1. Differential Approximation (Tangent Line Approximation).
   a. Use the tangent line to $f(x) = \sin(x)$ at $x = 0$ to approximate $f(\pi/60)$. (Note: The phrase "use the tangent line" could be replaced with "use differentials." The phrase "at $x = 0$" could actually be omitted since $\pi/60$ is close to 0, and we know the function very well at 0.)
   b. Use differentials to approximate $(26)^{1/3}$.

(a) $f(\pi/60) \approx f(0) + f'(0)(\pi/60 - 0)$

Note: $f(x) = \sin(x) \Rightarrow f(0) = 0$

$f'(x) = \cos(x) \Rightarrow f'(0) = 1$

$\therefore f(\pi/60) \approx 0 + 1 \cdot \left(\frac{\pi}{60}\right) = \frac{\pi}{60}$.

Remark: sin(pi/60) = 0.0523359... and pi/60 = 0.052359... So, the estimate is fairly close!

(b) Set $f(x) = x^{1/3}$. Note that we know $f(27) = 3$, and 27 is "close" to 26.

$\therefore (26)^{1/3} = f(26) \approx f(27) + f'(27)(26-27)$

$f'(x) = \frac{1}{3}x^{-2/3} \Rightarrow f'(27) = \frac{1}{3} \cdot \frac{1}{9} = \frac{1}{27}$

$\Rightarrow (26)^{1/3} \approx 3 + \frac{1}{27}(-1) = \frac{80}{27}$.

Remark: Tangent line approximation gives a very rough estimate. However, if you check your calculator, you will find (26)^(1/3) = 2.962496... and 80/27 = 2.96296... So, in this case, the estimate is very good.

2. The mean value theorem (for derivatives).

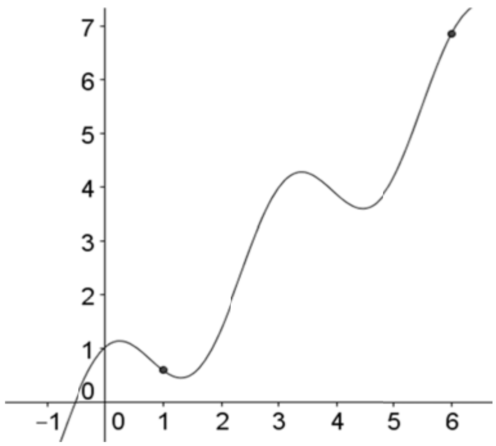
   a. State both the mean value theorem and Rolle's theorem.

   b. Explain how Rolle's theorem is a consequence of the mean value theorem.

   c. Verify the mean value theorem for the function $f(x) = x^3 - 4x^2 + x + 6$ on the interval $[-1, 2]$.

   d. Let $f(x)$ be a given differentiable function. Suppose $f(1) = 2$, and the value c that satisfies the mean value theorem on the interval $[1, 5]$ satisfies $f'(c) = 2$. Give the value for $f(5)$.

   e. Consider the graph below. Determine the number of values c that satisfy the conclusion of the mean value theorem on the interval $[1, 5]$, and give approximate values for each of these.



a

Mean Value Theorem: If f(x) is a continuous function on [a,b] and f'(x) exists on (a,b), then there is a value c so that a<c<b and f'(c) = (f(b)-f(a))/(b-a).

Rolle's Theorem: Suppose f(x) is a continuous function on [a,b] and f'(x) exists or (a,b). If f(a)=f(b)=0, then there is a value c so that a<c<b and f'(c) = 0.

b From the mean value theorem, there is a value c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

However, Rolle's theorem assumes $f(a) = f(b) = 0$.

$\therefore$ the right hand side above is 0.

$$\Rightarrow f'(c) = 0.$$

(c) Verify the mean value theorem for the function $f(x)=x^3-4x^2+x+6$ on the interval $[-1,2]$.

Note that $f(x)$ is a polynomial, so it is certainly continuous on $[-1,2]$, and $f'(x)$ also exists on $(-1,2)$.

Also, $f'(x) = 3x^2-8x+1$, $f(2)=0$, $f(-1)=0$.

Let's solve

$$f'(c) = \frac{f(2)-f(-1)}{2--1} = 0$$

$$\Leftrightarrow \quad 3c^2-8c+1 = 0$$

$$c = \frac{8 \pm \sqrt{64-12}}{6} = \frac{8 \pm \sqrt{52}}{6}$$

$$= \frac{8 \pm 2\sqrt{13}}{6} = \frac{4}{3} \pm \frac{\sqrt{13}}{3}.$$

Note that $\frac{4}{3} - \frac{\sqrt{13}}{3}$ lies in the interval $(-1,2)$.

$\therefore$ $c = \frac{4}{3} - \frac{\sqrt{13}}{3}$ verifies the mean value theorem.

d

Let $f(x)$ be a given differentiable function. Suppose $f(1)=2$ and the value c that satisfies the mean value theorem on the interval $[1,5]$ satisfies $f'(c)=2$. Give the value for $f(5)$.

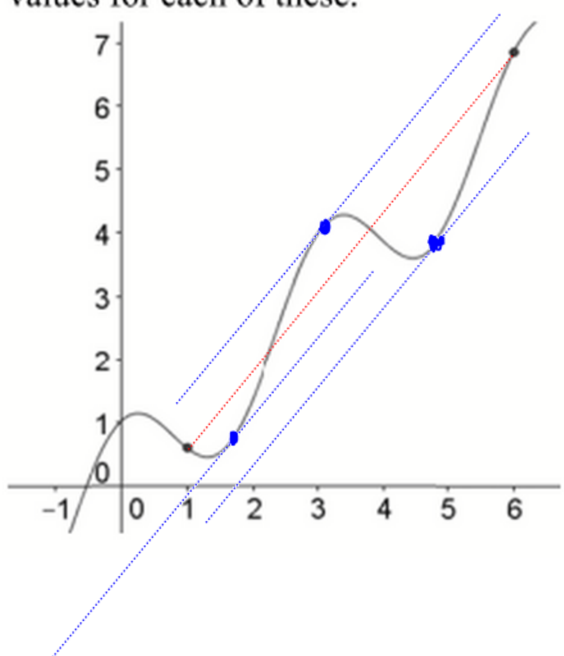
$$f'(c) = \frac{f(5)-f(1)}{5-1}$$

$$2 = \frac{f(5)-2}{4}$$

$$\Rightarrow f(5) = 10.$$

e

Consider the graph below. Determine the number of values c that satisfy the conclusion of the mean value theorem on the interval $[1,6]$, and give approximate values for each of these.



Note that a geometric interpretation of the mean value theorem is that there is at least one value c between a and b so that the tangent line at x=c is parallel to the line through (a,f(a)) and (b,f(b)).

In this case, a=1 and b=6. I have used a dotted red line to connect (1,f(1)) and (6,f(6)). The blue lines are the ones that are tangent to the curve within the interval from 1 to 6, and parallel to the dotted red line.

So, it appears that there are 3 values of c that satisfy the conclusion of the mean value theorem on the interval [1,6].

