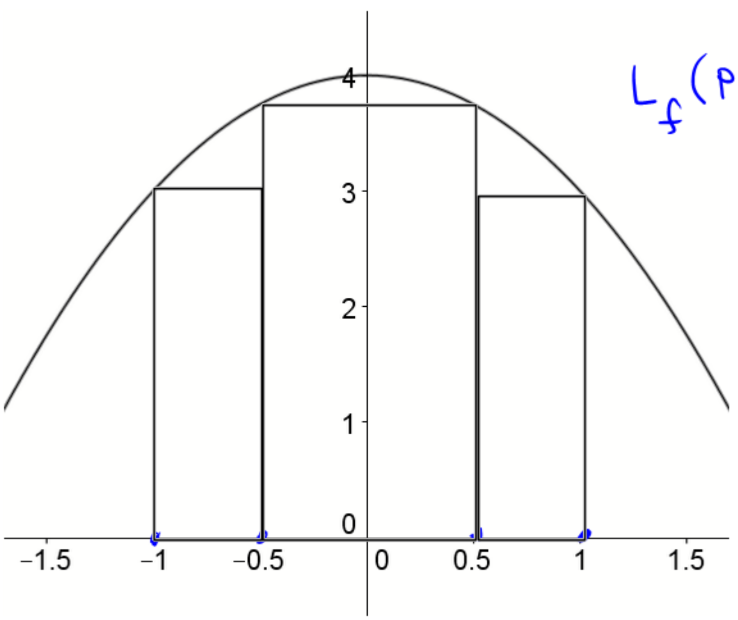


b. Give the lower sum for $f(x) = 4 - x^2$ on the interval [-1,1] with respect to the partition $P = \left\{-1, -\frac{1}{2}, \frac{1}{2}, 1\right\}$.

The lower sum for f(x) with respect to P is

$L_f(P) = f(-1) \cdot \frac{1}{2} + f\left(\frac{1}{2}\right) \cdot 1 + f(1) \cdot \frac{1}{2}$

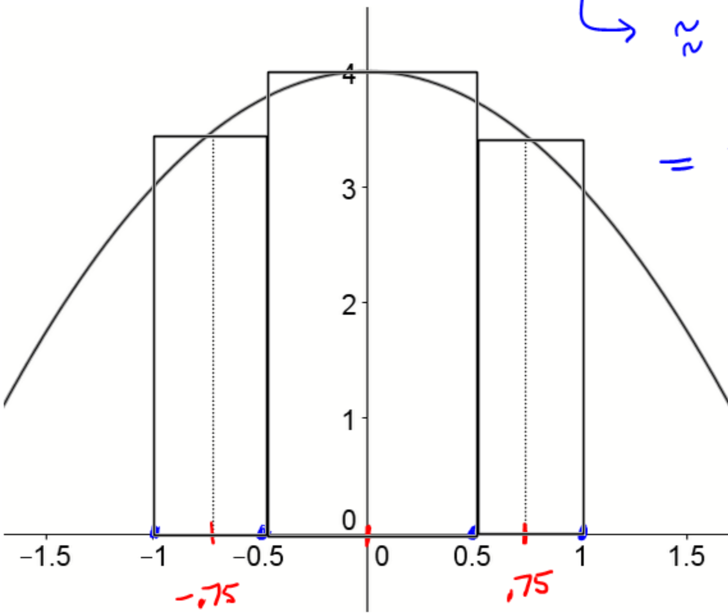
$= 6.75$



c. Give the midpoint approximation for $\int_{-1}^{1}\left(4-x^2\right)dx$ with respect to the partition

$$P=\left\{-1,-\frac{1}{2},\frac{1}{2},1\right\}.$$

$$\approx f(-.75)\cdot\frac{1}{2}+f(0)\cdot 1+f(.75)\cdot\frac{1}{2}$$



$$= 7.4375$$

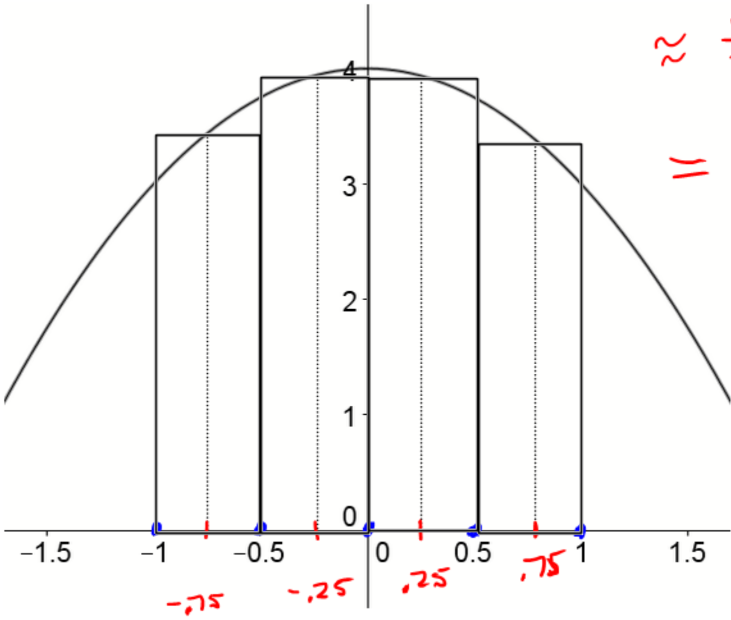
Note: $\int_{-1}^{1}(4-x^2)dx = \frac{22}{3} = 7.3\overline{3}$

