

II. PROBABILITY OF AN EVENT

As indicated above, ‘*probability*’ is a quantification, or a mathematical model, of a random experiment. This quantification is a measure of the likelihood that a given event will occur when the experiment is performed.

A *probability function* is a function P defined on the subsets of a sample space S . To define this function we proceed as follows:

Probability of a Simple Event:

Let $S = \{e_1, e_2, \dots, e_n\}$ be the sample space of an experiment. To each simple event e_i we assign a real number $P(e_i)$, called the *probability of the event* e_i , such that the following two conditions are satisfied:

(i) The probability of each simple event is a number between 0 and 1, inclusive. That is,

$$0 \leq P(e_i) \leq 1, \quad i = 1, 2, \dots, n$$

(ii) The sum of the probabilities of the simple events e_1, e_2, \dots, e_n in the sample space is 1. That is,

$$P(e_1) + P(e_2) + \dots + P(e_n) = 1$$

Any assignment that satisfies conditions (i) and (ii) is an *acceptable probability assignment*.

Examples 2.1:

1. $S = \{e_1, e_2, e_3, e_4, e_5\}$ is the sample space of an experiment. Which of the following are acceptable probability assignments? If the assignment is not acceptable, explain why.

(a) $P(e_1) = 0.3, P(e_2) = 0.2, P(e_3) = 0.15, P(e_4) = 0.1, P(e_5) = 0.05$.

(b) $P(e_1) = 0.3, P(e_2) = 0.2, P(e_3) = -0.15, P(e_4) = 0.3, P(e_5) = 0.35$.

(c) $P(e_1) = 0.3, P(e_2) = 0.2, P(e_3) = 0.15, P(e_4) = 0.1, P(e_5) = 0.25$.

2. Roll a die. Find the sample space and determine two different acceptable probability assignments.

Solutions:

1. (a) Condition (i) is satisfied; each value is a number between 0 and 1. Checking condition (ii):

$$0.3 + 0.2 + 0.15 + 0.1 + 0.05 = 0.80 \neq 1.$$

The sum is not 1; the assignment is not acceptable.

(b) $P(e_3) = -0.15$; condition (i) is not satisfied; the assignment is not acceptable.

(c) Condition (i) is satisfied; each value is a number between 0 and 1. Checking condition (ii):

$$0.3 + 0.2 + 0.15 + 0.1 + 0.25 = 1.$$

This is an acceptable probability assignment.

2. The sample space : $S = \{1, 2, 3, 4, 5, 6\}$. There are infinitely many solutions to this problem. We'll give 3:

$$\text{Assignment \#1: } P(1) = \frac{1}{6}, P(2) = \frac{1}{6}, P(3) = \frac{1}{6}, P(4) = \frac{1}{6}, P(5) = \frac{1}{6}, P(6) = \frac{1}{6}.$$

This is the assignment we would make if we know that the die is fair; each number is as likely to turn up as any other number.

For another assignment, we simply have to insure that conditions (i) and (ii) are satisfied; there are infinitely many choices. Here's one:

$$\text{Assignment \#2: } P(1) = \frac{1}{9}, P(2) = \frac{2}{9}, P(3) = \frac{1}{9}, P(4) = \frac{2}{9}, P(5) = \frac{1}{9}, P(6) = \frac{2}{9}.$$

This assignment would indicate that the die is loaded in such a way that an even number is twice as likely to come up as an odd number.

$$\text{Assignment \#3: } P(1) = P(2) = P(4) = P(5) = P(6) = 0, P(3) = 1.$$

This is the assignment we would make if we knew that the die was loaded and that a "3" always comes up.