$c \approx 1.7$, $c \approx 3$, $c \approx 4.9$

3. Find the critical numbers, the interval(s) of increase, and the interval(s) of decrease for the function $g(x) = -x^3 + 6x^2 + 15x - 2$.

$$g'(x) = -3x^2 + 12x + 15.$$

Critical numbers occur at values where the derivative either does not exist, or the derivative is 0. The derivative above exists for every value of x, so the only critical numbers are those for which g'(x) = 0.

$$g'(x) = 0 \iff -3x^2 + 12x + 15 = 0$$

$$x^2 - 4x - 5 = 0$$

$$(x-5)(x+1) = 0$$

$$x = 5, \quad x = -1.$$

$\therefore$ the critical numbers of $g(x)$ are $x = -1$ and $x = 5$.

Notice that $g'(x)$ is continuous, so we can determine increase and decrease by forming a "slope chart."

$g'(x)$	$- - -$	$- -$	0	$+ +$	$+ + +$	0	$-$	$- -$
x		-2	-1	0		5	6	

$g'(-2) < 0$ $\qquad$ $g'(0) > 0$ $\qquad$ $g'(6) < 0$

$\therefore$ from the slope chart, $g(x)$ is decreasing on the intervals $(-\infty, -1]$ and $[5, \infty)$, and $g(x)$ is decreasing on the interval $[-1, 5]$.

Remark: Some people have trouble with the values -1 and 5 because the derivative is zero at those values. However, recall that a function is decreasing on an interval if and only if f(a) > f(b) whenever a and b are in the interval and a < b. Also, a function is increasing on an interval if and only if f(a) < f(b) whenever a and b are in the interval and a < b. You can verify that this occurs on the intervals given above.

However, with all of that said, and although the answers above ARE CORRECT, the following answer is typically accepted as correct.

$g(x)$ is decreasing on the intervals $(-\infty, -1)$ and $(5, \infty)$, and increasing on the interval $(-1, 5)$.

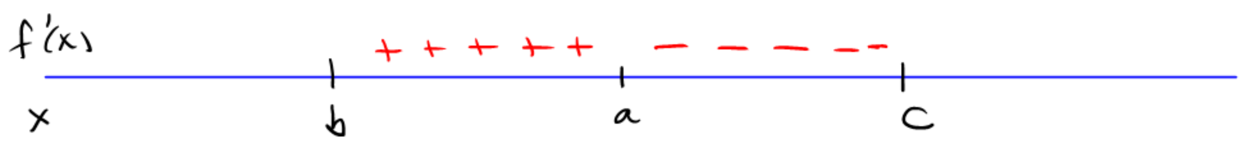
4. Classifying critical numbers.
   a. What does it mean to say that $x = a$ is a critical number for a function $f(x)$?
   b. Describe the first derivative test for classifying a critical number for a function.
   c. Describe the second derivative test for classifying a critical number for a function.
   d. Give an example of a function and a critical value for which the second derivative test fails.
   e. Classify the critical number $x = 0$ for the function $f(x) = x^2 \cos(x)$.
   f. Find and classify all of the critical numbers for the function given in problem 3 using the first derivative test.
   g. Repeat the classification of critical numbers (if possible) from part ~~b~~ f, using the second derivative test.

x=a is in the domain of f(x), and

a) x=a is a critical number for f(x) if and only if f'(a) does not exist or f'(a)=0.

b) Suppose x=a is the critical number, and f'(x) exists in intervals (b,a) and (a,c).

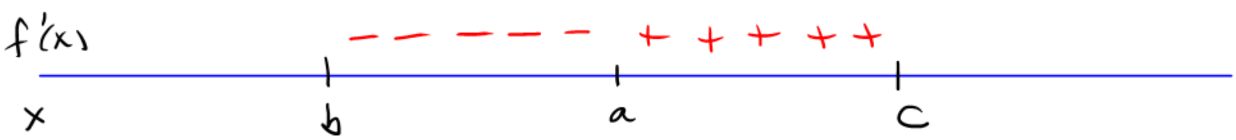
If f'(x)>0 for all b<x<a and f'(x)<0 for all a<x<c, then f(x) has a local maximum at x=a.



rough shape. for f(x) near x=a

Note: If f'(a)=0, then the shape is more like

If f'(x)<0 for all b<x<a and f'(x)<0 for all a<x<c, then f(x) has a local minimum at x=a.



rough shape. for f(x) near x=a

Note: If f'(a)=0, then the shape is more like

(c) Describe the second derivative test for classifying a critical number for a function.

Suppose $x=a$ is a critical number for $f(x)$ AND there is an interval (b,c) so that $b<a<c$ and $f''(x)$ exists for all $b<x<c$.

If $f''(a)>0$ then $f(x)$ has a local min at $x=a$.

If $f''(a)<0$ then $f(x)$ has a local max at $x=a$.

(d) Give an example of a function and a critical value for which the second derivative test fails.

I'll give 3 examples.

Function	Critical Number		Comment
$f(x)=x^3$	$x=0$.	$f''(0)=0$	f(x) has neither a local max nor a local min at x=0
$g(x)=x^4$	$x=0$	$g''(0)=0$	f(x) has a local min at x=0
$h(x)=-x^4$	$x=0$.	$h''(0)=0$	f(x) has a local max at x=0

(e) Classify the critical number $x=0$ for the function $f(x)=x^2\cos(x)$.

$$f'(x) = x^2(-\sin(x)) + 2x\cos(x)$$
$$= -x^2\sin(x) + 2x\cos(x)$$

Note that x=0 is definitely a critical number since f'(0)=0.

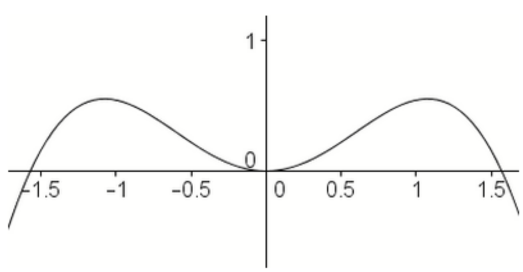
Also, we can see that f''(x) exists, so let's try to classify this critical number using the second derivative test.

$$f''(x) = -x^2\cos(x) - 2x\sin(x) + 2\cos(x) + 2x(-\sin(x))$$

$$\Rightarrow f''(0) = 2 > 0$$

$\therefore$ f(x) has a local minimum at $x=0$.

A graph of f(x) is shown below to verify our findings.