So the midpoint approximation is more accurate than the approximations in a and b.

However, the average of a and b is $\frac{1}{2}(7.75+6.75) = 7.25$, which is a decent approximation.

In general, the average of the upper and lower sums will give a reasonable approximation of the definite integral, as will the midpoint approximation. The upper and lower sums typically do not do as good a job of approximating the definite integral.

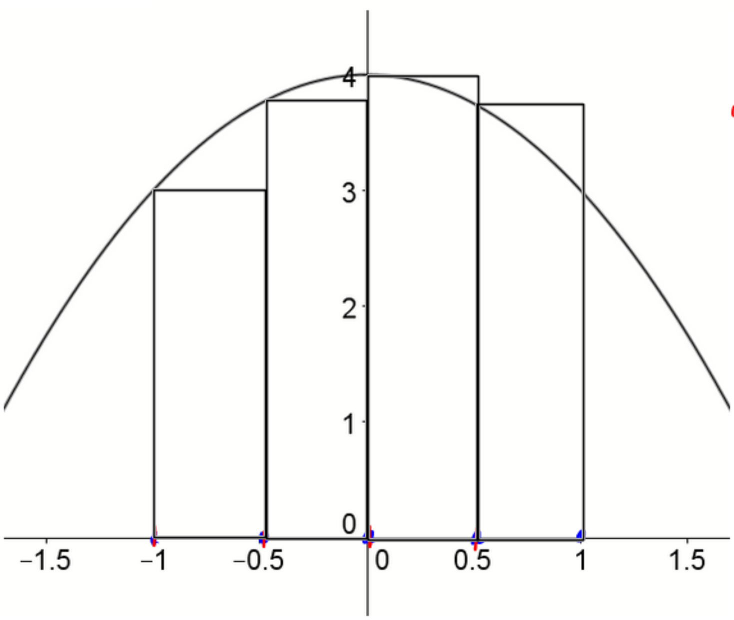
d. Give the midpoint approximation for $\int_{-1}^{1}\left(4-x^2\right)dx$ using $n=4$.



$\approx \frac{1}{2}\left(f(-.75)+f(-.25)+f(.25)+f(.75)\right)$

$= 7.375$

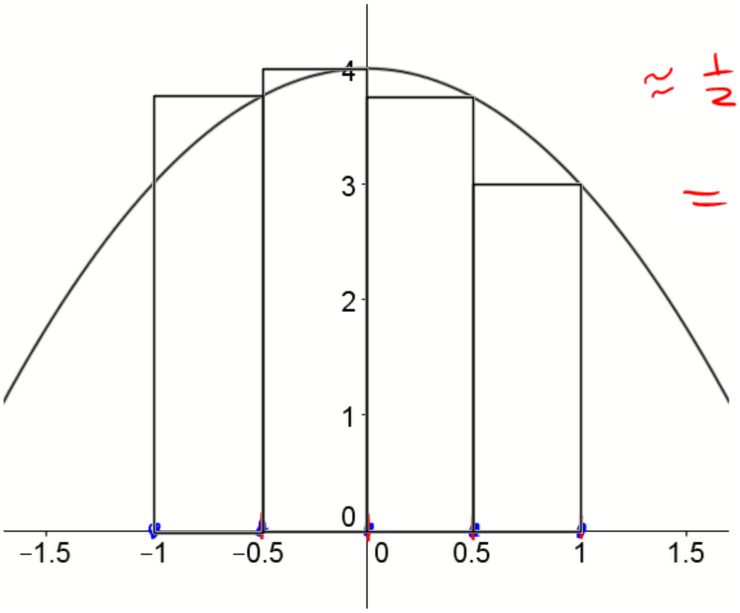
e. Give the left hand endpoint approximation for $\int_{-1}^{1}\left(4-x^2\right)dx$ using $n=4$.



