CHAPTER 2
COUNTING

I. BASIC PRINCIPLES OF COUNTING

This chapter is concerned with "counting"; counting the number of elements in a given set or in some specified subset of a given set, counting the possible outcomes of some experiment, or counting the number of ways to perform a sequence of operations. Our objective is to provide systematic approaches to, and methods for answering the question: How many?

Examples 1.1:

1. Let $A = \{a, b, c\}$. How many subsets of A are there?

2. If we toss a penny and a nickel, how many different possible outcomes are there?

3. An experiment consists of flipping a coin and then rolling a die. How many different possible outcomes are there?

Solutions:

1. The subsets of $\{a, b, c\}$ are:

$$\varnothing,\ \{a\},\ \{b\},\ \{c\},\ \{a,b\},\ \{a,c\},\ \{b,c\},\ \{a,b,c\}.$$

There are 8 subsets.

2. The possible outcomes are

$$HH,\ HT,\ TH,\ TT$$

where, for example, *HH* signifies "head on the penny" and " head on the nickel", etc. There are 4 possible outcomes.

3. The coin comes up in one of two ways; *H* or *T*, the die comes up in any one of six ways; 1, 2, 3, 4, 5, 6. The possible outcomes are $\{H1, H2, \ldots, H6, T1, T2, \ldots, T6\}$. There are $2 \cdot 6 = 12$ possible outcomes.

In the examples above, we were able to answer the "how many" question because the number of elements in each set was small and we were able to list the elements explicitly.

Here is another, less obvious, example:

Example 1.2:

In a group of 100 college freshmen, it is found that 50 students are taking English, 30 students are taking mathematics and 10 students are taking both English and mathematics.

> (*a*) How many students are taking either English or mathematics? That is, how many students are taking at least one of the two courses?
>
> (*b*) How many students are taking neither English nor mathematics?

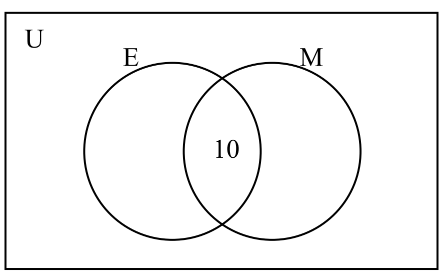
Solution:

Let E represent the students who are taking English and let M represent the students who are taking mathematics. We might be tempted to answer question (*a*) by

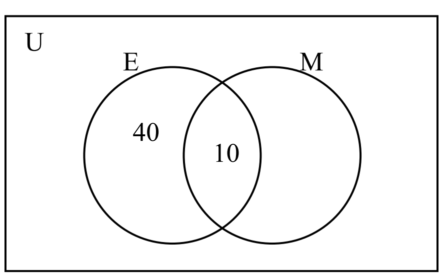
$$n(E \cup M) = n(E) + n(M) = 50 + 30 = 80\,.$$

However, as we now show, this is not correct. Consider following sequence of Venn diagrams:

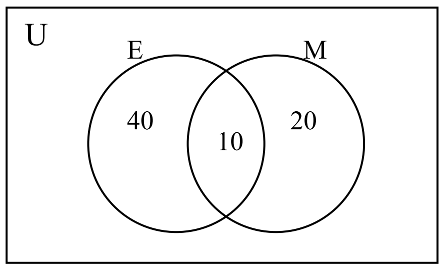
1. We know that there are 10 students taking both English and mathematics:



2. Now, since $n(E) = 50$ and 10 of those have already been accounted for, there must be 40 students who are taking English but not mathematics:



3. Similarly, $n(M) = 30$ and 10 have been accounted for in $E \cap M$, so there must be 20 students who are taking mathematics but not taking English:

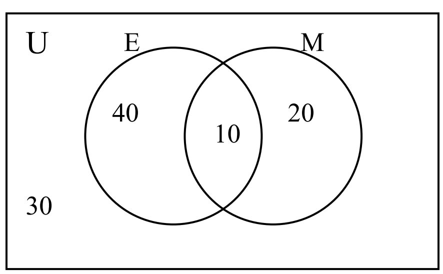


We can now answer the first question: The number of students who are *either* taking English *or* mathematics is given by

$$n(E \cup M) = 40 + 10 + 20 = 70 .$$

Note that, $n(E \cup M) = 50 + 30 - 10 = n(E) + n(M) - n(E \cap M)$.