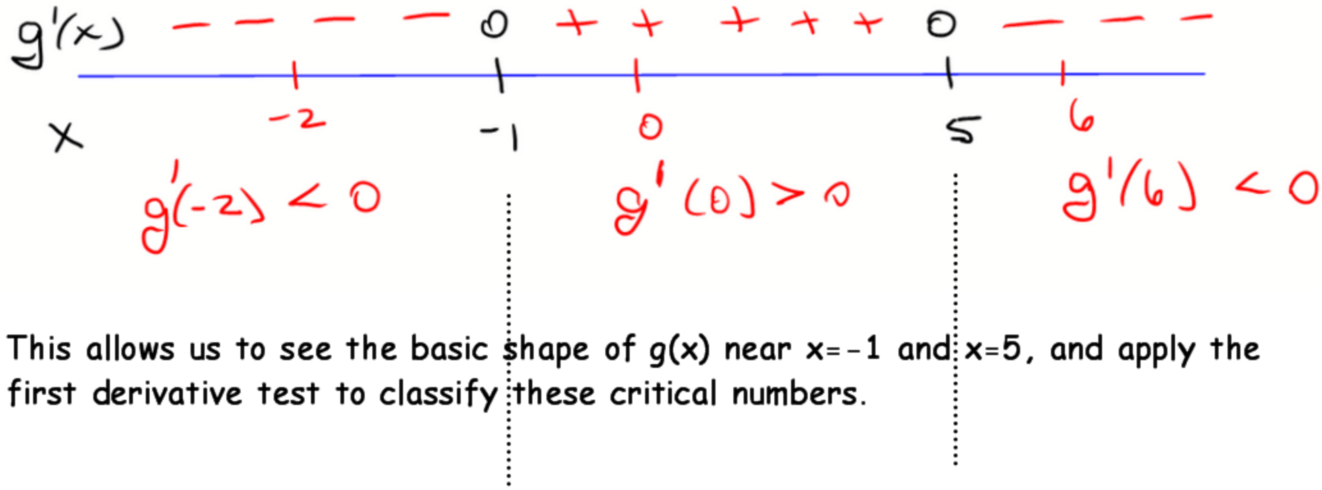


(f) Find and classify all of the critical numbers for the function given in problem 3 using the first derivative test.

$$g(x) = -x^3 + 6x^2 + 15x - 2$$

and $$g'(x) = -3x^2 + 12x + 15$$
$$= -3(x-5)(x+1)$$

We noted that the only critical numbers of g(x) are at x=-1 and x=5, and we created the following slope chart.

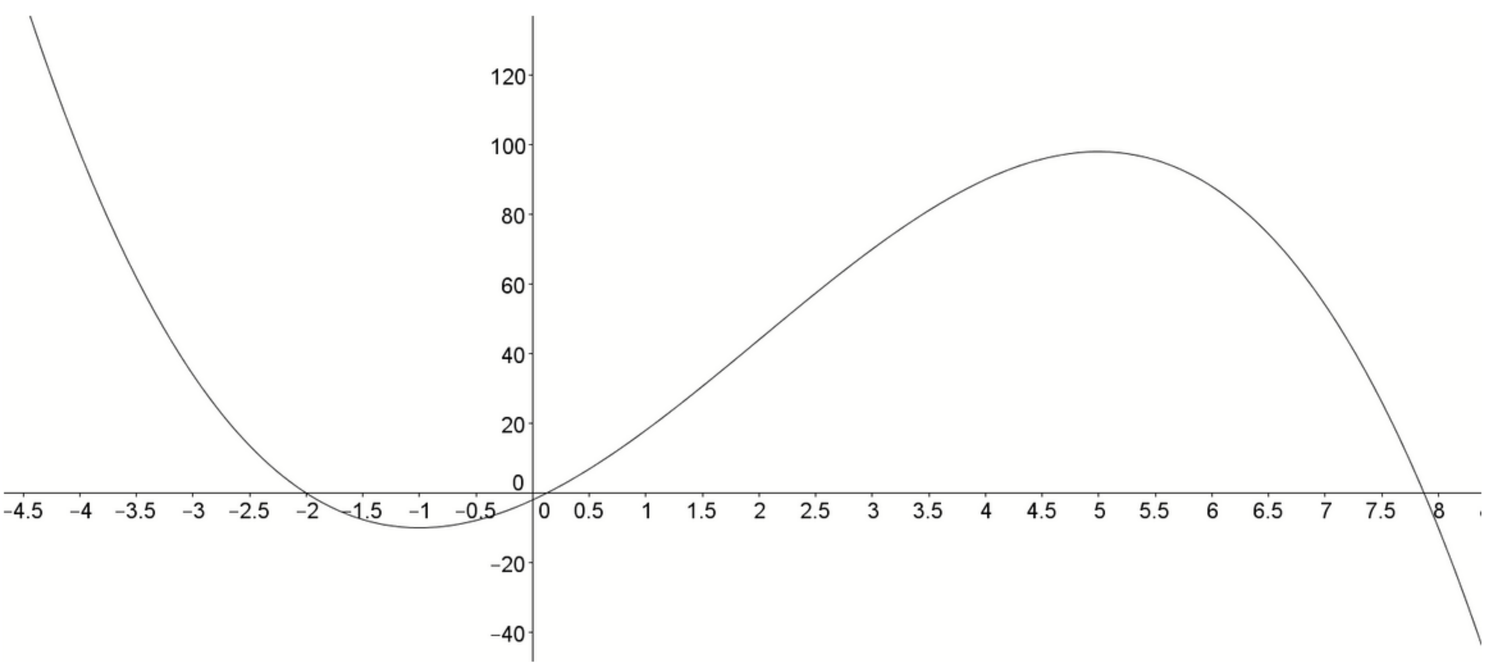


This allows us to see the basic shape of g(x) near x=-1 and x=5, and apply the first derivative test to classify these critical numbers.

Basic shape

∴ g(x) has a local min at x = -1 and a local max at x = 5.

These findings are illustrated in the graph of g(x) below.



(g) Repeat the classification of critical numbers (if possible) from part ~~b~~ f, using the second derivative test.

$$g(x) = -x^3 + 6x^2 + 15x - 2$$

and $$g'(x) = -3x^2 + 12x + 15$$

As shown earlier, the critical numbers are x=-1 and x=5. Also, g''(x) exists, so we can try to apply the second derivative test.

$$g''(x) = -6x + 12$$

$\Rightarrow$ $g''(-1) = 18 > 0$ $\Rightarrow$ local min at $x=-1$

$g''(5) = -18 < 0$ $\Rightarrow$ local max at $x=5$

Therefore, by the second derivative test, g(x) has a local minimum at x=-1 and a local maximum at x=5.

5. Absolute extrema (minimums and maximums).
   a. State the extreme value theorem.
   b. Explain the process of finding the extreme values of a continuous function $f(x)$ on an interval $[a,b]$.
   c. Find the absolute maximum and minimum values for the function $g(x) = -x^3 + 6x^2 + 15x - 2$ on the interval $[-1,2]$.
   d. Find the absolute minimum value of the function $f(x) = x^4 - 8x^2 + 3$.

a

Extreme Value Theorem: If f(x) is a continuous function on the interval [a,b], then there are values c and d in [a,b] so that f(c)≤f(x)≤f(d) for all x in [a,b].

In other words, a continuous function on a closed bounded interval achieves both its absolute minimum and absolute maximum values.

b

We can find the absolute extreme values as follows:
1. Evaluate f(a) and f(b).
2. Evaluate f(x) at every critical number in [a,b].
3. Compare the values found in parts 1 and 2. The smallest value is the absolute minimum value of f(x), and the largest value is the absolute maximum value of f(x).

c

$g(x) = -x^3 + 6x^2 + 15x - 2$ is a polynomial, so it is continuous on $[-1,2]$.

Let's apply the process above.

