CHAPTER 3 PROBABILITY

I. WHAT IS PROBABILITY?

Random Experiments

The weatherman on 10 o'clock news program states that there is a 20% chance that it will snow tomorrow, a 65% chance that it will rain and a 15% chance that the weather will be clear. What does this mean? What are the chances that it will not rain tomorrow? What are the chances that there will be some form of precipitation? Assuming that it will either snow, rain or be clear, which of the three <u>will</u> occur?

The table below shows the results of a recent poll of 1000 Americans on the President's economic policies:

Strongly approve	32%
Mildly approve	17%
Mildly disapprove	14%
Strongly disapprove	23%
No opinion	14%

If a person is selected at random from the set of people who were polled, what is the likelihood that the person approved of the President's economic policies? What is the likelihood that the person did not disapprove?

An ordinary die is rolled. What is the likelihood that the number that turns up is a prime number? What are the chances that the number that turns up is not greater than 4?

A two-headed coin is flipped. What is the likelihood that the coin will come up heads? Tails?

Processes in which there are an observable set of possible outcomes are called *experiments*.

There are two basic types of experiments. In the first three examples above, we do not know in advance what the outcome will be. We do not know for certainty what the weather will be tomorrow. If we pick a person at random from the poll group, we would have no way of knowing in advance what the person's preference will turn out to be. In contrast, if we flip a two-headed coin we do know for sure what the result will be, heads! An experiment in which any one of number of possible outcomes may result is called a *random experiment* or *probability experiment*. In contrast, a process in which the outcome is known in advance (e.g., flipping a two-headed coin) is called a *deterministic experiment*.

Probability is the branch of mathematics that is concerned with "modeling" random experiments. That is, probability attempts to provide mathematical formulations (mathematical models) of random experiments. Since we will be concerned only with random experiments in the work that follows, the word *experiment* will mean random experiment.

Sample Spaces

Suppose we perform an experiment. The set S of possible outcomes is called the *sample space* for the experiment.

Examples 1.1: List, or describe, the sample space for each experiment.

1. Roll an ordinary die and record the number of dots on the upper face.

2. Draw a card from a standard 52-card deck and record its suit.

3. Draw a card from a standard deck.

4. Toss a fair coin. If the result is "heads", stop. If the result is "tails", toss the coin a second time.

5. A dartboard is in the shape of a square of side length 12 inches. Place a coordinate system on the board with the origin at the center of the square. Throw a dart at the board and record its position as an ordered pair of real numbers (a,b)

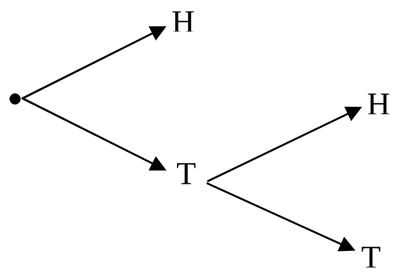
Solutions:

1. Since any one of the six faces can land up, we set $S = \{1,2,3,4,5,6\}$.

2. There are four suits: hearts (H), spades (S), diamonds (D), and clubs (C). Thus we set $S = \{H, S, D, C\}$.

3. There are 52 possible outcomes: ace of hearts (AH), king of hearts (KH),…two of hearts(2H), and so on. Thus, $S = \{AH, KH, ..., 2H, AS, KS, ...2C\}$. Alternatively, $S = \{c \mid c \text{ is a card from a standard 52-card deck}\}$.

4. A tree diagram is a convenient way to picture the outcomes of this experiment.



Thus, $S = \{H, TH, TT\}$.

5. Since the square has side length 12 inches and the origin is at the center, the coordinates of an arbitrary point (a,b) on the board must satisfy $-6 \le a \le 6$, $-6 \le b \le 6$. Therefore,

$$S = \{(a,b) \mid a \text{ and } b \text{ are real numbers, and } -6 \le a \le 6, -6 \le b \le 6\}.$$

Finite and infinite sample spaces:

Look at examples in Examples 1.1. There is a significant difference between the sample spaces for Examples 1 – 4 and Example 5, namely, the sample spaces in 1 – 4 have only finitely many elements, while the sample space in example 5 has infinitely many elements. *We will be concerned only with the finite case* in this treatment of probability.