4. Finally, $n(U) = 100$ and 70 students have been accounted for, so there must be 30 students who are taking neither English nor mathematics.



Addition Principle:

Example 1.2 illustrates our first principle of counting, the *addition principle*. For any two sets A and B,

(1) $$n(A \cup B) = n(A) + n(B) - n(A \cap B) .$$

It is important to remember that, in general, $n(A \cup B) \neq n(A) + n(B)$. However, if A and B are disjoint sets ($A \cap B = \emptyset$), then it follows immediately from (1) that

$$n(A \cup B) = n(A) + n(B) .$$

Complement Principle:

Suppose A is a set in a universal set U. Another consequence of the addition principle is the so-called *complement principle*. Since

$$A \cup A^c = U \quad \text{and} \quad A \cap A^c = \varnothing ,$$

It follows that $n(U) = n(A) + n(A^c)$ and therefore,

(2) $$n(A^c) = n(U) - n(A) .$$

In words, the number of elements in the *complement* of A is the number of elements in the universal set U *minus* the number of elements in A.

Examples 1.3:

1. Let A and B be sets where $n(U) = 50$, $n(A) = 11$, $n(B) = 41$, and $n(A \cap B) = 7$. Find

(a) $n(A \cup B)$ (b) $n(B^c)$ (c) $n(A \cap B^c)$ (d) $n(A \cup B)^c$

2. . Let A and B be sets where $n(U) = 50$, $n(A) = 11$, $n(B) = 41$, and $n(A \cup B) = 43$. Find

(a) $n(A \cap B)$ (b) $n(A \cup B)^c$ (c) $n(A)$ (d) $n(B \cap A^c)$

3. A certain city has two daily newspapers , the *Chronicle* and the *Times*. The results of a survey of 100 residents are: 35 people subscribe to the *Chronicle*, 60 subscribe to the *Times*, and 20 subscribe to both papers.

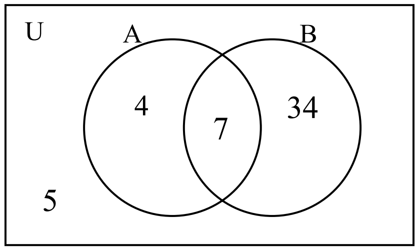
(a) How many people subscribe to the *Chronicle*, but not to the *Times*?

(b) How many people subscribe to the *Times*, but not to the *Chronicle*?

(c) How many people subscribe to neither paper?

Solutions:

1. The Venn diagram is:



$(a)\ n(A \cup B) = 4 + 7 + 34 = 45$. $(b)\ n(B^c) = 50 - 41 = 9$.

$(c)\ n(A \cap B^c) = 4$. $(d)\ n(A \cup B)^c = 50 - 45 = 5$.

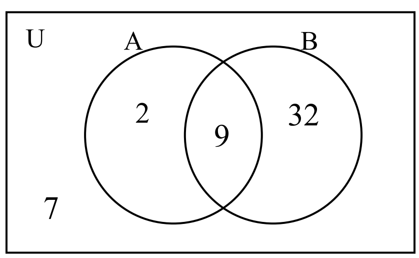
2. (a) Using the addition principle, we have

$$43 = 11 + 41 - n(A \cap B) = 52 - n(A \cap B)$$

Therefore,

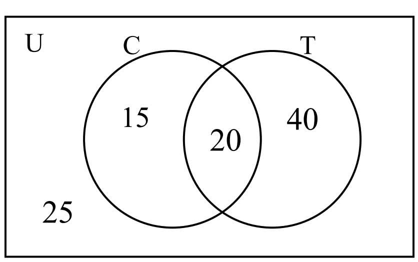
$$n(A \cap B) = 52 - 43 = 9.$$

Now, draw a Venn diagram:



$(b)\ n(A \cup B)^c = 7$. $(c)\ n(A) = 11$. $(d)\ n(B \cap A^c) = 32$.