1. $\boxed{g(-1) = -10}$ and $\boxed{g(2) = 44}$

2. $g'(x) = -3x^2 + 12x + 15$
$= -3(x^2 - 4x - 5)$
$= -3(x+1)(x-5)$

2. continued

$g'(x)$ exists for all x

and $g'(x)=0$ iff $x=-1$ or $x=5$

$\therefore$ the critical numbers for $g(x)$

at $x=-1$ and $x=5$, but only

$x=-1$ lies in the interval $[-1,2]$.

$\boxed{g(-1) = -10.}$

3. Now we compare the values from 1 and 2. -10 is the smallest value and 44 is the largest value.

Therefore, on the interval [-1,2],

g(x) has its absolute minimum value at x=-1, and the absolute minimum value is g(-1)=-10.

g(x) has its absolute maximum value at x=2, and the absolute maximum value is g(2)=44.

d. Find the absolute minimum value of the function $f(x) = x^4 - 8x^2 + 3$.

This problem is a little different, because the function is not given on a closed bounded interval. We will need to use shape information from the first or second derivative to help answer the question.

$$f'(x) = 4x^3 - 16x$$

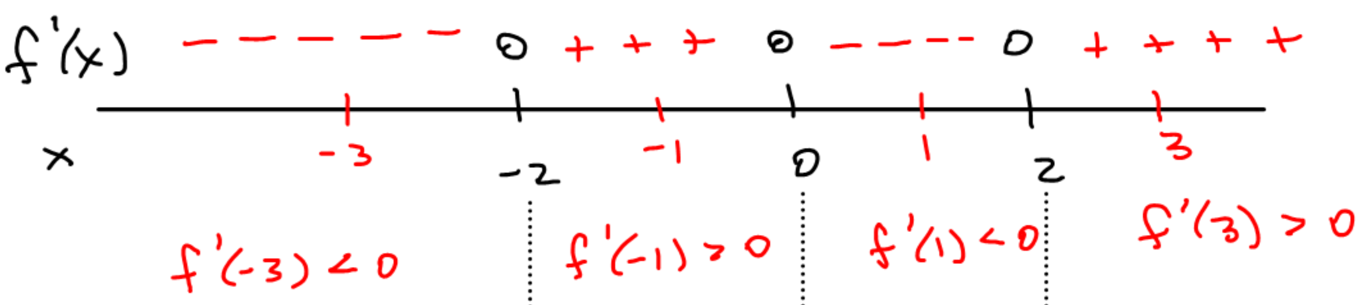
Notice that $f'(x)$ exists for all x. So the only critical numbers will be those where $f'(x)=0$.

$$f'(x)=0 \iff 4x^3-16x=0$$

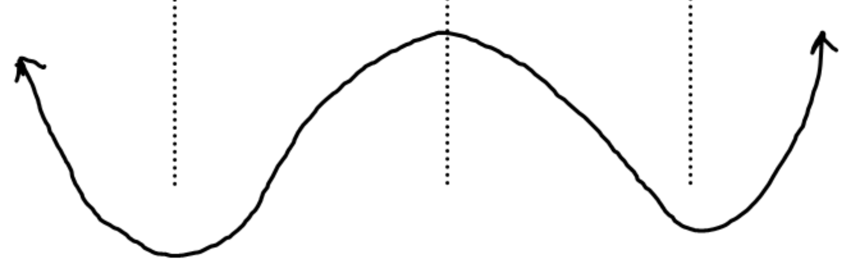
$$4x(x^2-4)=0$$

$$x=0,\ x=-2,\ x=2.$$

Therefore, the critical numbers for $f(x)$ are at $x=-2$, $x=0$, and $x=2$.
We create a slope chart below.



From the slope chart, we can create the following rough sketch of the graph of $f(x)$, and conclude that $f(x)$ has its absolute minimum value either at $x=-2$ or $x=2$.

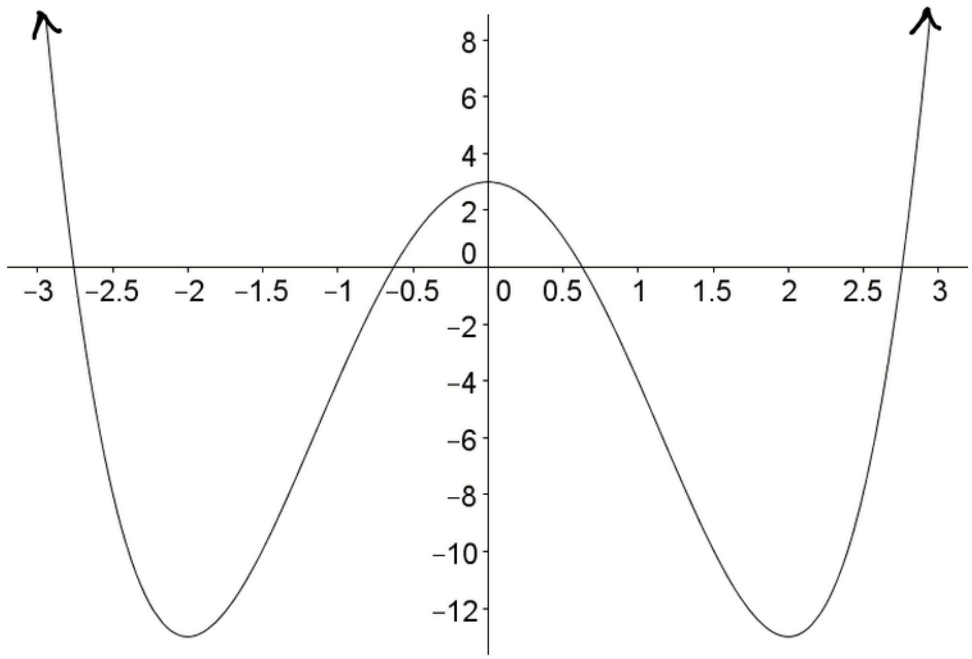


$$\boxed{f(x)=x^4-8x^2+3.}$$

$f(-2)=-13$ and $f(2)=-13$.

$\therefore$ The absolute minimum value of $f(x)$ occurs at both $x=-2$ and $x=2$, and the absolute minimum value is -13.

A graph of $f(x)$ is shown below.



6. Definitions.

   a. What does it mean to say that a function $f(x)$ is increasing on an interval?

   b. What does it mean to say that a function $f(x)$ is decreasing on an interval?

   c. What does it mean to say that a function $f(x)$ is concave up on an interval?

   d. What does it mean to say that a function $f(x)$ is concave down on an interval?

   e. What does it mean to say that a function $f(x)$ as inflection at a value $x = a$?

a) f(x) is increasing on an interval I if and only if f(a)<f(b) whenever a and b are in I and a<b.

b) f(x) is decreasing on an interval I if and only if f(a)>f(b) whenever a and b are in I and a<b.

c) f(x) is concave up on an interval I if and only if f'(x) is increasing on the interval I.

d) f(x) is concave down on the interval I if and only if f'(x) is decreasing on the interval I.

e) f(x) has inflection at the value x=a if and only if f(a) exists and the concavity to the left of x=a is different from the concavity to the right of x=a. i.e. f(x) has a change in concavity at x=a.