$\approx \frac{1}{2}\left(f(-1)+f(-.5)+f(0)+f(.5)\right)$

$= 7.25$

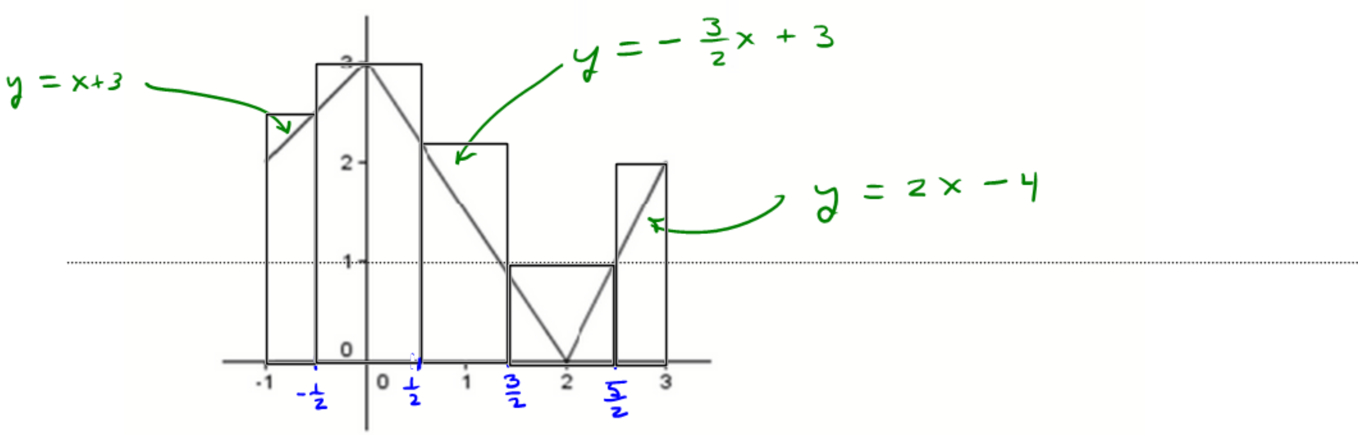
f. Give the right hand endpoint approximation for $\int_{-1}^{1}(4-x^2)\,dx$ using $n=4$.



$\approx \frac{1}{2}\left(f(-.5)+f(0)+f(.5)+f(1)\right)$

$= 7.25$

g. Use the graph of $f(x)$ below.



Give the upper sum for $f(x)$ on the interval [-1,3] with respect to the partition $P=\left\{-1,-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2},3\right\}$.

$U_f(P) \approx f(-\tfrac{1}{2})\cdot\tfrac{1}{2} + f(0)\cdot 1 + f(\tfrac{1}{2})\cdot 1 + f(\tfrac{5}{2})\cdot 1 + f(3)\cdot\tfrac{1}{2}$

$= \frac{5}{2}\cdot\frac{1}{2} + 3\cdot 1 + 2.25\cdot 1 + 1\cdot 1 + 2\cdot\frac{1}{2}$

$= 8.5$

2. Fundamental theorem of calculus.

a. $\frac{d}{dx}\int_{\pi}^{x}\sin(t)\,dt =$ $\sin(x)$

b. $\frac{d}{dx}\int_{3x}^{\pi}\sin(t^2)\,dt =$ $-\sin((3x)^2)\cdot 3 = -3\sin(9x^2)$

c. $\frac{d}{dx}\int_{3x}^{2x}\sin(t^2)\,dt =$ $\sin((2x)^2)\cdot 2 - \sin((3x)^2)\cdot 3$

$= 2\sin(4x^2) - 3\sin(9x^2)$

d. $\frac{d}{dx}\int_{1-2x}^{2x^3}\sin(t^2)\,dt =$

$= \sin((2x^3)^2)\cdot 6x^2 - \sin((1-2x)^2)\cdot(-2)$

$= 6x^2\sin(4x^6) + 2\sin((1-2x)^2)$.

3. Basic Integration.

a. $\int_2^7 x\sqrt{x^2+2}\,dx =$

See below.

b. $\int \left(3\sec^2(r) - 2\sqrt{r-1}\right) dr =$

c. $\int_0^1 \frac{\cos(x)}{2+\sin(x)}\,dx =$

d. $\int_0^1 e^{-2x}\,dx =$

e. $\int_0^{\pi/4} \frac{2}{1+x^2}\,dx =$

f. $\int_0^{\pi/4} \sec(x)\tan(x)\,dx =$

g. $\int \frac{3t}{t^4+1}\,dt =$

(a) $\int_2^7 x\sqrt{x^2+2}\,dx = \frac{1}{2}\int_2^7 \sqrt{\underbrace{x^2+2}_{u}} \cdot \underbrace{2x\,dx}_{du}$