3. A Venn diagram for the problem is:



$(a)\ n(C \cap T^c) = 15$. $(b)\ n(T \cap C^c) = 40$. $(c)\ n(C^c \cap T^c) = 25$.

Exercises 2.1.1:

1. A and B are subsets of a universal set U, and $n(A)=80$, $n(B)=50$ $N(A \cap B)=20$, $n(U)=200$. Find
(a) $n(A \cup B)$ (b) $n(A^c \cap B)$ (c) $n(A^c \cap B^c)$ (d) $n[(A \cup B)^c]$

2. A and B are subsets of a universal set U, and $n(A)=25$, $n(B)=55$ $N(A \cup B)=60$, $n(U)=100$. Find
(a) $n(A \cap B)$ (b) $n(A \cap B^c)$ (c) $n(A^c \cap B^c)$ (d) $n[(A \cup B)^c]$

In Problems 3 and 4 use the given information to complete the table

$\cap$	A	A^c	Totals
B	?	?	?
B^c	?	?	?
Totals	?	?	?

3. $n(A)=60$, $n(B)=80$, $n(A \cap B)=20$, $n(U)=150$

4. $n(A)=40$, $n(B)=55$, $n(A \cup B)=80$, $n(U)=100$

5. A group of 100 people touring Europe includes 46 people who speak French, 52 who speak German and 20 who speak neither language.

   a. How many people speak at least one of the languages?
   b. How many people speak both languages?
   c. How many people speak French but not German?
   d. How many people speak German but not French?

6. A cable TV company has 5000 subscribers in a small community. The company offers two premium channels, HBO and Showtime. If 2050 subscribers receive HBO, 1480 receive Showtime and 2230 receive neither:

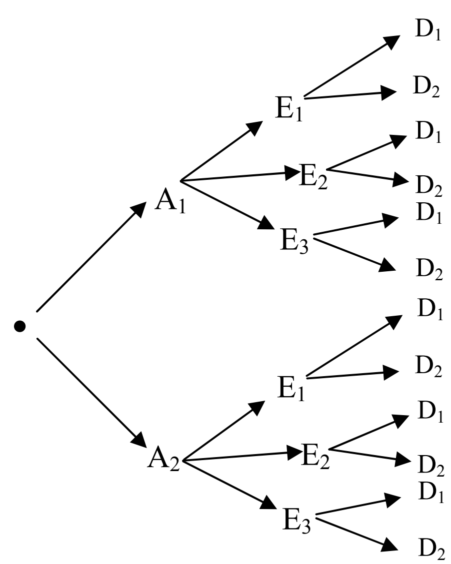
   a. How many subscribers receive both channels?
   b. How many subscribers receive only HBO?
   c. How many subscribers receive only Showtime?

We have seen that the addition principle can be used to determine the number of elements in unions, intersections and complements of sets. Now we consider sets whose elements are determined by a sequence of operations and we will see that multiplication is used to count the number of elements in sets formed in this manner. We start with an example.

Example 1.4:

A local restaurant advertises a fixed price dinner that includes your choice of appetizer, entree, and dessert. If there are 2 appetizers, 3 entrees, and 2 desserts, how many different dinner combinations are there?

Solution: Let A_1, A_2 denote the appetizers, E_1, E_2, E_3 the 3 entrees, and D_1, D_2 the 2 desserts. A tree diagram illustrating the possible dinner combinations is given by:



Therefore, there are 12 different dinner combinations. We can analyze this result as follows: We have a sequence of three “operations”: Operation 1- select an appetizer; Operation 2 – select an entree; Operation 3- select a dessert. Operation 1 can be done in 2 ways, operation 2 can be done in 3 ways, Operation 3 can be done in 2 ways. The total number of different ways the sequence of operations can be performed is: $2 \times 3 \times 2 = 12$.

