1. Riemann sums. Sketch the region associated with the approximation, and then given the requested value.
a. Give the upper sum for $f(x)=4-x^{2}$ on the interval $[-1,1]$ with respect to the partition

See next

$$
P=\left\{-1,-\frac{1}{2}, \frac{1}{2}, 1\right\} .
$$

b. Give the lower sum for $f(x)=4-x^{2}$ on the interval $[-1,1]$ with respect to the partition

$$
P=\left\{-1,-\frac{1}{2}, \frac{1}{2}, 1\right\} .
$$

c. Give the midpoint approximation for $\int_{-1}^{1}\left(4-x^{2}\right) d x$ with respect to the partition $P=\left\{-1,-\frac{1}{2}, \frac{1}{2}, 1\right\}$.
d. Give the midpoint approximation for $\int_{-1}^{1}\left(4-x^{2}\right) d x$ using $n=4$.
e. Give the left hand endpoint approximation for $\int_{-1}^{1}\left(4-x^{2}\right) d x$ using $n=4$.
f. Give the right hand endpoint approximation for $\int_{-1}^{1}\left(4-x^{2}\right) d x$ using $n=4$.
g. Use the graph of $f(x)$ below.


Give the upper sum for $f(x)$ on the interval $[-1,3]$ with respect to the partition $P=\left\{-1,-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, 3\right\}$.

1. Riemann sums. Sketch the region associated with the approximation, and then given the requested value.
a. Give the upper sum for $f(x)=4-x^{2}$ on the interval $[-1,1]$ with respect to the partition $P=\left\{-1,-\frac{1}{2}, \frac{1}{2}, 1\right\}$. The upper sum for $f(x)$ with respect to $P$ is

b. Give the lower sum for $f(x)=4-x^{2}$ on the interval $[-1,1]$ with respect to the partition

$$
P=\left\{-1,-\frac{1}{2}, \frac{1}{2}, 1\right\} .
$$

The lower sum for $f(x)$ with respect to $P$ is



$$
P=\left\{-1,-\frac{1}{2}, \frac{1}{2}, 1\right\}
$$

$$
\Gamma
$$

$$
\longrightarrow \approx f(-.75) \cdot \frac{1}{2}+f(0) \cdot 1+f(.75) \cdot \frac{1}{2}
$$



Note: $\int_{-1}\left(4-x^{2}\right) d x=\frac{22}{3}=7.3 \overline{3}$
so the midpoint approximation is more accurate than the approximations in (a) and (b).

However, the average of (a) and

$$
\text { is } \frac{1}{2}(7.75+6.75)=7.25
$$

which is a decent approximation.

In general, the average of the upper and lower sums will give a reasonable approximation of the definite integral, as will the midpoint approximation. The upper and lower sums typically do not do as good a job of approximating the definite integral.
d. Give the midpoint approximation for $\int_{-1}^{1}\left(4-x^{2}\right) d x$ using $n=4$.

e. Give the left hand endpoint approximation for $\int_{-1}^{1}\left(4-x^{2}\right) d x$ using $n=4$.

f. Give the right hand endpoint approximation for $\int_{-1}^{1}\left(4-x^{2}\right) d x$ using $n=4$.

g. Use the graph of $f(x)$ below.


Give the upper sum for $f(x)$ on the interval $[-1,3]$ with respect to the partition

$$
\begin{aligned}
& P=\left\{-1,-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, 3\right\} \\
& \begin{aligned}
\cup_{f}(P) & \approx f\left(-\frac{1}{2}\right) \cdot \frac{1}{2}+f(0) \cdot 1+f\left(\frac{1}{2}\right) \cdot 1+f\left(\frac{5}{2}\right) \cdot 1+f(3) \cdot \frac{1}{2} \\
& =\frac{5}{2} \cdot \frac{1}{2}+3 \cdot 1+2.25 \cdot 1+1 \cdot 1+2 \cdot \frac{1}{2} \\
& =8.5
\end{aligned}
\end{aligned}
$$