$u = x^2+2$

$du = 2x\,dx$

$x = 7 \Rightarrow u = 51$

$x = 2 \Rightarrow u = 6$

$= \frac{1}{2}\int_6^{51} \sqrt{u}\,du$

$= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_6^{51}$

$= \frac{1}{3}\left(51^{3/2} - 6^{3/2}\right)$

$= 17\sqrt{51} - 2\sqrt{6} \approx 116.505\ldots$

(b) $\int\left(3\sec^2(r)-2\sqrt{r-1}\right)dr = 3\tan(r) - 2\cdot\frac{2}{3}(r-1)^{3/2} + C$

$$= 3\tan(r) - \frac{4}{3}(r-1)^{3/2} + C.$$

(c) $\int_0^1 \frac{\cos(x)}{2+\sin(x)}dx = \int_0^1 \frac{1}{\underbrace{\boxed{2+\sin(x)}}_{u}}\cdot\underbrace{\cos(x)\,dx}_{du}$

$u = 2+\sin(x)$

$du = \cos(x)\,dx$

$x=1 \Rightarrow u = 2+\sin(1)$

$x=0 \Rightarrow u = 2$

$$= \int_2^{2+\sin(1)} \frac{1}{u}\,du$$

$$= \ln|u| \Big|_2^{2+\sin(1)}$$

$$= \ln(2+\sin(1)) - \ln(2)$$

$$= \ln\left(1+\frac{1}{2}\sin(1)\right) \approx 0.351\ldots$$

Note: $2+\sin(1) > 0$

$2 > 0$

(d) $\int_0^1 e^{-2x}dx = -\frac{1}{2}e^{-2x}\Big|_0^1$

$$= -\frac{1}{2}\left(e^{-2} - e^0\right)$$

$$= \frac{1}{2} - \frac{1}{2e^2} \approx 0.432\ldots$$

(e) $\int_0^{\pi/4} \frac{2}{1+x^2}\,dx = 2\int_0^{\pi/4} \frac{1}{1+x^2}\,dx$

$$= 2\arctan(x)\Big|_0^{\pi/4} = 2\left(\arctan\left(\frac{\pi}{4}\right) - \arctan(0)\right)$$

$\boxed{\text{Note: } \arctan(x) \equiv \tan^{-1}(x)}$

$$= 2\arctan(\pi/4)$$

$$\approx 1.331\ldots$$

(f) $\int_0^{\pi/4} \sec(x)\tan(x)\,dx = \sec(x)\Big|_0^{\pi/4} = \sec\left(\frac{\pi}{4}\right) - \sec(0)$

$$= \sqrt{2} - 1$$

$\boxed{\text{Note: } \frac{d}{dx}\sec(x) = \sec(x)\tan(x)}$

$$\approx 0.414\ldots$$

(g) $\int \frac{3t}{t^4+1}\,dt = \int \frac{3t}{(t^2)^2+1}\,dt = \frac{1}{2}\cdot 3\int \frac{1}{(\underset{u}{t^2})^2+1}\cdot \underbrace{2t\,dt}_{du}$

$u = t^2$

$du = 2t\,dt$

$$= \frac{3}{2}\int \frac{1}{u^2+1}\,du$$

$$= \frac{3}{2}\arctan(u) + C$$

$$= \frac{3}{2}\arctan(t^2) + C$$

4. Area. Graph each region and find the requested area.

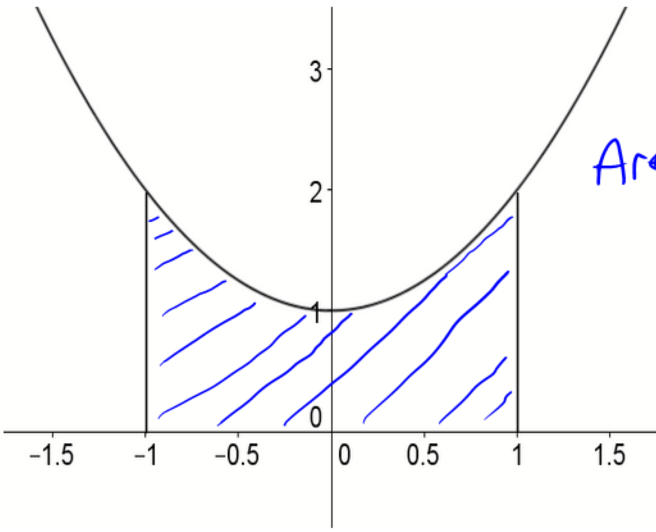
   a. Find the area bounded by the graph of $f(x) = 1 + x^2$ and the x-axis over the interval [-1,1].

   b. Find the area bounded by the graph of $f(x) = 1 - e^x$ and the x-axis over the interval [0,1].

   c. Find the area bounded by the graph of $f(x) = \sin(x)$ and the x-axis over the interval $[\pi/2, \pi]$.

   d. Give the area bounded between the x-axis and the graph of $f(x) = x^2 + 2x - 3$ over the interval [-2,2].