Multiplication Principle (also called the **Fundamental Principle of Counting**):

Suppose that a "task" or an "experiment" consists of a sequence of steps $S_1, S_2, \ldots, S_k$ and suppose that there are n_1 ways to complete the first step, n_2 ways to complete the second step, … , and n_k ways to complete the k^{th} step. Then the total number of ways to complete the "task" or "experiment" is:

(3) $$n_1 \times n_2 \times \ldots \times n_k \, .$$

Examples used to illustrate the multiplication principle typically involve "experiments" such as drawing colored or numbered marbles from a box, drawing cards from a deck of cards, and so forth. Before proceeding to more examples, we need to distinguish between two types of such experiments: **With replacement vs without replacement** (also said "with repetition vs without repetition").

Example 1.5:

Suppose a box contains 4 balls numbered 1, 2, 3, 4, and suppose that we randomly select one of the balls, note its number, and then select a second ball. How many different outcomes (i,j) (the number i on the first ball, the number j on the second ball) are possible? Before answering this question we need to know:

(*a*) do we put the first ball back in the box before drawing the second ball, or

(*b*) do we set the first ball aside and then go on to draw the second ball?

In the first case, we say that we are drawing the two balls *with replacement*, and in the second case, we are drawing two balls *without replacement*. The two cases yield different answers to the question: How many different outcomes (i, j) are possible?

***(a)* With replacement:**

Step1: There are 4 possible numbers for the first ball. We put the ball back in the box.

Step 2: There are 4 possible numbers for the second ball.

Thus, there are $4 \times 4 = 16$ possible outcomes (i, j).

***(b)* Without replacement:**

Step1: There are 4 possible numbers for the first ball. We set that ball aside.

Step 2: There are 3 balls left in the box so there are 3 possible numbers for the second ball.

Thus, there are $4 \times 3 = 12$ possible outcomes (i, j).

Examples 1.6:

1. How many different 4-letter code words are possible from the first five letters of the alphabet if:

(a) No letter is repeated? (Here letters are chosen *without repetition.*)

(b) Letters can be repeated? (Here letters are chosen *with repetition.*)

(c) Letters can be repeated but adjacent letters must be different?

2. The car license plates in a certain state consist of 3 letters followed by 3 digits (from 0 to 9).

(a) How many license plates have no letters repeated?

(b) How many license plates have the letters begin with a vowel, the numbers begin with a prime digit, and the letters and numbers can be repeated?

Solutions:

1. We choose 4 letters from $S = \{a, b, c, d, e, f\}$.

(a) No letters repeated: $4 \cdot 3 \cdot 2 \cdot 1 = 24$.

(b) Letters can be repeated: $4 \cdot 4 \cdot 4 \cdot 4 = 4^4 = 256$.

(c) Letters can be repeated, adjacent letters are different: $4 \cdot 3 \cdot 3 \cdot 3 = 168$

2. (a) No letters repeated; numbers can be repeated:

$$26 \cdot 25 \cdot 24 \cdot 10 \cdot 10 \cdot 10 = 15,600,000.$$

(b) Letters begin with a vowel, numbers begin with a prime digit, letters and numbers can be repeated:

$$5 \cdot 26 \cdot 26 \cdot 4 \cdot 10 \cdot 10 = 1,352,000.$$

Exercises 2.1.2:

1. A new car model is available with 6 choices of color, 3 choices of transmission, 5 types of interior, and 2 types of engine. How many different variations of this car are possible?

2. A man has 4 sports jackets, 7 ties and 3 vests. How many different outfits can he wear?

3. The license plates in a certain state consist of 5 digits chosen from the 10 digits 0, 1, 2, …, 9.

   a. How many different license plates are there if digits can be repeated?
   b. How many different license plates are there if digits cannot be repeated?
   c. How many different license plates are there if the first digit cannot be 0 and repetitions are not allowed?

4. The call letters of a certain radio station must have 4 letters and the first letter must be either a K or a W.

   a. How many different call letters can be made if repetitions are permitted?
   b. How many different call letters can be made if repetitions are not permitted?
   c. How many different call letters are there if exactly one of the letters must be a vowel and repetitions are not permitted?