7. Concavity and inflection.

a. Determine any intervals of concave up or concave down, and give any values where inflection occurs for the function $g(x) = -x^3 + 6x^2 + 15x - 2$.

b. Determine any intervals of concave up or concave down, and give any values where inflection occurs for the function $f(x) = x^4 - 8x^2 + 3$.

ⓐ

$$g'(x) = -3x^2 + 12x + 15$$

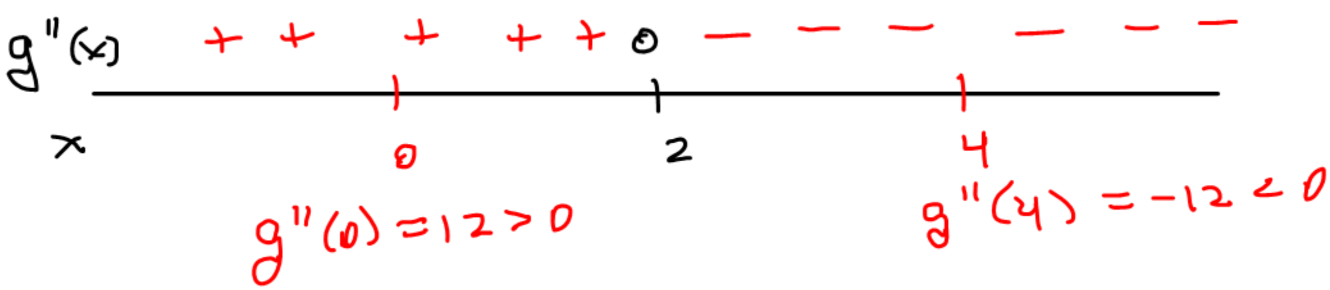
$$g''(x) = -6x + 12$$

Notice that g''(x) exists for all values of x, so if g(x) has inflection, then it will be found at values where g''(x)=0.

$$g''(x) = 0 \iff -6x + 12 = 0$$

$$x = 2$$

So, x=2 is a candidate for inflection. We need to check the concavity on either side of x=2 to see if the concavity changed. We do this with a concavity chart.



From the chart, g(x) is concave up to the left of x=2, and concave down to the right of x=2.

i.e. g(x) is concave up on $(-\infty, 2)$

and g(x) is concave down on $(2, \infty)$.

Therefore, g(x) has a change of concavity at x=2, implying g(x) has inflection at x=2.

Note: There is a subtle point associated with the work that we just did. The answers we provided are completely acceptable. However, in all honesty, we can say a little more.

Recall that a function is concave up on an interval if and only if its derivative is increasing on the interval. From the calculations above, $g''(x)>0$ for $x<2$, and $g''(2)=0$. Therefore, $g'(x)$ is increasing on $(-\infty, 2]$.

So $g(x)$ is concave up on the interval $(-\infty, 2]$.

Similarly, $g(x)$ is concave down on the interval $[2, \infty)$.

b

Determine any intervals of concave up or concave down, and give any values where inflection occurs for the function $f(x) = x^4 - 8x^2 + 3$.

$$f'(x) = 4x^3 - 16x$$

$$f''(x) = 12x^2 - 16$$

$f''(x)$ exists for all values of x, so if inflection occurs, it will occur at a value where $f''(x)=0$.

$$f''(x) = 0 \iff 12x^2 - 16 = 0$$

$$\iff x = \pm \frac{2}{\sqrt{3}} = \pm \frac{2\sqrt{3}}{3}$$

Comment: It is not necessary to rationalize the denominator above.

We can determine whether f(x) has inflection at ~~x=2~~ $x = \pm \frac{2\sqrt{3}}{3}$ and also determine the concavity of f(x) by creating a concavity chart.

$f''(x)$	$+$	$+\ +\ +\ +$	0	$-\ -$	$-\ -\ -\ -$	0	$+\ +$	$+\ +$
x	-3		$-\frac{2\sqrt{3}}{3}$		0	$\frac{2\sqrt{3}}{3}$		3

$f''(-3) = 92 > 0$

$f''(0) = -16 < 0$

$f''(3) = 92 > 0$

We can see the concavity and change in concavity that from the chart. Therefore,

f(x) is concave up on the intervals $\left(-\infty, -\frac{2\sqrt{3}}{3}\right)$ and $\left(\frac{2\sqrt{3}}{3}, \infty\right)$ and f(x) is concave down on the interval $\left(-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right)$.

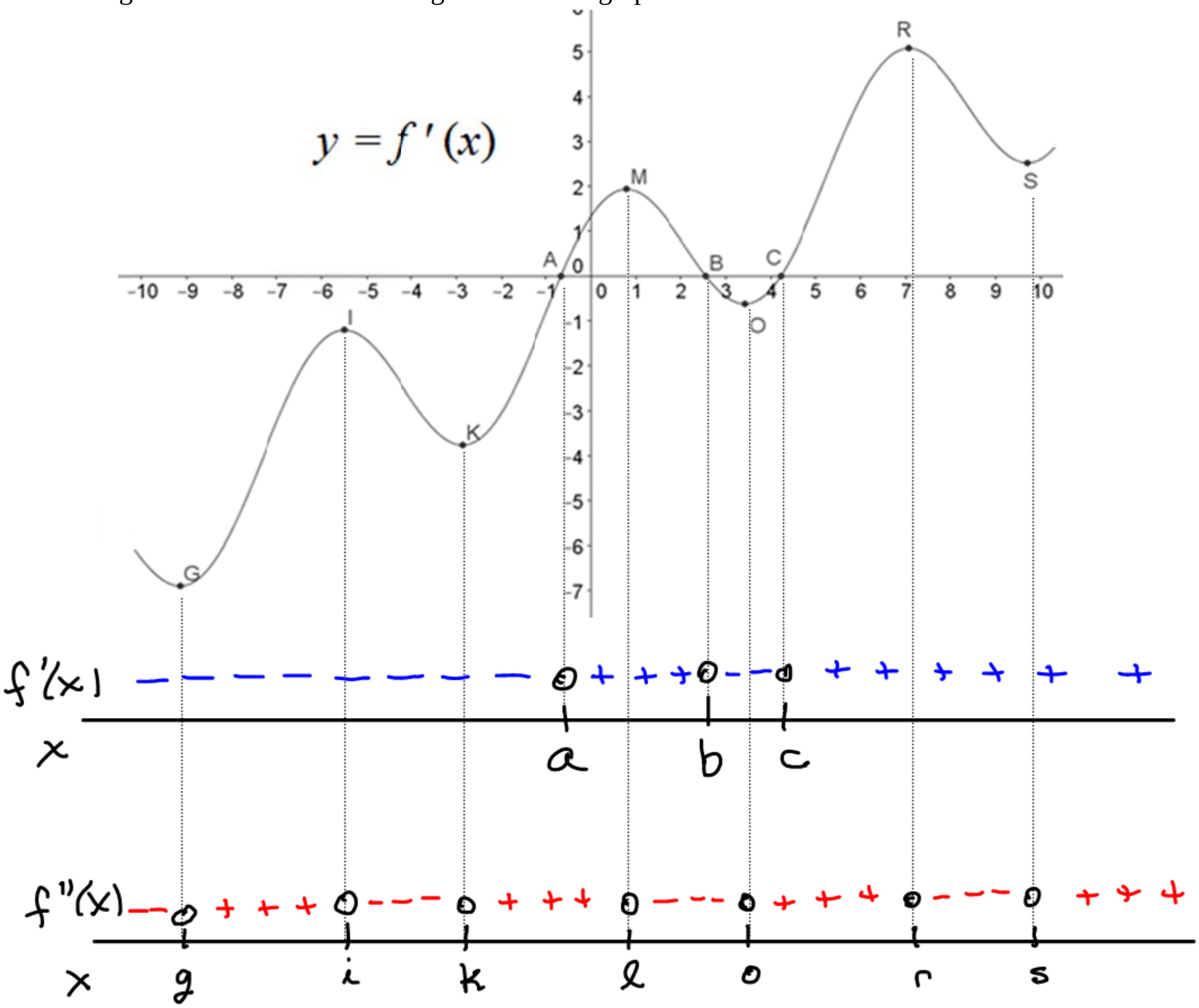
Also, from the change in concavity that occurs, we can see that