2. Fundamental theorem of calculus.
a. $\frac{d}{d x} \int_{\pi}^{x} \sin (t) d t=\sin (x)$
b. $\frac{d}{d x} \int_{3 x}^{\pi} \sin \left(t^{2}\right) d t=-\sin \left((3 x)^{2}\right) \cdot 3=-3 \sin \left(9 x^{2}\right)$
c. $\frac{d}{d x} \int_{3 x}^{2 x} \sin \left(t^{2}\right) d t=\sin \left((2 x)^{2}\right) \cdot 2-\sin \left((3 x)^{2}\right) \cdot 3$
d. $\frac{d}{d x} \int_{1-2 x}^{2 x^{3}} \sin \left(t^{2}\right) d t=$

$$
=2 \sin \left(4 x^{2}\right)-3 \sin \left(9 x^{2}\right)
$$

$$
\begin{aligned}
C & =\sin \left(\left(2 x^{3}\right)^{2}\right) \cdot 6 x^{2}-\sin \left((1-2 x)^{2}\right) \cdot(-2) \\
& =6 x^{2} \sin \left(4 x^{6}\right)+2 \sin \left((1-2 x)^{2}\right)
\end{aligned}
$$

3. Basic Integration.
a. $\int_{2}^{7} x \sqrt{x^{2}+2} d x=$
see
below.
b. $\int\left(3 \sec ^{2}(r)-2 \sqrt{r-1}\right) d r=$
c. $\int_{0}^{1} \frac{\cos (x)}{2+\sin (x)} d x=$
d. $\int_{0}^{1} e^{-2 x} d x=$
e. $\int_{0}^{\pi / 4} \frac{2}{1+x^{2}} d x=$
f. $\int_{0}^{\pi / 4} \sec (x) \tan (x) d x=$
g. $\int \frac{3 t}{t^{4}+1} d t=$
(a) $\int_{2}^{7} x \sqrt{x^{2}+2} d x=\frac{1}{2} \int_{2}^{7} \sqrt[\underbrace{x^{2}+2}_{u}]{1} \cdot \underbrace{2 x d x}_{d u}$

$$
\left.\begin{array}{rl}
u=x^{2}+2 \\
d u & =2 x d x \\
x=2 \Rightarrow u=51
\end{array}\right)=\frac{1}{2} \int_{6=6} \sqrt{u} d u
$$

(b)

$$
\begin{aligned}
\int\left(3 \sec ^{2}(r)-2 \sqrt{r-1}\right) d r & =3 \tan (r)-2 \cdot \frac{2}{3}(r-1)^{3 / 2}+C \\
& =3 \tan (r)-\frac{1}{3}(r-1)^{3 / 2}+C .
\end{aligned}
$$

> (C)
$\int_{0}^{1} \frac{\cos (x)}{2+\sin (x)} d x=$

$$
\int_{0}^{1} \frac{1}{\frac{1}{2+\sin (x)}} \cdot \underbrace{\cos (x) d x}_{d u}
$$

$$
\left[\begin{array}{rl}
u=2+\sin (x) \\
d u=\cos (x) d x \\
x=1 \Rightarrow u=2+\sin (1)
\end{array}\right)=\int_{2}^{2+\sin (1)} \frac{1}{u} d u
$$

Note: $2+\sin (1)>0$

$$
2>0
$$

(d) $\int_{0}^{1} e^{-2 x} d x=-\left.\frac{1}{2} e^{-2 x}\right|_{0} ^{1}$

$$
\begin{aligned}
& =-\frac{1}{2}\left(e^{-2}-e^{0}\right) \\
& =\frac{1}{2}-\frac{1}{2 e^{2}} \approx 0.432 \cdots
\end{aligned}
$$

(e)

$$
\begin{aligned}
\int_{0}^{\pi / 4} \frac{2}{1+x^{2}} d x & =2 \int_{0}^{\pi / 4} \frac{1}{1+x^{2}} d x \\
& =\left.2 \arctan (x)\right|_{0} ^{\pi / 4}=2\left(\arctan \left(\frac{\pi}{4}\right)-\arctan (0)\right)
\end{aligned}
$$

Note: $\arctan (x) \equiv \tan ^{-1}(x)$

$$
\begin{aligned}
& =2 \arctan (\pi / 4) \\
& \approx 1.331 \cdots
\end{aligned}
$$