(a)



Note: $1+x^2 \geq 0$ on $[-1,1]$

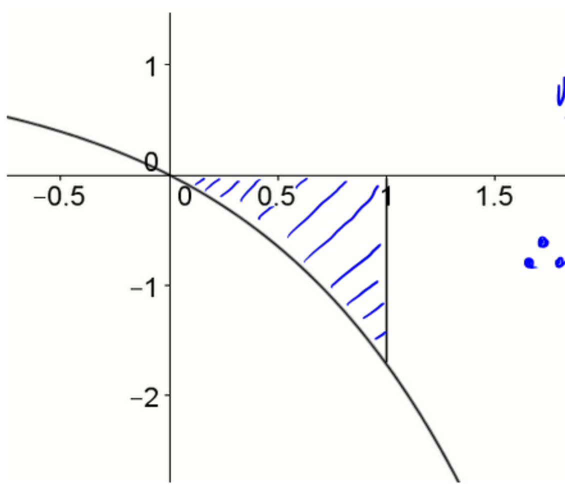
$$\text{Area} = \int_{-1}^{1} (1+x^2)\,dx$$

$$= \left(x + \frac{1}{3}x^3\right)\Big|_{-1}^{1}$$

$$= \left(1+\frac{1}{3}\right) - \left(-1-\frac{1}{3}\right)$$

$$= \frac{8}{3}$$

(b)



Note: $1-e^x \leq 0$ on $[0,1]$.

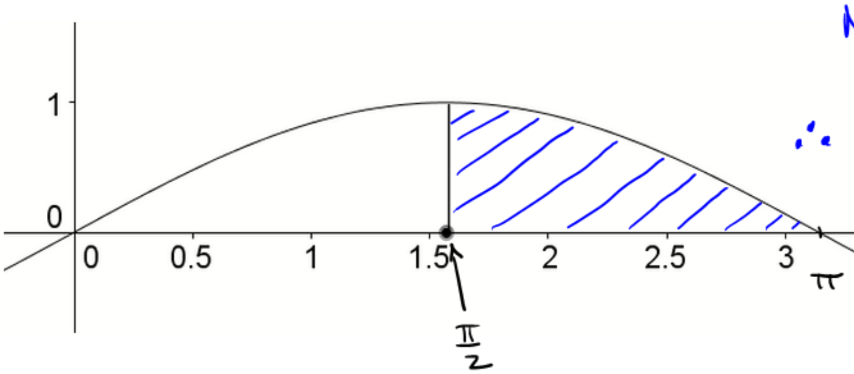
$$\therefore \text{Area} = -\int_0^1 (1-e^x)\,dx$$

$$= -(x-e^x)\Big|_0^1$$

$$= -[(1-e)-(0-1)]$$

$$= e-2 \approx 0.718\ldots$$

(c) Find the area bounded by the graph of $f(x)=\sin(x)$ and the x-axis over the interval $[\pi/2, \pi]$.



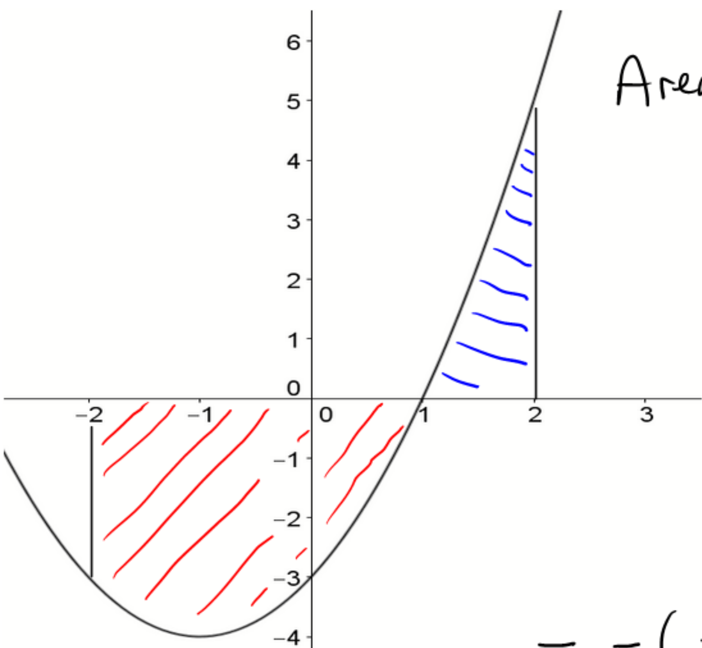
Note: $\sin(x) \geq 0$ on $\left[\frac{\pi}{2}, \pi\right]$

$$\therefore \text{ Area} = \int_{\frac{\pi}{2}}^{\pi} \sin(x)\,dx$$

$$= -\cos(x)\Big|_{\pi/2}^{\pi}$$

$$= -(-1-0) = 1.$$

(d) Give the area bounded between the x-axis and the graph of $f(x)=x^2+2x-3$ over the interval [-2,2].