f(x) has inflection at both $x = -\frac{2\sqrt{3}}{3}$ and $x = \frac{2\sqrt{3}}{3}$.

Remark: Similar to the comment made in part b, although the concavity information given above would be accepted as correct, in actuality, we have the following:

f(x) is concave up on the intervals $\left(-\infty, -\frac{2\sqrt{3}}{3}\right]$ and $\left[\frac{2\sqrt{3}}{3}, \infty\right)$ and f(x) is concave down on the interval $\left[-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right]$.

8. The graph of $f'(x)$ is shown below on the interval $[-11, 11]$. Give the intervals of increase, decrease, concave up, and concave down for $f(x)$. Also, find and classify any critical numbers for $f(x)$, and list any inflection for $f(x)$. Your answers should be given in terms of the labels given with the graph.



The slope chart and concavity chart where created directly from the graph of f'(x).

Note: We are using a, b, c, g, i, k, m, o, r and s as the x-coordinates of the points A, B, C, G, I K, M O, R and S.

f(x) has critical numbers at x=a, x=b and x=c. From the slope chart (first derivative test), f(x) has local minimums at x=a and x=c, and f(x) has a local maximum at x=b.

f(x) is decreasing on the intervals [-11,a) and (b,c).

f(x) is increasing on the intervals (a,b) and (c,11] .

f(x) is concave down on the intervals [-11,g), (i,k), (l,o) and (r,s).

f(x) is concave up on the intervals (g,i), (k,l), (o,r), and (s,11] .

From the change in concavity, f(x) has inflection at x=g, i, k, l, o, r and s.

Remark: Although the answers given above would be graded as correct, it is actually true that similar to our earlier comments, all of the intervals above can be given as closed.

9. Determine the domain, any asymptotes, intervals of increase, intervals of decrease, intervals of concave up, intervals of concave down, critical numbers, and inflection for the function $f(x) = \dfrac{x-1}{x^2-5x+6}$. Then, graph the function.

Domain: All x except where $x^2-5x+6=0$.

$$(x-3)(x-2)=0$$

$$x=3, \quad x=2$$

$\therefore$ Domain $= (-\infty, 2) \cup (2,3) \cup (3, \infty)$

Asymptotes:

Vertical - Check $x=2$ and $x=3$.

$$\lim_{x\to 2^+} \frac{x-1}{x^2-5x+6} = \lim_{x\to 2^+} \frac{\overset{\nearrow 1}{(x-1)}}{\underset{\searrow 0^+}{(x-2)}\underset{\searrow -1}{(x-3)}} = -\infty$$

$$\lim_{x\to 2^-} \frac{x-1}{x^2-5x+6} = \lim_{x\to 2^-} \frac{\overset{\nearrow 1}{(x-1)}}{\underset{\searrow 0^-}{(x-2)}\underset{\searrow -1}{(x-3)}} = \infty$$

Therefore, f(x) has a vertical asymptote at x=2, and the behavior is shown above.

$$\lim_{x\to 3^+} \frac{x-1}{x^2-5x+6} = \lim_{x\to 3^+} \frac{\overset{\nearrow 2}{(x-1)}}{\underset{\searrow 1}{(x-2)}\underset{\searrow 0^+}{(x-3)}} = \infty$$

$$\lim_{x\to 3^-} \frac{x-1}{x^2-5x+6} = \lim_{x\to 3^-} \frac{\overset{\nearrow 2}{(x-1)}}{\underset{\searrow 1}{(x-2)}\underset{\searrow 0^-}{(x-3)}} = -\infty$$

Therefore, f(x) has a vertical asymptote at x=3 and the behavior is shown above.

Horizontal:

$$\lim_{x \to \infty} \frac{x-1}{x^2-5x+6} = 0$$

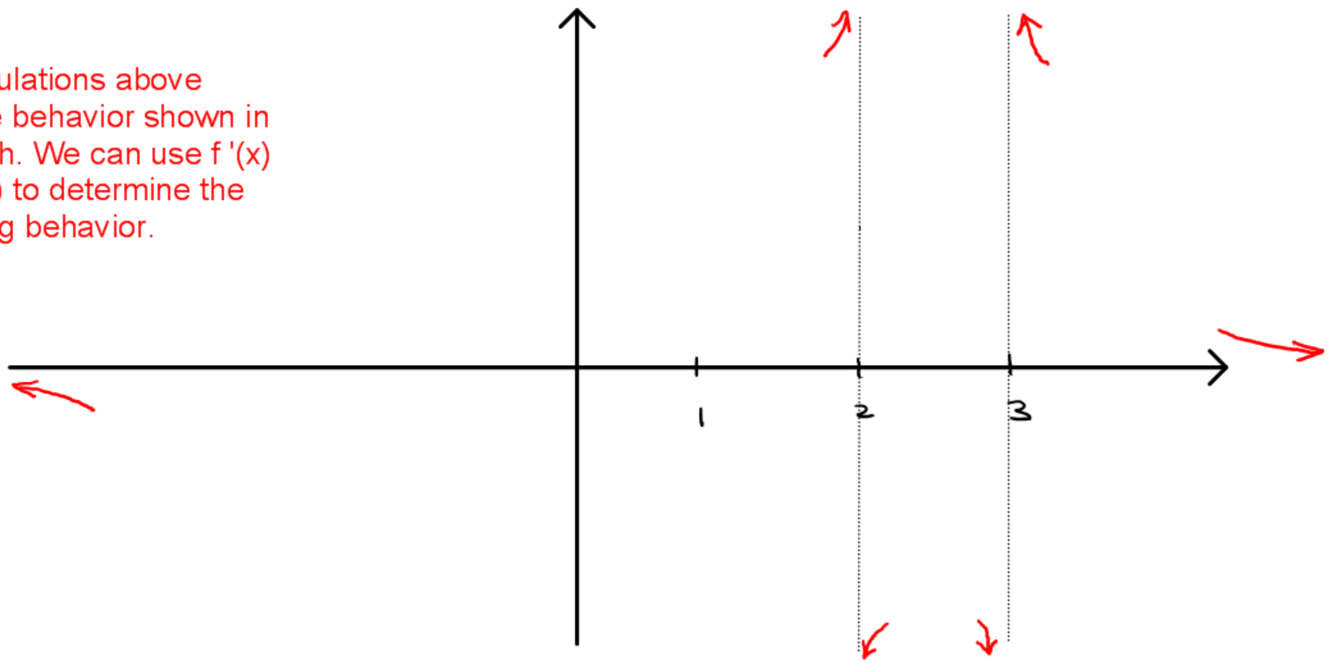
(Note: $f(x) \to 0$ through positive values as $x \to \infty$.

$$\lim_{x \to -\infty} \frac{x-1}{x^2-5x+6} = 0$$

(Note: $f(x) \to 0$ through negative values as $x \to -\infty$.