In addition to being acceptable, we want the probability assignment to accurately reflect the experiment; we want the assignment of probabilities to be *reasonable*. For example, suppose we flip a fair coin. The sample space is $S = \{H, T\}$, and the simple events are H – the coin comes up heads and T – the coin comes up tails. While the assignment

$$P(H) = 3/4, \quad P(T) = 1/4$$

is acceptable, it is not reasonable. Since the coin is fair, we reasonably assume that a head is just as likely to turn up as a tail. This implies that we should set $P(H) = P(T)$ from which it follows that

$$P(H) = 1/2, \quad P(T) = 1/2$$

is the assignment we should make.

Examples 2.2:

1. A box contains 11 marbles: 6 red, 3 blue, and 2 white. After shaking the box thoroughly, you draw a marble out of the box. What is the sample space for this experiment and what probability would you assign to the simple events.
2. A coin is biased in such a way that a head is twice as likely to come up as a tails. Determine a probability assignment for the simple events of the sample space $S = \{H, T\}$ that reflects this bias.
3. A die is rolled 1000 times with the results given in the table.

Outcome	Number of times
1	166
2	168
3	167
4	170
5	165
6	164

Give two probability assignments for the simple events.

Solutions:

1. The marble will either be red (R), blue (B), or white (W). Therefore, $S = \{R, B, W\}$. Since 6 of the 11 marbles are red, 3 are blue and 2 are white, we assign

$$P(R) = \frac{6}{11}, P(B) = \frac{3}{11}, P(W) = \frac{2}{11}.$$

2. We'll let $P(T) = x$, $0 \leq x \leq 1$, be the probability that a tail comes up. Then the probability that a head comes up is $P(H) = 2x$. Now, by condition (ii),

$$\begin{aligned}x + 2x &= 1 \\3x &= 1 \\x &= \frac{1}{3}.\end{aligned}$$

Thus, $P(T) = \frac{1}{3}$, $P(H) = \frac{2}{3}$.

3. For the first assignment we'll use the results given in the table:

$$\begin{aligned}P(1) &= \frac{166}{1000} = 0.166, & P(2) &= \frac{170}{1000} = 0.170, & P(3) &= \frac{169}{1000} = 0.169 \\P(4) &= \frac{170}{1000} = 0.170, & P(5) &= \frac{165}{1000} = 0.165, & P(6) &= \frac{160}{1000} = 0.160.\end{aligned}$$

These numbers are close to $\frac{1}{6}$ so it appears that the die is fair. If we make that assumption, then we set

$$P(1) = \frac{1}{6}, P(2) = \frac{1}{6}, P(3) = \frac{1}{6}, P(4) = \frac{1}{6}, P(5) = \frac{1}{6}, P(6) = \frac{1}{6}.$$

Examples 2.1 and 2.2 illustrate two typical ways in which acceptable and reasonable probability assignments are made for the simple events in a sample space S .

1. *Theoretical.* We use assumptions and analytical reasoning (the die is fair so the outcomes have equal probability; the coin is biased in a certain way), and impose the two conditions that the probability assignment must satisfy. No experiments are actually performed.
2. *Empirical.* For one of our assignments in Examples 2.2.3 we used the results of an actual experiment.

Empirical probability can be defined more precisely as follows: Suppose we perform an experiment n times and the simple event e_i occurs with *frequency* $f(e_i)$. That is, the outcome e_i occurs $f(e_i)$ times. Then the ratio $f(e_i)/n$ is called the *relative frequency* or *empirical probability* of e_i .

$$P(e_i) = \frac{\text{frequency of occurrence of } e_i}{\text{number of trials}} = \frac{f(e_i)}{n}.$$

We leave it as an exercise to verify that this assignment is acceptable.

The connection between theoretical and empirical probability assignments for a given random experiment is contained in a theorem which states that the empirical probabilities approach the theoretical probabilities as n gets larger and larger, provided the theoretical probability assignment is correct. Look at Example 2.2.3 again. The empirical probability assignments to the outcomes are all close to $1/6$. Based on this evidence, if we rolled the die 10,000 times or a 100,000 times, we would expect that a “1” would occur essentially $1/6$ of the time, a “2” would occur essentially $1/6$ of the time, and so on. So, we assume that the die is fair and assign (theoretically) the probabilities $P(1)=1/6$, $P(2)=1/6$, ... , $P(6)=1/6$.

