V. RANDOM VARIABLES, PROBABILITY DISTRIBUTIONS, EXPECTED VALUE

A game of chance featured at an amusement park is played as follows: You pay $1 to play. A penny and a nickel are flipped. You win $2 if either 2 heads or 2 tails turn up, and you get your dollar back. You lose $2 if a head and a tail turn up, and you do not get your dollar back.

Another way to describe this game is: Two coins are flipped. You win $2 if either 2 heads or 0 heads turn up; you lose $3 if exactly one head turns up.

Suppose you can play this game as many times as you like. Would you play over and over again? Does either side (i.e., you or the amusement park) have an advantage? If so, what is it?

The objective of this section is to provide the mathematical methods to answer these questions.

Random Variable:

The probability experiment involved in the game described above is: Flip a penny and a nickel. The sample space of equally likely simple events is:

$$S = \{HH, HT, TH, TT\}.$$

For the purposes of our game, we are not interested in the particular pattern of the coins. Rather we want to know the number of heads that turns up. So, we associate 2 with *HH*, 1 with *HT* and with *TH*, and 0 with *TT*. That is, we have defined a function that associates a real number with each of the simple events in S.

Given a probability experiment with sample space S. A *random variable* X is a function that assigns a numerical value to each simple event in S.

Examples 5.1:

1. Flip a penny, a nickel and a dime. Let S be the sample space of equally likely simple events and let X be "the number of heads that turn up". Give the values of X.

Sample Space S	Number of Heads $X(e_i)$
e_1: *HHH*	3
e_2: *HHT*	2
e_3: *HTH*	2

e_4: *THH*	2
e_5: *HTT*	1
e_6: *THT*	1
e_7: *TTH*	1
e_8: *TTT*	0

2. In the game of "craps", 2 dice are tossed (If it helps you, imagine that one die is red and the other is green). The interest in this game is not on the particular number, say *i*, on the red die and the particular number, say *j*, on the green die that turns up. Instead, we are interested in the some of the two numbers $i + j$. Thus, to each of the 36 equally likely simple events (*i*,*j*) in the sample space, we assign the number $i + j$. This association defines the random variable X –"the sum of the numbers that turn up". Some of the values of X are:

$$X[(1,1)] = 2, \quad X[(3,4)] = 7, \quad X[(6,3)] = 9, \quad X[(6,6)] = 12 .$$

Probability Distribution for a Random Variable:

In Example 5.1.1 we defined the random variable X whose values were 0, 1, 2, 3. We now consider the questions: If we toss 3 coins, (you can think of them as a penny, a nickel and a dime), what is the probability that X = 0? That X = 1? That X = 2? That X = 3?

The value $X = 0$ corresponds to the event E_0 = "exactly 0 heads turn up" ; $E_0 = \{TTT\}$ and