Choosing a sample space:

A sample space for an experiment is the set of possible outcomes. However, there may be different appropriate sample spaces for exactly the same experiment. The choice of the sample space depends on what you want to observe, on what questions you want to answer. Here are some simple examples.

Examples 1.2:

1. a. Flip a fair coin twice.

   b. Flip a fair coin twice and record the total number of heads that come up.

   c. Flip a fair coin twice and record whether the results match or not.

2. a. Roll an ordinary die.

   b. Roll an ordinary die and note whether the number that turns up is odd or even.

3. a. Draw a card from a standard deck.

b. Draw a card from a standard deck and record its suit.

c. Draw a card from a standard deck and note its color (red, black)

Solutions:

1. a. The possible outcomes are: first toss heads, second toss heads, denoted HH, first toss heads, second toss tails, denoted HT, etc. $S = \{HH, HT, TH, TT\}$.

   b. We can see from (*a*) that the total number of heads, N, is either 2, 1 or 0. Thus, $N = \{2,\ 1,\ 0\}$.

   c.The two tosses either match, Y , or they don't, *N*. Thus, $M = \{Y, N\}$.

2. a. $S = \{1, 2, 3, 4, 5, 6\}$

   b. Let *O* denote "odd" and *E* denote "even". Then $S = \{O, E\}$.

3. a. As noted above, $S = \{AH, KH, ..., 2H, AS, KS, ...2C\}$.

   b. If we are interested in the suit of the card drawn, then $S = \{H, S, D, C\}$.

   c. If we are only interested in the color of the card drawn, then $S = \{R, B\}$.

Look again at Example 1.2.1. Notice that the sample space *S* in (*a*) contains more information than either *N* or *M*. More importantly, notice that, if we know which outcome in *S* has occurred, then we know which outcome in *N* and which outcome in *M* has also occurred. But the reverse is not true. For example, knowing that the coins matched does not tell us whether both were heads or both were tails; knowing that there was one head does not distinguish between $\{HT\}$ and $\{TH\}$. Similar illustrations can be made up using Examples 1.2.2 and 1.2.3. Thus, there can be more than one "correct" sample space for a given experiment.

As a general rule, when specifying a sample space for an experiment, include as much detail as is necessary to answer all of the questions of interest about the experiment. If in doubt, choose a sample space with more elements rather than less.

Events

Suppose we perform an experiment and determine a sample space *S* with possible outcomes $e_1, e_2, e_3, ..., e_n$.

An *event E* is a subset of S (including the empty set and the whole set S).

A *simple event* is a one-element subset of S; the simple events represent each of the possible outcomes of the experiment. For example, if we roll an ordinary die, then $S = \{1, 2, 3, 4, 5, 6\}$, and $e_1 = \{1\}$ is the (simple) event that the number 1 turns up, $e_2 = \{2\}$ represents the event that the number 2 turns up, and so on.

If E contains more than one element, then it is a *compound event*.

It is important to note that, in considering random experiments, the sample space S plays the role of the universal set introduced in the Chapter 1 – Sets.

Example 1.3:

1. An experiment consists of recording the possible boy-girl compositions of a three-child family.

   a. How many simple events are there?

   b. Use a tree diagram to determine a sample space for the experiment.

   c. Let E be the event “at least 2 girls”. What is E?

   d. Let F be the event “four boys”. What is F?

2. An experiment consists of rolling a pair of dice, one red, the other green, and recording the sum of the numbers on the top faces.

   a. Give a sample space for the experiment. How many outcomes (simple events) are there?

   b. Let E be the event “the sum is a prime number”. What is E?

   c. Let F be the event “the sum is greater than 1”. What is F?

3. Give the sample space for each of the following experiments coin tossing experiments and find the event E = “exactly one head turns up”.

   a. Toss a fair coin twice.