Area = Area( ) + Area( )

Note: $f(x) \leq 0$ on $[-2, 1]$

$f(x) \geq 0$ on $[1, 2]$

$$= -\int_{-2}^{1}(x^2+2x-3)\,dx + \int_{1}^{2}(x^2+2x-3)\,dx$$

$$= -\left(\frac{x^3}{3}+x^2-3x\right)\Bigg|_{-2}^{1} + \left(\frac{x^3}{3}+x^2-3x\right)\Bigg|_{1}^{2}$$

$$= -\left[\left(\frac{1}{3}+1-3\right)-\left(-\frac{8}{3}+4+6\right)\right] + \left[\left(\frac{8}{3}+4-6\right)-\left(\frac{1}{3}+1-3\right)\right]$$

$$= \frac{34}{3}$$

5. Anti-derivatives.

a. Give the general anti-derivative for $g(x) = x^3 + 2x - \sqrt{x}$.

b. $F(x)$ is the anti-derivative for the function $x\sqrt{x^2+3}$ that satisfies $F(-1) = 2$. Give $3\left(F(0) - \sqrt{3}\right) + 1$.

c. $F''(x) = x^2 - \dfrac{2}{\sqrt{x}} + 1$, $F'(1) = -3$ and $F(1) = 2$. Give $F(x)$.

d. Find a formula for $f(x)$, given that $2x^3 - 3x^2 + x - 1 = \int_{-1}^{x} f(t)dt$.

e. Suppose *f* (*x*) is an anti-derivative of *r* (*x*), and *g* (*x*) is an anti-derivative of *s* (*x*). We are given the data in the table about the functions *f*, *g*, *r* and *s*. $\int_1^3 \left(3r(x) - 2s(x)\right)dx =$

x	1	2	3	4
$f(x)$	3	2	1	4
$r(x)$	1	4	2	3
$g(x)$	2	1	4	3
$s(x)$	4	2	3	1

a) Give the general anti-derivative for $g(x) = x^3 + 2x - \sqrt{x}$. $= \int (x^3 + 2x - \sqrt{x})dx$

$$= \frac{1}{4}x^4 + x^2 - \frac{2}{3}x^{3/2} + C$$

b) $F(x)$ is the anti-derivative for the function $x\sqrt{x^2+3}$ that satisfies $F(-1) = 2$. Give $3\left(F(0) - \sqrt{3}\right) + 1$.

The general anti-derivative of $x\sqrt{x^2+3}$ is

$$\int x\sqrt{x^2+3}\,dx = \frac{1}{2}\int \sqrt{x^2+3}\cdot 2\cdot x\,dx = \frac{1}{2}\cdot\frac{2}{3}(x^2+3)^{3/2} + C$$

$\therefore \quad F(x) = \frac{1}{3}(x^2+3)^{3/2} + C$ and $F(-1)=2$

So, $\quad 2 = \frac{1}{3}((-1)^2+3)^{3/2} + C$

$2 = \frac{8}{3} + C \Rightarrow C = -\frac{2}{3}$

$\therefore \quad F(x) = \frac{1}{3}(x^2+3)^{3/2} - \frac{2}{3}.$

$\Rightarrow \quad 3(F(0) - \sqrt{3}) + 1 = 3\left(\left(\frac{1}{3}3^{3/2} - \frac{2}{3}\right) - \sqrt{3}\right) + 1$

$= 3\left(\sqrt{3} - \frac{2}{3} - \sqrt{3}\right) + 1$

$= -1.$

(c) $F''(x) = x^2 - \frac{2}{\sqrt{x}} + 1$, $F'(1) = -3$ and $F(1) = 2$. Give $F(x)$.

$F'(x) = \int\left(x^2 - \frac{2}{\sqrt{x}} + 1\right)dx = \frac{1}{3}x^3 - 4\sqrt{x} + x + C$

and $F'(1) = -3.$

$\therefore \quad -3 = \frac{1}{3} - 4 + 1 + C \Rightarrow C = -\frac{1}{3}$

$\Rightarrow F'(x) = \frac{1}{3}x^3 - 4\sqrt{x} + x - \frac{1}{3}$

$\Rightarrow F(x) = \int\left(\frac{1}{3}x^3 - 4\sqrt{x} + x - \frac{1}{3}\right)dx$

$= \frac{1}{12}x^4 - \frac{8}{3}x^{3/2} + \frac{1}{2}x^2 - \frac{1}{3}x + \tilde{C}$

and $F(1) = 2$

$\Rightarrow \quad 2 = \frac{1}{12} - \frac{8}{3} + \frac{1}{2} - \frac{1}{3} + \tilde{C}$

$\Rightarrow \tilde{C} = \frac{53}{12}$

$\therefore \quad F(x) = \frac{1}{12}x^4 - \frac{8}{3}x^{3/2} + \frac{1}{2}x^2 + \frac{17}{3}x + \frac{53}{12}.$

(d) Find a formula for $f(x)$, given that $2x^3 - 3x^2 + x - 1 = \int_{-1}^{x} f(t)\,dt$.