$$P(X = 0) = P(E_0) = \frac{n(E_0)}{n(S)} = \frac{1}{8} .$$

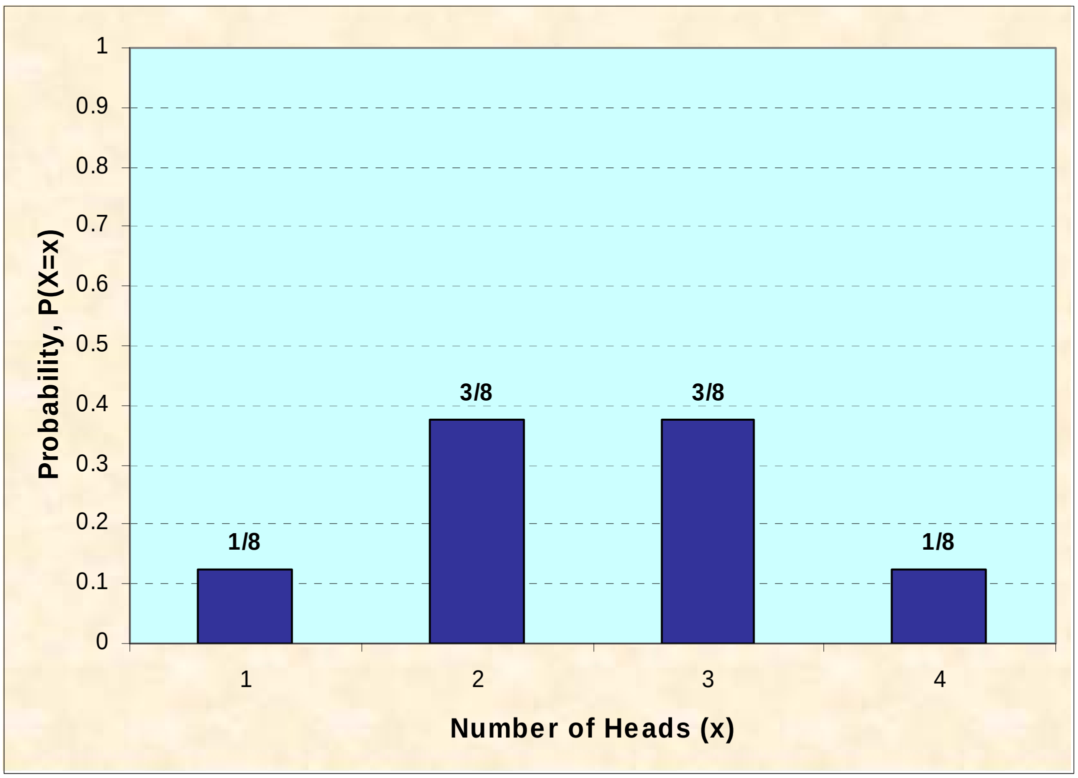
The value $X = 1$ corresponds to the event E_1 = "exactly 1 head turns up"; $E_1 = \{HTT, THT, TTH\}$ and

$$P(X = 1) = P(E_1) = \frac{n(E_1)}{n(S)} = \frac{3}{8} .$$

Similarly, $P(X = 2) = \frac{3}{8}$ and $P(X = 3) = \frac{1}{8}$. These results are summarized in the table below. This table is called a *probability distribution for the random variable X.*

Number of Heads x	0	1	2	3
Probability $P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

A graphical representation of this probability distribution is shown in the following figure, called a *histogram*.



Note from the table that

(1) $0 \le P(X = x) \le 1$ ($P(X = x)$ is the probability of an event)

(2) $P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = 1$.

These are the properties that characterize the probability distribution of a random variable X associated with any probability experiment:

Let X be a random variable with values $\{x_1, x_2, ..., x_k\}$. The probability function $P(X = x), \quad x \in \{x_1, x_2, ..., x_k\}$, is a *probability distribution of the random variable* X if

(1) For each $x \in \{x_1, x_2, ..., x_k\}$, $0 \le P(X = x) \le 1$

(2) $P(X = x_1) + P(X = x_2) + ... + P(X = x_k) = 1$.

Examples 5.2:

1. Two dice are rolled and X is the random variable "the sum of the numbers that turn up". Find the probability distribution of X.

2. A box of 10 flashbulbs contains 3 defective bulbs. A random sample of 2 is selected and tested. Let X be the number of defective bulbs in the sample. Find the probability distribution of X.

Solutions:

1. Based on the results of preceding examples, the probability distribution of X is

X	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

2. Let X be the random variable: "the number of defective bulbs". Then the values of X are 0, 1, and 2.

The total number of ways to select 2 objects from a collection of 10 without regard to the order of selection is $C_{10,2}$.