The Equally Likely Assumption:

If we toss an unbiased coin, we would expect that a heads is as likely to turn up as a tails. There are two possible outcomes, they are equally likely to occur, and so we assign the value, $1/2$, to each outcome; $P(H) = P(T) = 1/2$. To say this another way, if we were to toss the coin a large number of times, we would expect that half (or at least very close to half) the tosses would result in heads, and half in tails. If we roll a fair die, there are six possible outcomes and any one number is as likely to turn up as any other. Thus, we assign the values $P(1) = P(2) = \dots = P(6) = 1/6$. If we draw a card from a well-shuffled, standard deck, then any one card is as likely to be drawn as any other and so we assign the value $P(e) = 1/52$ to each of the 52 simple events.

In general, if a sample spaces S has n elements,

$$S = \{e_1, e_2, \dots, e_n\}$$

and if we determine that each of the outcomes e_i is as likely to occur as any other, then we assign the probability value $1/n$ to each. That is,

$$P(e_i) = \frac{1}{n}, \quad i = 1, 2, 3, \dots, n$$

Examples 2.3:

1. A red die and a green die are rolled. The dice are fair. Let e_{ij} denote the outcome i on the red die, j on the green die. What is $P(e_{ij})$?

2. Two identical die are rolled and the sum of the numbers that turn up is noted. What is the sample space? Are the outcomes equally likely?
3. A combination lock has 3 dials, each labeled with the 9 digits from 0 to 9. An opening combination is a sequence of 3 digits with no digit repeated. How many opening combinations are there? What is the probability of a person guessing the right opening combination?
4. A 5-card hand is dealt from a standard, well shuffled, 52-card deck. How many simple events are in the sample space? Are they equally likely? If so, what are the probability assignments for the simple events?

Solutions:

1. The dice are fair; each outcome is as likely to occur as any other; there are 36 possible outcomes (6 possibilities on the red die, 6 possibilities on the green die, $6 \times 6 = 36$). Therefore,

$$P(e_{i,j}) = \frac{1}{36}.$$

2. The sample space is $S = \{2, 3, 4, 5, \dots, 11, 12\}$. There are 11 outcomes. If we assume that the outcomes are equally likely, then we would assign $\frac{1}{11}$ to each outcome. While this assignment is acceptable, is it reasonable? Note that there is only one way to get a “2”, both dice have to come up “1”. Similarly, there is only one way to get a “12”, both dice have to come up “6”. On the other hand, there are several ways to get a “7” – a “1” and a “6”, a “2” and a “5”, etc. Therefore a “7” is more likely to occur than a “2” or a “12”. We conclude that the outcomes are not equally likely and so the equally likely assignment is not appropriate. We will finish this example below.

3. The first digit can be any one of the 10 numbers. Since no digit can be repeated, the second digit can be any one of the remaining 9 and the third any one of the 8 that remain after the first 2 numbers have been chosen. Thus, by the Multiplication Principle, the number of opening combinations is

$$10 \times 9 \times 8 = 720$$

Note that the number of opening combinations is the number of permutations of 10 things taken 3 at a time, $P_{10,3}$. The sample space is the set of possible opening combinations. The probability of guessing the correct one is $1/720$.

4. The number of 5-card hands that can be dealt from a standard deck is the number of combinations of 52 things taken 5 at a time; $C_{52,5}$. Each hand is a likely to occur as any other so each hand has probability $1/C_{52,5}$.

Probability of an Event:

Now suppose that we have been given a random experiment, determined the sample space S , and devised an acceptable probability assignment for the simple events. How do we define the probability of an arbitrary event E ?