Differentiate both sides with respect to x.

$$\frac{d}{dx}\left(2x^3 - 3x^2 + x - 1\right) = \frac{d}{dx}\int_{-1}^{x} f(t)\,dt$$

$$\boxed{6x^2 - 6x + 1 = f(x)}$$

(e) Suppose $f(x)$ is an anti-derivative of $r(x)$, and $g(x)$ is an anti-derivative of $s(x)$. We are given the data in the table about the functions f, g, r and s. $\int_{1}^{3}\left(3r(x) - 2s(x)\right)dx =$

x	1	2	3	4
$f(x)$	3	2	1	4
$r(x)$	1	4	2	3
$g(x)$	2	1	4	3
$s(x)$	4	2	3	1

$$\int_{1}^{3}\left(3r(x) - 2s(x)\right)dx = \left(3f(x) - 2g(x)\right)\Big|_{1}^{3}$$

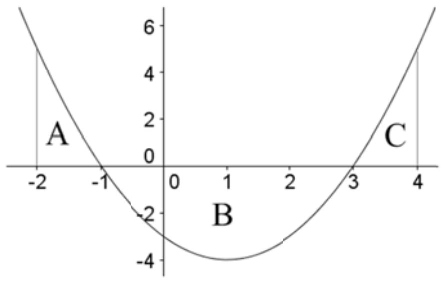
$$= \left(3f(3) - 2g(3)\right) - \left(3f(1) - 2g(1)\right)$$

$$= \left(3\cdot 1 - 2\cdot 4\right) - \left(3\cdot 3 - 2\cdot 2\right)$$

$$= (3-8) - (9-4)$$

$$= -5 - 5 = -10.$$

6. **Use the following information in parts a-c**. The graph of $f(x)$ is shown below, and $f(-2)=5$. The area of region A is 7/3, the area of region B is 34/3, and the area of region C is 7/3.



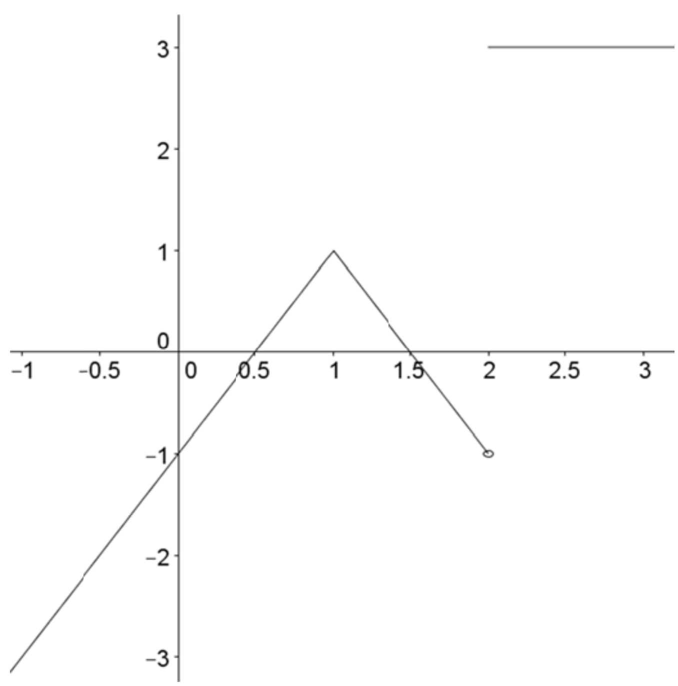
Next page

a. Give the area of the region bounded between the graph of $f(x)$ and the *x*-axis on the interval [-2 , 4].

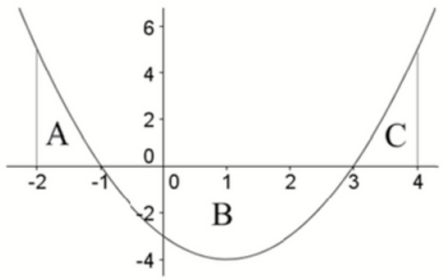
b. $\int_{-1}^{4} f(x)dx =$

c. $\int_{-1}^{3/2}\left(3x-\frac{d}{dx}f(2x)\right)dx =$

d. The graph of $y = g'(x)$ is shown below, and $g(1)=1$. Give the values for $g(0)$, $g(2)$ and $g(3)$.



Use the following information in parts a-c. The graph of $f(x)$ is shown below, and $f(-2)=5$. The area of region A is 7/3, the area of region B is 34/3, and the area of region C is 7/3.



a. Give the area of the region bounded between the graph of $f(x)$ and the x-axis on the interval [-2 , 4].