The probability of 0 defective bulbs: the number of ways to select 2 good bulbs is $C_{7,2}$ and

$$P(0) = \frac{C_{7,2}}{C_{10,2}} = \frac{21}{45} = \frac{7}{15}.$$

The probability of 1 defective bulb(1 good bulb, 1 bad bulb): the number of ways to select 1 good bulb and 1 defective bulb is $C_{7,1} \cdot C_{3,1}$ and

$$P(1) = \frac{C_{7,1} \cdot C_{3,1}}{C_{10,2}} = \frac{7 \cdot 3}{45} = \frac{7}{15}.$$

The probability of 2 defective bulbs: the number of ways to select 2 defective bulbs is $C_{3,2}$ and

$$P(2) = \frac{C_{3,2}}{C_{10,2}} = \frac{3}{45} = \frac{1}{15}.$$

The probability distribution of X is:

X	0	1	2
$P(X = x)$	$\frac{7}{15}$	$\frac{7}{15}$	$\frac{1}{15}$

Expected Value:

We continue with our probability experiment of tossing 3 coins and letting X be the number of heads that turn up. Suppose we repeat the experiment n times, recording the number of heads that turn up each time. If we then add up the number of heads that occur in each of the n tosses and divide by n, we will get the average number of heads per toss. What would we expect the average number of heads per toss to be? According to the probability distribution given in the table above, when n is large

0 heads will occur in $\frac{1}{8}$ of the tosses,

1 head will occur in $\frac{3}{8}$ of the tosses,

2 heads will occur in $\frac{3}{8}$ of the tosses,

3 heads will occur in $\frac{1}{8}$ of the tosses.

Therefore, for large n, the total number of heads that will occur is:

$$0\left(\tfrac{1}{8}\right)n + 1\left(\tfrac{3}{8}\right)n + 2\left(\tfrac{3}{8}\right)n + 3\left(\tfrac{1}{8}\right)n = [0\left(\tfrac{1}{8}\right) + 1\left(\tfrac{3}{8}\right) + 2\left(\tfrac{3}{8}\right) + 3\left(\tfrac{1}{8}\right)]\, n .$$

Dividing by n we get the average number of heads per toss:

$$0\left(\frac{1}{8}\right)+1\left(\frac{3}{8}\right)+2\left(\frac{3}{8}\right)+3\left(\frac{1}{8}\right)=\frac{12}{8}=1.5\,.$$

This number is called the *expected value of the random variable* X. Note that the expected value of X is not necessarily a value that will occur in a single trial of the experiment (In this particular case, you can not get 1.5 heads if you flip 3 coins). The expected value is the *average* of what occurs over a large number of trials of the experiment.

In general, let X be a random variable with probability distribution given by the table

X	x_1	x_2	…	x_k
$P(X = x)$	p_1	p_2	…	p_k

The *expected value of* X, denoted by $E(X)$, is given by the formula

$$E(X)=x_1p_1+x_2p_2+...+x_kp_k\,.$$

Examples 5.3:

1. An experiment consists of rolling a single die. Let the random variable X be the number that turns up. What is the expected value of X?

2. Two dice are rolled and X is the random variable "the sum of the numbers that turn up". Find $E(X)$.

3. A box of 10 flashbulbs contains 3 defective bulbs. A random sample of 2 is selected and tested. Let X be the number of defective bulbs in the sample. Find $E(X)$.

Solutions:

1. Let X = "the number that turns up". The probability distribution of X is

X	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Therefore,

$$E(X) = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = \frac{21}{6} = 3.5 .$$

2. From Example 5.2.1,

$$E(X) = 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right) + 7\left(\frac{6}{36}\right) +$$
$$8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right) = \frac{252}{36} = 7$$

3. From Example 5.2.2,

$$E(X) = 0\left(\frac{7}{15}\right) + 1\left(\frac{7}{15}\right) + 2\left(\frac{1}{15}\right) = \frac{3}{5} = 0.6 .$$

Expected Value of a Game:

Now we use the expected value of a random variable to find the expected value of a game of chance.