5. A corporation executive must select a manager and an assistant manager for each of his two stores, the first store having 10 employees and the second store having 8 employees.

   a. How many ways can this be done if the managers and assistant managers can come from either store?
   b. How many ways can this be done if the manager and assistant managers must be selected from the store in which they are currently employed?

6. Four-digit numbers are to be made from the 6 digits 0, 1, 2, …, 5. (Assume that the first digit cannot be 0.)

   a. How many numbers can be made with no number repeated?
   b. Allowing repetitions, how many even numbers can be made?
   c. Allowing repetitions, how many numbers can be made that are divisible by 4?

7. Each of 2 countries sends 5 delegates to a negotiating conference. A rectangular table is used with 5 chairs on each long side. If each country is assigned a long side of the table, how many different seating arrangements are possible?

8. How many different ways can 5 people, *A, B, C, D,* and *E*, sit in a row at the theater if:

   a. *A* and *B* must sit together?
   b. *C* must sit to the right of, but not necessarily next to *B*?
   c. *D* and *E* will not sit next to each other?

II. PERMUTATIONS AND COMBINATIONS

Let $S = \{a, b, c, d\}$. We consider two questions:

(1) How many *different* ways can we arrange 3 of the 4 letters?

Another way to say this is: If no letter can be repeated, how many 3-letter "words" can be formed from the 4 letters? (Here a 3-letter "word" is simply a particular arrangement of the 4 letters, it doesn't have to be a "real" word. For example, *abc* is one "word", *cba* is another, and these are different "words".). It is understood in this context that no letter can be repeated; if the 4 letters are on slips of paper in a box, then we are drawing 3 slips of paper from the box in succession and *without replacement*.

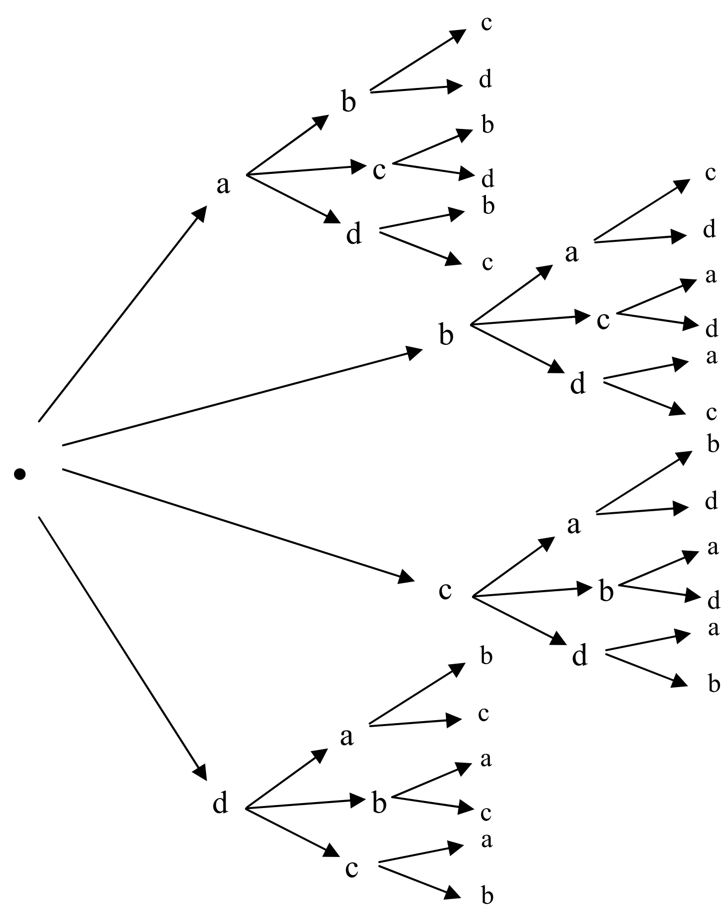
(2) How many *different* 3-element subsets of S are there?

The answer to the first question is simply an application of the Multiplication Principle. Our "task" is to select 3 letters from the set of 4.