Therefore y=0 is the only horizontal asymptote, and the behavior is indicated above.

The calculations above imply the behavior shown in this graph. We can use f '(x) and f "(x) to determine the remaining behavior.



Increase/Decrease: $f(x) = \dfrac{x-1}{x^2-5x+6}$

$$f'(x) = \frac{x^2-5x+6-(x-1)(2x-5)}{(x^2-5x+6)^2} = \frac{x^2-5x+6-2x^2+7x-5}{(x^2-5x+6)^2}$$

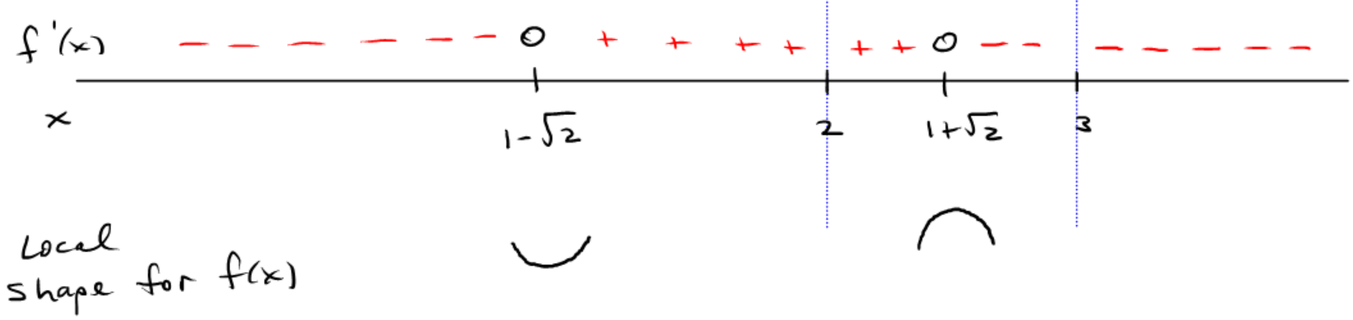
$$= \frac{-x^2+2x+1}{(x^2-5x+6)^2}$$

Note: $f'(x)$ d.n.e at $x=2$ and $x=3$ (and recall $f(x)$ has vertical asymptotes at these values)

$f'(x)=0$ iff $x^2-2x-1=0$

iff $x = \dfrac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$

Slope chart: From f '(x) and the asymptotic information, we have the slope chart below.



$f(x)$ has a local min at $x = 1-\sqrt{2}$

$f(x)$ has a local max at $x = 1+\sqrt{2}$

$f(x)$ is increasing on $(1-\sqrt{2}, 2)$ and $(2, 1+\sqrt{2})$

$f(x)$ is decreasing on $(-\infty, 1-\sqrt{2})$, $(1+\sqrt{2}, 3)$ and $(3, \infty)$.

Concavity and inflection:

Recall: $$f'(x) = \frac{-x^2+2x+1}{(x^2-5x+6)^2}$$

$$\Rightarrow f''(x) = \frac{(x^2-5x+6)^2(-2x+2) - (-x^2+2x+1)\cdot 2(x^2-5x+6)\cdot(2x-5)}{(x^2-5x+6)^4}$$

$$= \frac{(x^2-5x+6)(-2x+2) + (x^2-2x-1)\cdot 2\cdot(2x-5)}{(x^2-5x+6)^3}$$

$$= \frac{2(x^3-3x^2-3x+11)}{(x^2-5x+6)^3}$$

Note: $f''(x)$ d.n.e. at $x=2$ and $x=3$ (and $f(x)$ has vertical asymptotes at these values)

$f''(x) = 0$ iff $x = -1.84732\ldots$

Concavity chart:

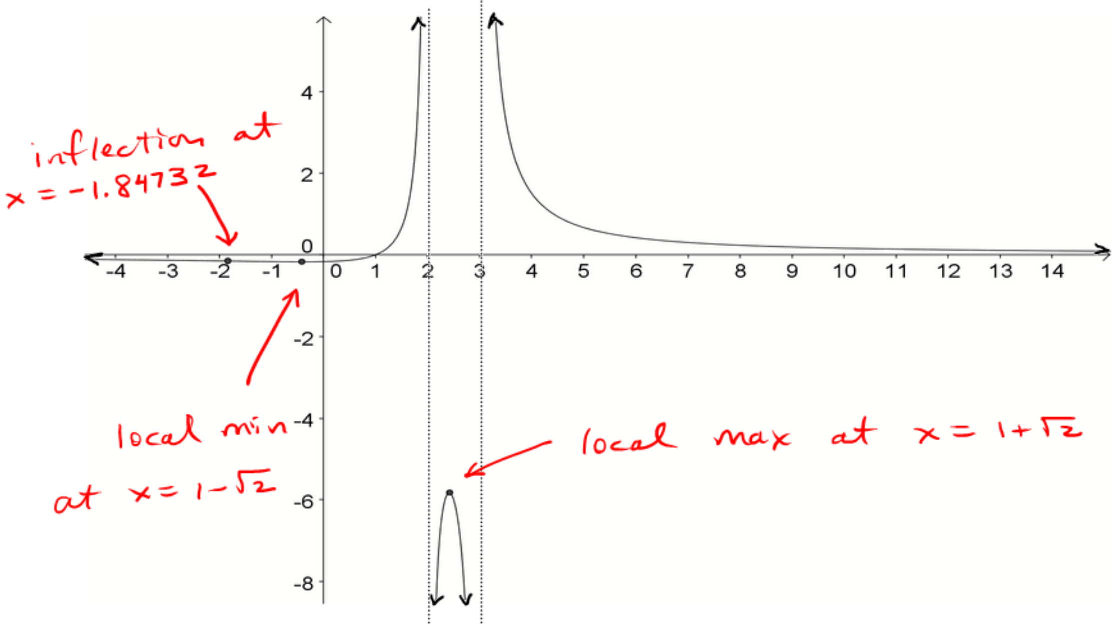
x	3	-1.84732	0	2	$\frac{5}{2}$	3	4
$f''(x)$	$- - - - -$	0	$+ + + + + +$		$- - - -$		$+ + +$

$f''(3) < 0$ $\quad$ $f''(0) > 0$ $\quad$ $f''(\frac{5}{2}) < 0$ $\quad$ $f''(4) > 0$

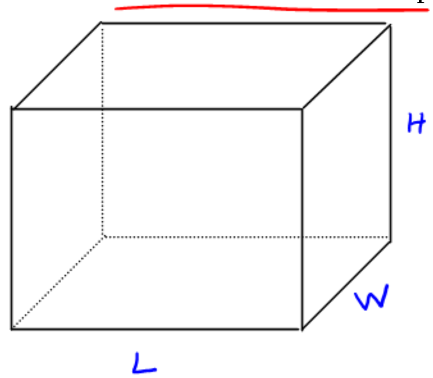
$f(x)$ is concave up on $(-1.84732, 2)$ and $(3, \infty)$.