We return to the example that began this section. You pay \$1. A penny and a nickel are flipped. You win \$2 if either 2 heads or 2 tails turn up, and your \$1 is returned; otherwise you lose \$2 and your \$1 is not returned. The sample space is:

$$S = \{0 \text{ heads}, 1 \text{ head}, 2 \text{ heads}\} .$$

This time our random variable X is the amount you "win" if 0 heads, 1 head, or 2 heads turn up. The probability distribution for X, called a *payoff table*, is:

Number of Heads	0	1	2
X	\$2	-\$3	\$2
Probability p	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

The expected value of the game is:

$$E(X) = \$2\left(\frac{1}{4}\right) + (-\$3)\left(\frac{1}{2}\right) + \$2\left(\frac{1}{4}\right) = \$1 - \$\frac{3}{2} = -\$\frac{1}{2} = -50¢ .$$

If you play this game over and over again, you will lose an average of 50¢ per game. The amusement park has the advantage in this game; the park wins an average of 50¢ each time the game is played.

A game is *fair* if and only if the expected value $E(X)=0$. That is, neither side has an advantage.

Our amusement park game is not fair. So this raises the question: For the game to be fair, how much should you lose, in addition to the $1 you paid to play, if exactly one head turns up? Denote that amount by u. Then the payoff table is:

Number of Heads	0	1	2
X	\$2	-\$(u + 1)	\$2
Probability p	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Since the game is fair, we have

$$E(X) = \$2\left(\frac{1}{4}\right) + [-\$(u+1)]\left(\frac{1}{2}\right) + \$2\left(\frac{1}{4}\right) = 0\,.$$

Solving this equation for u we get,

$$1 - \frac{1}{2}u - \frac{1}{2} = 0 \quad \text{which gives} \quad u = 1\,.$$

Thus, the game will be fair if it is modified so that we lose $1 if exactly one head turns up.

Examples 5.4:

1. A roulette wheel has 38 equally likely spaced slots numbered $00, 0, 1, 2, \ldots, 36$. A player bets $1 on one of the numbers and wins $35 (and gets the $1 back) if the ball comes to rest on the chosen number; otherwise the player loses the $1. the random variable X assigns the value $35 to having the ball land on the chosen number and $-\$1$ to having the ball land on any other number. What is the expected value of roulette?

2. One thousand tickets are sold at $1 each for a charity raffle. Tickets are to be drawn at random and cash prizes are to be awarded as follows: 1 prize of $100, 3 prizes of $50, and 5 prizes of $20. What is the expected value of this raffle if you buy 1 ticket?

3. In the original Texas Lotto game you paid $1 to select 6 numbers from the set $\{1, 2, 3, ..., 50\}$.

(a) What is your expected value if you buy 1 ticket and the jackpot is $4 million?

(b) What is your expected value if you buy 1 ticket and the jackpot is $40 million?

(c) At what jackpot level is Texas Lotto a fair game?

Solutions:

1. The payoff table for roulette is:

X	$35	–$1
p	$\frac{1}{38}$	$\frac{37}{38}$

The expected value of the game is:

$$E(X) = 35\left(\frac{1}{38}\right) + (-1)\left(\frac{37}{38}\right) = -\$\frac{2}{38} = -\$\frac{1}{19} \approx -5¢ .$$

2. You have a:

$\frac{1}{1000} = 0.001$ chance of winning $99 [$100 – $1 (the cost of the ticket)]

$\frac{3}{1000} = 0.003$ chance of winning $49

$\frac{5}{1000} = 0.005$ chance of winning $19

$\frac{991}{1000} = 0.991$ chance of "winning" – $1.

Therefore, the payoff table is:

X	$99	$49	$19	–$1
p	0.001	0.003	0.005	0.991

The expected value is:

$$E(X) = 99(.001) + 49(.003) + 19(.005) - 0.991 = -0.65.$$

Thus $E(X) = -\$0.65 = -65¢$.