(f) $\int_{0}^{\pi / 4} \sec (x) \tan (x) d x=\left.\sec (x)\right|_{0} ^{\pi / 4}=\sec \left(\frac{\pi}{4}\right)-\sec (0)$

Nore: $\frac{d}{d x} \sec (x)=\sec (x) \tan (x)$

$$
\begin{aligned}
& =\sqrt{2}-1 \\
& \approx 0.414 \ldots
\end{aligned}
$$

(9)

$$
\begin{aligned}
& \int \frac{3 t}{t^{4}+1} d t=\int \frac{3 t}{\left(t^{2}\right)^{2}+1} d t=\frac{1}{2} \cdot 3 \int \frac{1}{\left(t_{\uparrow}^{2}\right)^{2}+1} \cdot \underbrace{2 t d t}_{u} \\
& \begin{aligned}
& u=t^{2} \\
& d u=2 t d t=\frac{3}{2} \int \frac{1}{u^{2}+1} d u \\
&=\frac{3}{2} \arctan (u)+C \\
&=\frac{3}{2} \arctan \left(t^{2}\right)+C
\end{aligned} .
\end{aligned}
$$

4. Area. Graph each region and find the requested area.
a. Find the area bounded by the graph of $f(x)=1+x^{2}$ and the $x$-axis over the interval [-1,1].
b. Find the area bounded by the graph of $f(x)=1-e^{x}$ and the $x$-axis over the interval [0,1].
c. Find the area bounded by the graph of $f(x)=\sin (x)$ and the $x$-axis over the interval $[\pi / 2, \pi]$.
d. Give the area bounded between the $x$-axis and the graph of $f(x)=x^{2}+2 x-3$ over the interval [-2,2].

Note: $1+x^{2} \geqslant 0$ on $[-1,1]$
(a)


$$
\begin{aligned}
& =\left.\left(x+\frac{1}{3} x^{3}\right)\right|_{-1} ^{1} \\
& =\left(1+\frac{1}{3}\right)-\left(-1-\frac{1}{3}\right)
\end{aligned}
$$

$$
=\frac{8}{3}
$$

(b)

$$
\begin{aligned}
& \text { Note: } 1-e^{x} \leq 0 \text { on }[0,1] \\
& \therefore \text { Area }=-\int_{0}^{1}\left(1-e^{x}\right) d x \\
&=-\left.\left(x-e^{x}\right)\right|_{0} ^{1} \\
&=-[(1-e)-(0-1)] \\
&=e-2 \approx 0.718 \cdots
\end{aligned}
$$

(C) Find the area bounded by the graph of $f(x)=\sin (x)$ and the $x$-axis over the interval $[\pi / 2, \pi]$.

Note: $\sin (x) \geqslant 0$ on $\left[\frac{\pi}{2}, \pi\right]$

(d) Give the area bounded between the $x$-axis and the graph of $f(x)=x^{2}+2 x-3$ over the interval [-2,2].


$$
=-\left[\left(\frac{1}{3}+1-3\right)-\left(-\frac{8}{3}+4+6\right)\right]+\left[\left(\frac{8}{3}+4-6\right)-\left(\frac{1}{3}+1-3\right)\right]
$$

$$
=\frac{34}{3}
$$

5. Anti-derivatives.
a. Give the general anti-derivative for $g(x)=x^{3}+2 x-\sqrt{x}$.
b. $\quad F(x)$ is the anti-derivative for the function $x \sqrt{x^{2}+3}$ that satisfies $F(-1)=2$. Give $3(F(0)-\sqrt{3})+1$.
c. $\quad F^{\prime}(x)=x^{2}-\frac{2}{\sqrt{x}}+1, \quad F^{\prime}(1)=-3$ and $F(1)=2$. Give $F(x)$.
d. Find a formula for $f(x)$, given that $2 x^{3}-3 x^{2}+x-1=\int_{-1}^{x} f(t) d t$.
e. Suppose $f(x)$ is an anti-derivative of $r(x)$, and $g(x)$ is an anti-derivative of $s(x)$. We are given the data in the table about the functions $f, g, r$ and $s . \int_{1}^{3}(3 r(x)-2 s(x)) d x=$

| $x$ | 1 | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | ---: |
| $f(x)$ | 3 | 2 | 1 | 4 |
| $r(x)$ | 1 | 4 | 2 | 3 |
| $g(x)$ | 2 | 1 | 4 | 3 |
| $s(x)$ | 4 | 2 | 3 | 1 |