Suppose $S = \{e_1, e_2, e_3, \dots, e_n\}$ is the sample space for an experiment, and suppose E is an arbitrary subset of S . We define the *probability of E* as follows:

- (i) If E is the empty set, then $P(E) = 0$.
- (ii) If E is a simple event, then $P(E)$ has already been assigned.
- (iii) $E = \{e_{i_1}, e_{i_2}, \dots, e_{i_k}\}$ is a compound event, then

$$P(E) = P(e_{i_1}) + P(e_{i_2}) + \dots + P(e_{i_k}).$$

That is, $P(E)$ is the sum of the probabilities of the simple events that comprise E .

- (iv) If $E = S$, then $P(E) = 1$. This follows from (iii) and the definition of the probability of simple event.

Note: An event E such that $P(E) = 0$ is called an *impossible event*. An event E such that $P(E) = 1$ is called a *certain event*.

Examples 2.4:

1. Toss a fair coin twice. What is the probability that at least one tail appears?
2. Roll a red die and a green die.
 - (a) What is the probability that the sum of the numbers that turn up is 7?
 - (b) What is the probability that the sum of the numbers that turn up is less than 4?
 - (c) What is the probability that the sum of the numbers is 13?

3. A company receives orders from women in the following age groups:

Age	Number of Orders
Under 20	10
20 - 29	22
30 - 39	47
40 - 49	18
50 and over	3
	100

A customer is selected at random from the set of 100 customers.













- (a) Give the sample space for the experiment and determine the probabilities of the simple events.
- (b) What is the probability that the customer selected is under 40?

Solutions:

1. The sample space is $S = \{HH, HT, TH, TT\}$. The outcomes are equally likely and so we assign probability $\frac{1}{4}$ to each. Let $E =$ “at least one tail”. Then $E = \{HT, TH, TT\}$ and

$$P(E) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

2. The sample space is illustrated in the figure.

		GREEN DIE					
							
RED DIE		(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
		(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
		(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
		(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
		(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
		(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

As we discussed previously, the outcomes are equally likely to occur and so we assign the value $\frac{1}{36}$ to each.

(a) Let $A =$ “the sum of the numbers is 7”. Then

$$A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

and

$$P(A) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{6}{36} = \frac{1}{6}.$$

(b) Let $B =$ “the sum of the numbers is less than 4”. Then

$$B = \{(1, 1), (1, 2), (2, 1)\}$$

and

$$P(B) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{3}{36} = \frac{1}{12}.$$

(c) Let $C =$ “the sum is 13”. Since the sum of the numbers on the dice is a number between 2 and 12. $C = \emptyset$ and $P(C) = 0$; C is an impossible event.

3. (a) Based on this data,

$$P(< 20) = 22/100, \quad P(20 - 29) = 22/100, \quad P(30 - 39) = 47/100, \\ P(40 - 49) = 18/100, \quad P(> 49) = 3/100$$

(b) The probability that she is under 40 is:

$$P(< 20) + P(20 - 29) + P(30 - 39) = 0.1 + 0.22 + 0.47 = 0.70.$$

It is clear that the critical step in determining the probabilities of events associated with a given experiment is the assignment of acceptable, reasonable probabilities to the simple events in the sample space S . Once that has been done, we can find the probability of any event connected with the experiment.

The equally likely case revisited: As illustrated above, probability experiments that result in equally likely simple events is the “easiest” case; each of the simple events is assigned probability $1/n$ where n is the total number of possible outcomes. While not all probability experiments have equally likely outcomes, many can be restated or refined so that the outcomes are equally likely and this is an effective approach to solving probability problems.

So now we consider the special case of the probability of an event E when the outcomes of the experiment are equally likely.