3. You are choosing 6 numbers from the set of 50 numbers. This can be done in $C_{50,6}$ different ways (the order of the selection is immaterial). Therefore , the probability of having the winning ticket is $1/C_{50,6} = 1/15{,}890{,}700 \approx 0.000000063$.

(a) The payoff table is:

X	\$3,999,999	-\$1
p	0.000000063	0.999999937

and

$$E(X) = (3{,}999{,}999)0.000000063 - 0.999999937 = -\$0.75$$

(b) The payoff table is

X	\$39,999,999	-\$1
p	0.000000063	0.999999937

and

$$E(X) = (39{,}999{,}999)0.000000063 - 0.999999937 = \$1.52$$

(c) Let N be the jackpot level at which the game is fair. Then

$$E(X) = N(0.000000063) - 0.999999937 = 0$$

and

$$N = \frac{0.999999937}{0.000000063} \approx 15{,}873{,}015 \; .$$

Exercises 3.5:

1. Roll a standard 6-sided die twice. Let X denote the maximum of the two numbers observed. Complete the table of the distribution of X given below.

x	1	2	3	4	5	6
P[X = x]	1/36					11/36

2. For the experiment described in #1, what is the expected value of X?

3. Let Y denote the minimum of the two rolls of the die. Find the distribution of the random variable X - Y.

4. In problem #3, find the expected value of X – Y.

5. The odds on your winning a certain bar bet are 4:1. If you put up $100, how much should your opponent put up to make it a fair bet?

6. In a certain lottery game, a sequence of three numbers from 1 to 9 is randomly chosen, with replacement. It costs $2 to play. What must the payoff to a winner be in order for the game to be fair?

7. You and Nick play a game called "gambler's ruin". You start with *x* dollars and Nick starts with *y* dollars. A fair coin is tossed repeatedly. On each head, Nick gives you $1 and on each tail you give him $1. You play until one of you is broke. It can be shown that the probability that you win is $\frac{x}{x+y}$. What is the expected payoff to you?

8. In Example 5.4.2, suppose prizes and ticket prices are expressed in pounds sterling instead of dollars and that one pound sterling is worth two dollars. What is the value of the game in pounds sterling? What general principle explains the answer?

9. In Example 5.4.2, suppose some benefactor agrees to give every player $1000, regardless of the outcome of the raffle. What is the expected value of the game? What general principle is at work?

10. A standard six-sided die is rolled once. X is the square of the number of spots. What is the expected value of X? What lesson does this teach us?

Answers to the Exercises – Chapter 3

Exercises 3.1:

1. a)

$$S = \{(i, j \mid i \text{ on the red die}, j \text{ on the green die}\} = \left\{\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix}\right\}$$

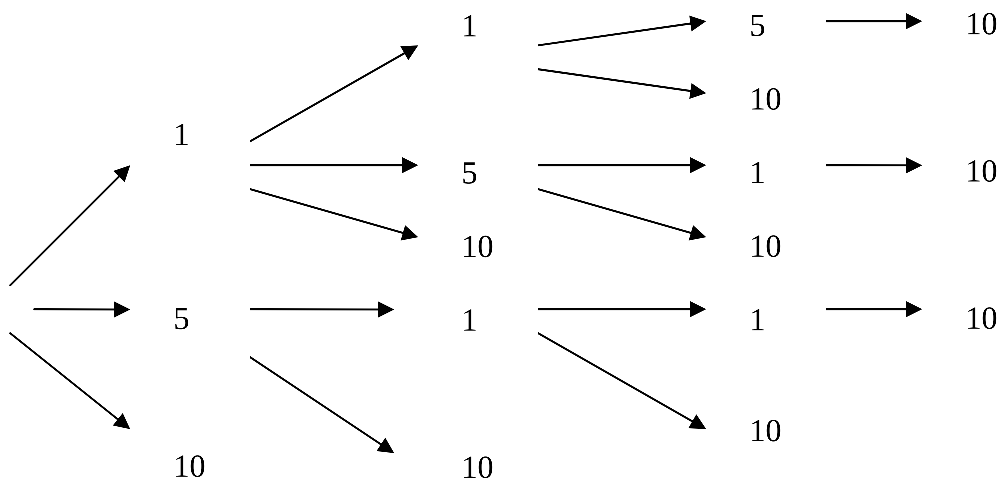
b)

$$S = \left\{\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ & & (3,3) & (3,4) & (3,5) & (3,6) \\ & & & (4,4) & (4,5) & (4,6) \\ & & & & (5,5) & (5,6) \\ & & & & & (6,6) \end{matrix}\right\}$$