(a) Give the general anti-derivative for $g(x)=x^{3}+2 x-\sqrt{x}=\int\left(x^{3}+2 x-\sqrt{x}\right) d x$

$$
=\frac{1}{4} x^{4}+x^{2}-\frac{2}{3} x^{3 / 2}+C
$$

D) $F(x)$ is the anti-derivative for the function $x \sqrt{x^{2}+3}$ that satisfies $F(-1)=2$. Give

$$
3(F(0)-\sqrt{3})+1
$$

The general anti-derivative of $x \sqrt{x^{2}+3}$ is

$$
\int x \sqrt{x^{2}+3} d x=\frac{1}{2} \int \sqrt{x^{2}+3} \cdot 2 \cdot x d x=\frac{1}{2} \cdot \frac{2}{3}\left(x^{2}+3\right)^{3 / 2}+C
$$

$$
\begin{aligned}
\therefore \quad F(x) & =\frac{1}{3}\left(x^{2}+3\right)^{3 / 2}+C \quad \text { and } F(-1)=2 \\
\text { So, } \quad 2 & =\frac{1}{3}\left((-1)^{2}+3\right)^{3 / 2}+C \\
2 & =\frac{8}{3}+C \quad C=-\frac{2}{3} \\
\therefore \quad F(x) & =\frac{1}{3}\left(x^{2}+3\right)^{3 / 2}-\frac{2}{3} \\
\Rightarrow \quad 3(F(0)-\sqrt{3})+1 & =3\left(\left(\frac{1}{3} 3^{3 / 2}-\frac{2}{3}\right)-\sqrt{3}\right)+1 \\
& =3\left(\sqrt{3}-\frac{2}{3}-\sqrt{3}\right)+1 \\
& =-1
\end{aligned}
$$

C. $F^{\prime \prime}(x)=x^{2}-\frac{2}{\sqrt{x}}+1, F^{\prime}(1)=-3$ and $F(1)=2$. Give $F(x)$.

$$
F^{\prime}(x)=\int\left(x^{2}-\frac{2}{\sqrt{x}}+1\right) d x=\frac{1}{3} x^{3}-4 \sqrt{x}+x+C
$$

and $F^{\prime}(1)=-3$.

$$
\begin{aligned}
\therefore \quad-3 & =\frac{1}{3}-4+1+C \Rightarrow C=-\frac{1}{3} \\
\Rightarrow & F^{\prime}(x)=\frac{1}{3} x^{3}-4 \sqrt{x}+x-\frac{1}{3} \\
\Rightarrow & F(x)=\int\left(\frac{1}{3} x^{3}-4 \sqrt{x}+x-\frac{1}{3}\right) d x \\
& =\frac{1}{12} x^{4}-\frac{8}{3} x^{3 / 2}+\frac{1}{2} x^{2}-\frac{1}{3} x+\tilde{C}
\end{aligned}
$$

and $f(1)=2$

$$
\begin{aligned}
& \text { and } \\
\Rightarrow \quad & 2=\frac{1}{12}-\frac{8}{3}+\frac{1}{2}-\frac{1}{3}+\tilde{C} \\
\Rightarrow & \tilde{C}=\frac{53}{12}
\end{aligned}
$$

$$
\therefore F(x)=\frac{1}{12} x^{4}-\frac{8}{3} x^{3 / 2}+\frac{1}{2} x^{2}+\frac{17}{3} x+\frac{53}{12}
$$

(d) Find a formula for $f(x)$, given that $2 x^{3}-3 x^{2}+x-1=\int_{-1}^{x} f(t) d t$.