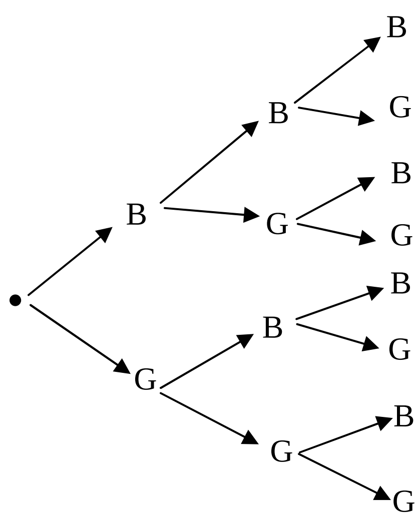
b. Toss a penny and a nickel.

c. Toss two identical fair coins.

Solutions:

1. a. Since the first child is either a boy or a girl , the second is either a boy or a girl, and the third is either a boy or a girl, the number of possible outcomes is $2 \cdot 2 \cdot 2 = 8$ by the Multiplication Principle.

b.



Thus, $S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$.

c. $E = \{BGG, GBG, GGB, GGG\}$

d. $F = \varnothing$

2. a. There are eleven possible outcomes.

b. $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

c. $E = \{2, 3, 5, 7, 11\}$

d. $F = S$.

3. a. & b. Tossing one coin twice and tossing two distinguishable coins (e.g., a penny and a nickel) are equivalent; we can “distinguish” what each does.

$$S = \{HH, HT, TH, TT\} \quad \text{and} \quad E = \{HT, TH\}$$

c. If we toss two identical coins, then we can only record what we might see:

$$S = \{HH, HT, TT\} \quad \text{and} \quad E = \{HT\}, \text{ a simple event.}$$

The outcomes HT and TH in the sample spaces above become one outcome here because we cannot tell which coin came up heads and which coin came up tails; the coins are indistinguishable. Another way to represent the sample space is in terms of the number of heads (or the number of tails) that turn up:

$$S = \{2, 1, 0\}.$$

Exercises 3.1:

1. Roll a pair of dice and note the numbers that turn up. Give the sample space if:

   a. One of the dice is red and the other is green.
   b. The dice are identical.

2. An experiment consists of tossing a fair coin and then rolling a fair die.

   a. How many possible outcomes are there?
   b. Give the sample space.
   c. Find the event E = “either the coin is heads or the die is odd”.

3. A couple decides to have children until they have at least one of each sex, up to a maximum of three children.

   a. Draw a tree diagram for this “experiment.”
   b. Give the sample space.
   c. Find the event E = “at least two girls”.

4. A box contains two \$1 bills, one \$5 bill and one \$10 bill. An experiment consists of drawing bills out of the box, without replacement, until the \$10 bill is drawn.

   a. Draw a tree diagram for this experiment.
   b. Give the sample space.
   c. Find the event E = “the sum of the bills drawn is \$17”.
   d. Find the event F = “the sum of the bills drawn is less than \$15.

5. An urn contains three balls numbered 1, 2, and 3.

a. The balls are drawn out in succession, without replacement, and the number on the ball is recorded. Draw a tree diagram for this experiment and find all possible 3-digit numbers.
b. Suppose three balls are drawn as follows: a ball is drawn, the number recorded, and then replaced before the next ball is drawn. How many elements are in the sample space?

6. Suppose a ball numbered 0 is added to the urn in Problem 4. The balls are drawn out in succession, without replacement, and the number on the ball is recorded.

   a. How many elements are in the sample space?
   b. How many 4-digit numbers are there? (Remember that a 4-digit number cannot begin with 0.)
   c. How many 4-digit numbers are odd?
   d. How many 4-digit numbers are less than 3000?

7. An experiment consists of rolling two fair dice. However, one of the dice has exactly one dot on two opposite faces, exactly two dots on two opposite faces, and exactly three dots on two opposite faces; the other one is a standard die.

   a. Give the sample space for this experiment.
   b. Let E = "the sum of the numbers that turn up is a prime number". How many elements are in E?
   c. Let F = "the sum of the numbers is 6". How many elements are in F?

8. An experiment consists of dealing a 5-card hand from a standard deck of 52 cards.

   a. How many different hands are there? That is, how many outcomes are in the sample space?
   b. How many hands have 4 aces?
   c. How many hands have all the cards in the same suit?
   d. How many hands have four cards of one kind (i.e., 4 kings, or 4 queens, and so on).
   e. How many hands have three cards of one kind and two of another kind?