2. **a)** There are 2 x 6 = 12 outcomes.
b) S = {H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6}
c) E = {H1 H2,H3,H4,H5,H6,T1,T3,T5}

3. **b)** $S = \{MMM,\ MMF, MF,\ FM, FFM, FFF\}$

c) $E = \{FFM,\ FFF\}$



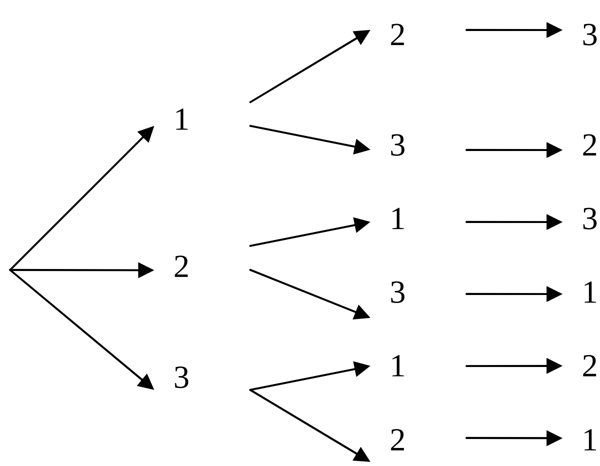
4. a) (above)

b) $S = \{(1,1,5,10)\ (1,1,10)\ (1,5,1,10)\ (1,5,10)\ (1,10)\ (5,1,1,10)\ (5,1,10)\ (5,10)\ (10)\}$

c) $E = \{(1,1,5,10),(1,5,1,10),(5,1,1,10)\}$

d) $F = \{(1,1,10)\ (1,10)\ (10)\}$

5. a)



S = (123 , 132 , 213 , 231 , 312 , 321 }

b) There are $3^3 = 27$ elements in the sample space.

6. a) There are $P_{4,4} = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ elements in the sample space.

b) There are $3 \cdot 3 \cdot 2 \cdot 1 = 18$ 4-digit numbers.

c) There are $2 \cdot 2 \cdot 1 \cdot 2 = 8$ odd four-digit numbers (the last digit must be 1 or 3, the first digit cannot be 0).

d) There are $2 \cdot 3 \cdot 2 \cdot 1 = 12$ four-digit numbers less than 3000.

7. a) S = { (1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,1),(3,2), (3,3),(3,4),(3,5),(3,6) }

b) E = { (1,1),.(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),(3,2),(3,4) }, $n(E) = 9$

c) F = { (1,5),(2,4),(3,3) }, $n(F) = 3$.

8. a) $C_{52,5} =$ 2,598,960 possible hands.

b) 48 possible hands.

c) $4 \times C_{13,5} = 4 \times 1287 = 5148$ possible hands.

d) There are 13 cards in each of the 4 suits, so the number of four-of-a-kind hands is: $13 \times C_{4,4} \times C_{48,1} = 13(1)(48) =$ 624 possible hands.

e) $13 \times C_{4,3} \times 12 \times C_{4,2} = 13 \times 4 \times 12 \times 6 = 3744$ possible hands.

Exercises 3.2

1. a. Event E cannot occur
   b. Event E is certain to occur.

2. (b), (e), (f), (g)

3. (a) is not acceptable (one of the probabilities is negative), (b) is not acceptable (the sum of the probabilities is not 1), (c) is acceptable.