Suppose that

$$S = \{e_1, e_2, e_3, \dots, e_n\}$$

is the sample space of an experiment and that the outcomes are equally likely. Then

$$P(e_i) = \frac{1}{n}, \quad i = 1, 2, 3, \dots, n.$$

If $E = \{e_{i_1}, e_{i_2}, \dots, e_{i_k}\}$, (E consists of k simple events), then

$$P(E) = P(e_{i_1}) + P(e_{i_2}) + \dots + P(e_{i_k}) = \underbrace{\frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}}_{k \text{ terms}} = \frac{k}{n}.$$

That is, the probability of the event E is simply

$$P(E) = \frac{\text{the number of elements in } E}{\text{the number of elements in } S} = \frac{n(E)}{n(S)}.$$

Examples 2.5:

1. Toss two identical coins.

- (a) What is the probability that the coins match (both heads or both tails)?
- (b) What is the probability that at least one head turns up?

2. Roll two identical dice (the completion of Example 2.3.2).

- (a) What is the probability that the sum of the numbers that turn up is 7?
- (b) What is the probability that the sum of the numbers that turn up is less than 4?
3. A committee has 6 female and 5 male members. A 5-member subcommittee is to be selected at random from the 11 committee members.
- (a) What is the probability that the subcommittee will consist of 3 females and 2 males?
- (b) What is the probability that the subcommittee will have at least 4 females?

Solutions:

1. If we toss two identical coins the set of possible outcomes is $S = \{HH, HT, TT\}$. Stated in terms of the number of heads that appear, $S = \{2, 1, 0\}$. However, these simple events are not equally likely; there is only one way to get 2 heads and only one way to get 0 heads, but exactly 1 head can occur in two ways. If the coins were distinguishable, then the sample space would be

$$R = \{HH, HT, TH, TT\}$$

and these events are equally likely with each having the probability $\frac{1}{4}$. Note that the compound event $\{HT, TH\}$ in R corresponds to the simple event HT (or exactly 1 head) in S . Since the coins don't know whether or not they are identical, we will assume that they are and then go on to determine the probability assignment for S :

$$P(HH) = \frac{1}{4}, \quad P(HT) = \frac{1}{2}, \quad P(TT) = \frac{1}{4}.$$

(a) Let $A =$ "the coins match". Then $A = \{HH, TT\}$ and $P(A) = \frac{1}{2}$.

(b) Let $B =$ "at least 1 head". Then $B = \{HH, HT, TH\}$ and $P(B) = \frac{3}{4}$.

2. The sample space is: $S = \{2, 3, 4, \dots, 12\}$. As we have seen, these outcomes are not equally likely; the assignment

$$P(2) = P(3) = P(4) = \dots = P(12) = \frac{1}{11}$$

is acceptable, but it is not reasonable.

As in part 1, we handle this situation by defining a sample space for the experiment in which the outcomes are equally likely. Assume that one of the dice is red and the other is green. There is no loss in generality here; the dice don't know what color they are. Now see Example 2.4.3.

(a) If $A =$ “the sum of the numbers is 7”, then $P(A) = \frac{1}{6}$.

(b) If $B =$ “the sum of the numbers is less than 4”, then $P(B) = \frac{1}{12}$.

3. The sample space is the set of 5-member subcommittees and, since the subcommittee will be selected at random, any particular 5-member subcommittee is as likely to be chosen as any other.

(a) Let $E =$ “the subcommittee has 3 female and 2 male members”. Then

$$P(E) = \frac{\text{no. of subcommittees with 3 females and 2 males}}{\text{no. of 5-member subcommittees}}.$$

The number of 5-member subcommittees is the combinations of 11 things taken 5 at a time, $C_{11,5}$. View the selection of a subcommittee with 3 female members and 2 male members as a 2-step process:

Step1: Select the 3 female members. This can be done in $C_{6,3}$ ways.

Step2: Select the 2 male member. This can be done in $C_{5,2}$ ways.