$f(x)$ is concave down on $(-\infty, -1.84732)$ and $(2, 3)$.

inflection occurs at $x = -1.84732$.



10. What are the dimensions of the base of the rectangular box of greatest volume that can be constructed from 100 square inches of cardboard if the base is to be twice as long as it is wide? Assume the box has a top.



$L = 2W$

$V = \text{Volume} = LWH$

$\Rightarrow \quad V = 2W^2 H$

Also, Base $= LW = 2W^2$

Top $= 2W^2$

Front Face $= LH = 2WH$

Back Face $= 2WH$

Left side $= WH$

Right side $= WH$

Total material $= 100$

$H > 0 \Leftrightarrow W < 5.$

$\Rightarrow \quad 4W^2 + 6WH = 100 \quad \Rightarrow \quad H = \dfrac{100 - 4W^2}{6W}$

$\Rightarrow \quad V = 2W^2 H = 2W^2 \dfrac{100 - 4W^2}{6W} = \dfrac{W}{3}(100 - 4W^2)$

V is a function of W.

$\Rightarrow \quad V(W) = \dfrac{100}{3} W - \dfrac{4}{3} W^3, \quad 0 < W < 5$

We need to maximize volume. Note that V extends to $0 \le W \le 5$.

1. $V(0) = 0, \quad V(5) = 0$

2. $V'(W) = \dfrac{100}{3} - 4W^2$

$V'(W) = 0$ iff $W^2 = \dfrac{25}{3}$

iff $W = \dfrac{5}{\sqrt{3}} = \dfrac{5\sqrt{3}}{3} = 2.88675\ldots$

(Recall $0 \le W \le 5$).

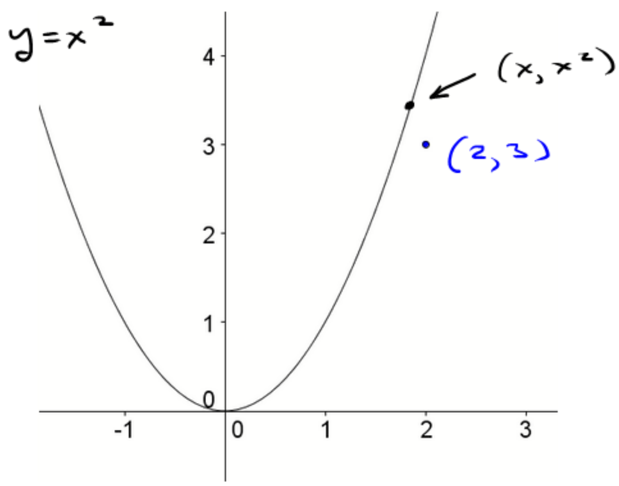
$$V\left(\frac{5\sqrt{3}}{3}\right) = 64.15003\ldots$$

3. $\therefore$ the maximum volume is 64.15003 cubic units. The dimensions are

$$L = 2\left(\frac{5\sqrt{3}}{3}\right), \quad W = \frac{5\sqrt{3}}{3}, \quad H = \frac{100 - 4\left(\frac{5\sqrt{3}}{3}\right)^2}{6\left(\frac{5\sqrt{3}}{3}\right)}$$

$$\Rightarrow L = 5.7735, \quad W = 2.88675, \quad H = 3.849$$

11. Find the point on the curve $y = x^2$ that is closest to the point $(2,3)$.



The distance from $(2,3)$ to (x, x^2) is

$$g(x) = \sqrt{(x-2)^2 + (x^2-3)^2}$$

Note: The closest point will occur for $x > 0$.

$$g'(x) = \frac{x-2 + (x^2-3)\cdot 2x}{\sqrt{(x-2)^2 + (x^2-3)^2}}$$

$g'(x)$ exists for all x.

$g'(x) = 0$ iff $x-2 + (x^2-3)\cdot 2x = 0$

iff $2x^3 - 5x - 2 = 0$

The only positive root is $x = 1.75233\ldots$

Slope chart:

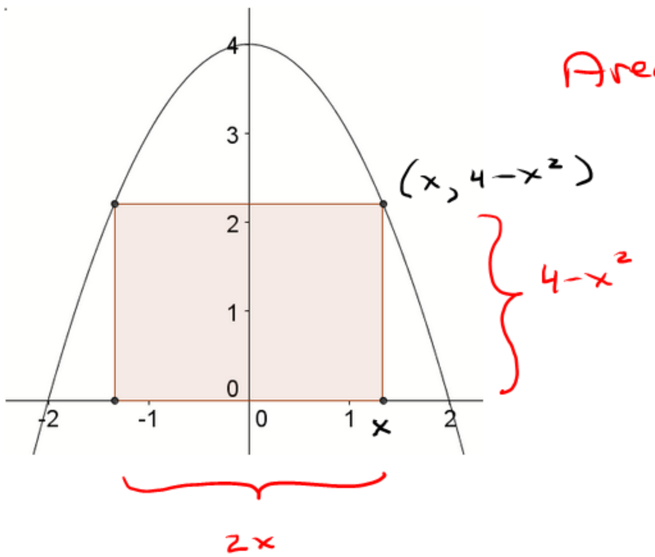
x	0		1		1.75233		3	
$g'(x)$		$-$	$-$	$-$	0	$+$	$+$	$+$

$g'(1) < 0$ $\quad$ $g'(3) > 0$

$\therefore$ the point on the curve $y = x^2$ that is closest to $(2,3)$ is

$(1.75233, (1.75233)^2) = (1.75233, 3.07066)$.

12. Give the dimensions of the rectangle with greatest area that has its base on the x-axis and its upper vertices on the parabola $y = 4 - x^2$.



Area $= A(x) = 2x(4-x^2)$, $0 \le x \le 2$.

1. $A(0) = 0$, $A(2) = 0$

2. $A(x) = 8x - 2x^3$

$A'(x) = 8 - 6x^2$

$A'(x) = 0$ iff $x = \pm \frac{2}{\sqrt{3}}$

Note: Only $x = \frac{2}{\sqrt{3}}$ is allowed since $0 \le x \le 2$.

$$A\left(\frac{2}{\sqrt{3}}\right) = \frac{16}{\sqrt{3}} - \frac{16}{3\sqrt{3}} = \frac{32}{3\sqrt{3}} = \frac{32\sqrt{3}}{9}$$

$\therefore$ The maximum area occurs when $x = \frac{2}{\sqrt{3}}$, and the maximum area is $\frac{32\sqrt{3}}{9}$ square units.

The dimensions of the rectangle with largest area are $\frac{4}{\sqrt{3}}$ by $4 - \left(\frac{2}{\sqrt{3}}\right)^2$

i.e. $\frac{4}{\sqrt{3}}$ by $\frac{8}{3}$.