Differentiate both sides with respect to $x$.

$$
\begin{aligned}
& \frac{d}{d x}\left(2 x^{3}-3 x^{2}+x-1\right)=\frac{d}{d x} \int_{-1}^{x} f(t) d t \\
& 6 x^{2}-6 x+1=f(x)
\end{aligned}
$$

Suppose $f(x)$ is an anti-derivative of $r(x)$, and $g(x)$ is an anti-derivative of $s(x)$. We are given the data in the table about the functions $f, g, r$ and $s . \int_{1}^{3}(3 r(x)-2 s(x)) d x=$

| $x$ | 1 | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | :--- |
| $f(x)$ | 3 | 2 | 1 | 4 |
| $r(x)$ | 1 | 4 | 2 | 3 |
| $g(x)$ | 2 | 1 | 4 | 3 |
| $s(x)$ | 4 | 2 | 3 | 1 |

$$
\begin{aligned}
& \int_{1}^{3}(3 r(x)-2 s(x)) d x=\left.(3 f(x)-2 g(x))\right|_{1} ^{3} \\
&=(3 f(3)-2 g(3))-(3 f(1)-2 g(1)) \\
&=(3 \cdot 1-2 \cdot 4)-(3 \cdot 3-2 \cdot 2) \\
&=(3-8)-(9-4) \\
&=-5-5=-10
\end{aligned}
$$

6. Use the following information in parts a-c. The graph of $f(x)$ is shown below, and $f(-2)=5$. The area of region A is $7 / 3$, the area of region B is $34 / 3$, and the area of region C is 7/3.

a. Give the area of the region bounded between the graph of $f(x)$ and the $x$-axis on the interval $[-2,4]$.
b. $\int_{-1}^{4} f(x) d x=$
c. $\int_{-1}^{3 / 2}\left(3 x-\frac{d}{d x} f(2 x)\right) d x=$
d. The graph of $y=g^{\prime}(x)$ is shown below, and $g(1)=1$. Give the values for $g(0), g(2)$ and $g(3)$.


Use the following information in parts atc. The graph of $f(x)$ is shown below, and $f(-2)=5$. The area of region A is $7 / 3$, the area of region B is $34 / 3$, and the area of region C is 7/3.

a. Give the area of the region bounded between the graph of $f(x)$ and the $x$-axis on the interval $[-2,4]$.
Area $=$ Area $(A)+\operatorname{Area}(B)+\operatorname{Area}(C)$

$$
=\frac{7}{3}+\frac{34}{3}+\frac{7}{3}=\frac{48}{3}=16 .
$$

b. $\int_{-1}^{4} f(x) d x=\int_{-1}^{3} f(x) d x+\int_{3}^{4} f(x) d x$

(c) $\int_{-1}^{3 / 2}\left(3 x-\frac{d}{d x} f(2 x)\right) d x=\int_{-1}^{3 / 2} 3 x d x-\int_{-1}^{3 / 2} \frac{d}{d x} f(2 x) d x$


From the graph,

$$
\begin{aligned}
& f(3)=0 \\
& f(-2)=5
\end{aligned}
$$

(d) The graph of $y=g^{\prime}(x)$ is shown below, and $g(1)=1$. Give the values for $g(0), g(2)$ and $g(3)$.

$$
\begin{aligned}
& f y=3 \\
& y=g^{\prime}(x) \quad g(1)-g(0)=\int_{0}^{1} g^{\prime}(x) d x \\
& \int_{0}^{1}=\int_{0}^{1}(2 x-1) d x \\
& =\left.\left(x^{2}-x\right)\right|_{0} ^{1} \\
& =0 \text {. } \\
& y=2 x-1 \\
& \text {-2. } \quad \Rightarrow \quad g(\theta)=g(1)=1 . \\
& \text {-3. } g(2)-g(1)=\int_{1}^{2} g^{\prime}(x) d x=\int_{1}^{2}(-2 x+3) d x=\left.\left(-x^{2}+3 x\right)\right|_{1} ^{2} \\
& =(-4+6)-(-1+3)=2-2=0 \text {. } \\
& \Rightarrow g(2)=g(1)=1 . \\
& g(3)-g(2)=\int_{2}^{3} g^{\prime}(x) d x=\int_{2}^{3} 3 d x=3 \Rightarrow g(3)=g(2)+3
\end{aligned}
$$