Therefore,

$$P(E) = \frac{C_{6,3}C_{5,2}}{C_{11,5}} = \frac{\frac{6!}{3!3!} \cdot \frac{5!}{2!3!}}{\frac{11!}{5!6!}} = \frac{20 \cdot 10}{462} = \frac{200}{462} = \frac{100}{231}.$$

(b) Let $F =$ “at least 4 females”. The number of subcommittees with at least 4 female members equals the number with exactly 4 female members plus the number with exactly 5 female members. Proceeding as above, the number of subcommittees with 4 female and 1 male member is given by $C_{6,4}C_{5,1} = 15 \cdot 5 = 75$. The number of subcommittees with 5 female members is given by $C_{6,5} = 6$. Thus,

$$P(F) = \frac{75+6}{462} = \frac{81}{462} = \frac{27}{154}.$$

Exercises 3.2:

- Let E be an event in a sample space S .
 - How would you interpret $P(E)=0$?
 - How would you interpret $P(E)=1$?
- A probability experiment is conducted. Which of the following cannot be considered a probability of an outcome.
 - $2/3$
 - -0.45
 - 75%
 - 0
 - 2.5
 - $-1/2$
 - 1.6
 - 1
- A probability experiment has 4 possible outcomes $A, B, C,$ and D . Which of the following is an acceptable probability assignment? Give reasons for your answer.
 - $P(A) = 0.15, P(B) = -0.35, P(C) = 0.50, P(D) = 0.70$
 - $P(A) = 0.32, P(B) = 0.28, P(C) = 0.24, P(D) = 0.30$
 - $P(A) = 0.26, P(B) = 0.14, P(C) = 0.42, P(D) = 0.18$
- A die is loaded in such a way that an even number is twice as likely to turn up as an odd number. Determine the probability assignments for the simple events.
- In a family with 3 children, excluding multiple births and assuming that a boy is as likely as a girl at each birth, what is the probability of having:
 - 2 boys and 1 girl in that order?
 - 2 boys and 1 girl in any order?
 - At least one child of each sex?
- An experiment consists of tossing 3 fair (i.e., not weighted) coins, except one of the coins is 2-headed. Find the probability of getting:
 - 1 head.
 - 2 heads.
 - 3 heads.
 - 0 heads.
 - At least one head.
 - More than 1 tail.
- An experiment consists of rolling 2 fair dice and adding the numbers on the faces that turn up. Find the probability that the sum is:
 - 12
 - 5
 - Greater than 8.
 - Less than 7.
 - An odd number.
 - A prime number.
 - Divisible by 3.

8. An experiment consists of rolling 2 fair dice and adding the numbers on the faces that turn up. However, each die has the number 1 on two opposite faces, the number 2 on two opposite faces and the number 3 on two opposite faces. Find the probability that the sum is:
- (a) 6 (b) 3 (c) An odd number. (d) 7
(e) Greater than 3.
9. Among 100 students in a college fraternity, 50 are majoring in the humanities majors, 30 are majoring in a science and 20 are engineering majors. If a student is selected at random, what is the probability that he is:
- (a) An engineering major?
(b) Neither a science nor an engineering major?
10. A roulette wheel has 38 spaces numbered 1, 2, ..., 36, 0 and 00. Find the probability of getting:
- (a) An odd number.
(b) A number greater than 24.
(c) A number less than 15.
(d) A prime number.
11. An experiment consists of dealing 5 cards from a standard deck of 52 cards. What is the probability of being dealt:
- (a) 5 hearts?
(b) 5 black cards?
(c) 4 of a kind (i.e., 4 aces, or 4 kings, or 4 queens, and so on)?
(d) 3 aces and 2 queens?
(e) 3 of one kind and two of another (a “full house”)?
12. A 4-person committee is to be composed of employees from two departments, department A with 15 employees and department B with 20 employees. If the 4 people are selected at random from the pool of 35 people, what is the probability that:
- (a) All are from department B?
(b) 3 are from B and 1 is from A?
(c) 2 are from A and 2 are from B?
(d) At least 3 are from department A?