Area = Area (A) + Area (B) + Area (C)

$= \frac{7}{3} + \frac{34}{3} + \frac{7}{3} = \frac{48}{3} = 16.$

b. $\int_{-1}^{4} f(x)dx = \int_{-1}^{3} f(x)\,dx + \int_{3}^{4} f(x)\,dx$

Notes: ① $f(x) \le 0$ on $[-1, 3]$

$\Rightarrow$ Area (B) $= -\int_{-1}^{3} f(x)dx$

$\Rightarrow \int_{-1}^{3} f(x)\,dx = -\frac{34}{3}$

② $f(x) \ge 0$ on $[3, 4]$

$\Rightarrow$ Area(C) $= \int_{3}^{4} f(x)dx$

$\Rightarrow \int_{3}^{4} f(x)dx = \frac{7}{3}.$

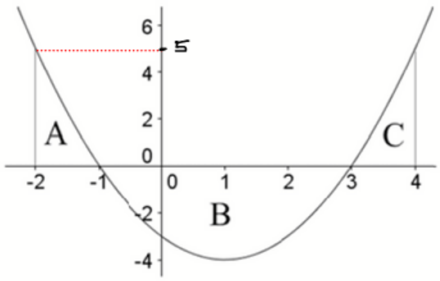
$= -\frac{34}{3} + \frac{7}{3} = \frac{-27}{3} = -9$

(c) $\int_{-1}^{3/2}\left(3x-\frac{d}{dx}f(2x)\right)dx = \int_{-1}^{3/2} 3x\,dx - \int_{-1}^{3/2}\frac{d}{dx}f(2x)\,dx$

$$= \frac{3}{2}x^2\Big|_{-1}^{3/2} - f(2x)\Big|_{-1}^{3/2}$$

$$= \frac{3}{2}\left(\frac{9}{4}-1\right) - \left(f(3)-f(-2)\right)$$

$$= \frac{15}{8} - 0 + 5 = \frac{55}{8}.$$

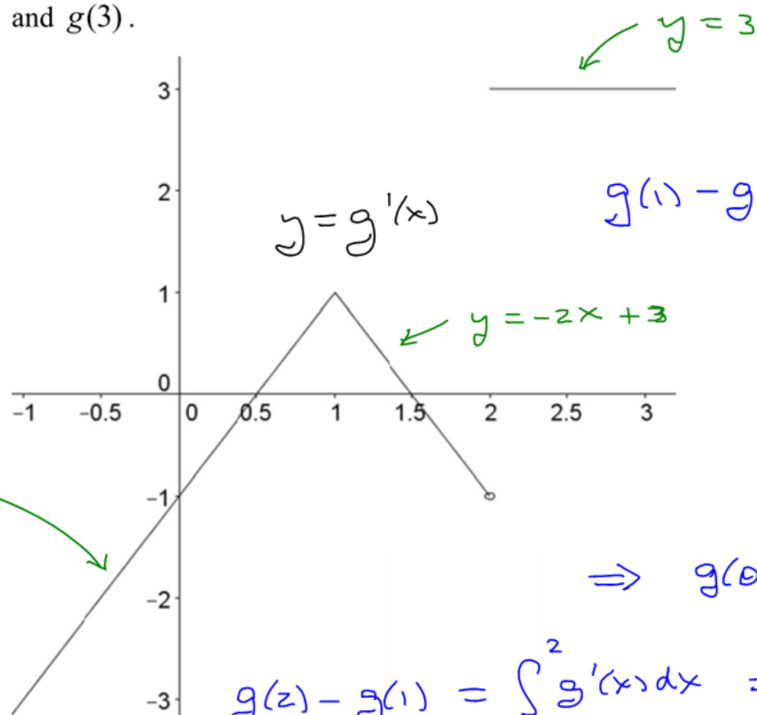


From the graph,

$f(3)=0$

$f(-2)=5$

(d) The graph of $y=g'(x)$ is shown below, and $g(1)=1$. Give the values for $g(0)$, $g(2)$ and $g(3)$.



$$g(1)-g(0) = \int_0^1 g'(x)\,dx$$

$$= \int_0^1 (2x-1)\,dx$$

$$= (x^2-x)\Big|_0^1$$

$$= 0.$$

$$\Rightarrow g(0)=g(1)=1.$$

$$g(2)-g(1) = \int_1^2 g'(x)\,dx = \int_1^2(-2x+3)\,dx = (-x^2+3x)\Big|_1^2$$

$$= (-4+6)-(-1+3) = 2-2=0.$$

$$\Rightarrow g(2)=g(1)=1.$$

$$g(3)-g(2) = \int_2^3 g'(x)\,dx = \int_2^3 3\,dx = 3 \Rightarrow g(3)=g(2)+3$$

$$= 4$$