4. *P(1)=P(3)=P(5)=1/9; P(2)=P(4)=P(6)=2/9.*

5. a. *P(BBG)=1/8*
   b. *P*(2 boys and 1 girl) = *3/8*
   c. *P*(at least one of each sex) = *3/4*

6. (a) *1/4* (b) *1/2* (c) *1/4*
   (d) *0* (e) 1 (f) *1/4*

7. (a) *1/36* (b) *1/9* (c) *5/18*
   (d) *5/12* (e) *1/2* (f) *5/12*
   (g) *1/3*

8. (a) *1/9* (b) *2/9* (c) 0
   (d) *5/9* (e) *5/9*

9. (a) *1/5* (b) *1/2*

10. (a) *9/19* (b) *6/19* (c) *8/19* (counting 00 < 15)
    (d) *11/38*

11. (a) $\frac{C_{13,5}}{C_{52,5}}$ (b) $\frac{C_{26,5}}{C_{52,5}}$ (c) $\frac{(13)(48)}{C_{52,5}}$

    (d) $\frac{C_{4,3}\cdot C_{4,2}}{C_{52,5}}$ (e) $\frac{(13)C_{4,3}(12)C_{4,2}}{C_{52,5}}$

12. (a) $\frac{C_{20,4}}{C_{35,4}}$ (b) $\frac{(15)C_{20,3}}{C_{35,4}}$ (c) $\frac{C_{15,2}C_{20,2}}{C_{35,4}}$

    (d) $\frac{C_{15,3}+C_{15,4}}{C_{35,4}}$

Exercises 3.3

1. $P(E)=\frac{1}{6}$
2. $P(E^c)=1-\frac{1}{6}=\frac{5}{6}$
3. Let E = "each person has a different birth month." Then

$$P(E)=\frac{(12)(11)(10)}{12^3}\approx 0.764\,.$$

E^c = "at least two people have the same birth month."

$P(E^c) = 1 - 0.764 = 0.236$.

4. 14%

5. $P(B^c \cup A^c) = 0.9$; $P(A \cap B^c) = 0.3$

6. 2 : 9

7. Let p be the probability that Slim Jim wins. Then

$$\frac{3}{10} + \frac{1}{2} + p = 1 \;\Rightarrow\; p = \frac{1}{5} \;.$$

Therefore, the odds on Slim Jim are 1 : 4.

Exercises 3.4

1. $P(H_2 \mid H_1) = \frac{12}{51}$; dependent

2. $P(H_2 \mid R) = \frac{25}{204}$; dependent

3. $P(S_2 \mid J) = \frac{1}{4}$; independent

4. $P(R_2 \mid H) = \frac{25}{51}$; dependent

5. $P(A \mid C^c) = \frac{1}{2}$. Let D = "guard says C will not be shot. $P(A \mid D) = \frac{1}{3}$.

6. H = "has defect," C = "correct diagnosis." $P(H \mid C) = 0.19$

Exercises 3.5:

1.

x	1	2	3	4	5	6
P[X=x]	1/36	3/36	5/36	7/36	9/36	11/36

2. $E(X) = 4.47$

3.

$X - y$	0	1	2	3	4	5
$P[X - Y]$	6/36	10/36	8/36	6/36	4/36	2/36

4. $E(X - Y) = 1.94$

5. $25

6. $1456

7. You win y dollars with probability $\frac{x}{x+y}$; you lose x dollars with probability $\frac{y}{x+y}$. Your expected payoff is: $y \cdot \frac{x}{x+y} - x \cdot \frac{y}{x+y} = 0.$

8. -0.325; if the prizes and prices are multiplied by a positive number α, then the value is multiplied by α.

9. $\$1000 - 0.65 = \$999.35$; if each value is changed by an amount S, then the expected value is changed by S.

10. $E(X) = 15.17$; the expected value of the squares is not the square of the original expected value.