Step 1: We can choose the first letter in any one of 4 ways. Once we have selected the first letter, 3 letters remain.
Step 2: We can pick the second letter in any one of 3 ways. Now there are 2 letters left.
Step 3: We can pick the third letter in one of 2 ways.

Thus, there are $4 \times 3 \times 2 = 24$ different 3-letter words can be formed from the 4 letters of S.

Here is a tree diagram that illustrates this result.



Each of the 3-letter "words" is called a *permutation* of the 4 letters taken 3 at a time. There are a total of 24 permutations of the 4 letters taken 3 at a time. It is important to note that a *permutation distinguishes the order of the objects* --- for example, *abc* and *cba* are *different* permutations.

Now we consider the second question. If we look at the tree diagram, we see that *abc* and *cba* are at the ends of different branches; these are different permutations of the 3 letters. However, as subsets of S, $\{a,b,c\}$ and $\{b, c, a\}$ are equal; they are the same subset. We do not distinguish *order* when we list elements of a set. So, if we take the tree diagram, put braces around the elements at the end of the branches to indicate the corresponding subsets, and then eliminate the duplications, we find that there are just 4

3-element subsets of S:

$$\{a,b,c\},\{a,b,d\},\{a,c,d\}, \quad \text{and} \quad \{b,c,d\}.$$

Each of these 3-element subsets is called a *combination* of the 4 letters taken 3 at a time. *In a combination, the order of listing the objects is not important* $-\ \{a,b,c\}$ and $\{c,a,b\}$ are the same combination.

Example 2.1:

Four people, Mr. Anderson (A), Ms. Baker (B), Mr. Clinton (C), and Ms. Doe (D) , constitute the Executive Committee of a software company.

(*a*) How many different ways can the committee select a Chair and a Vice-Chair? (Assume, of course, that one person cannot hold both positions so we are choosing people without repetition.)

(*b*) The Executive Committee decides to form a 2-member subcommittee to oversee the finances of the company. How many different subcommittees are possible?

Solution:

(*a*) Two steps are involved in selecting a Chair and a Vice-Chair.

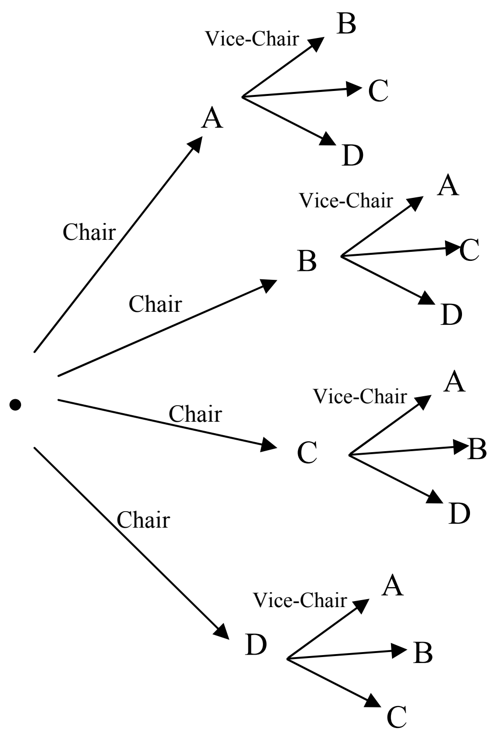
Step 1: Select the Chair: The Chair can be any one of the 4 people- 4 ways to select the Chair.

Step 2: Select the Vice-Chair. The Vice-Chair can be any one of the remaining 3 people – 3 ways to select the Vice-Chair.

Thus, the total number of the ways to select the Chair and the Vice-Chair is: $4 \times 3 = 12$.

Note that AB (Anderson-Chair, Baker-Vice-Chair) is a different slate of officers from BA (Baker-Chair, Anderson- Vice-Chair). Each of the possible slates is a permutation of the 4 objects taken 2 at a time.

Here is a tree diagram that illustrates the process of selecting a Chair and Vice-Chair.



(*b*) To form a 2-member subcommittee, we simply select 2 of the 4 members; the order of selection is unimportant – {A,B} and {B,A} are the same subcommittee. We can use the tree diagram above to help us to determine the 2-element subsets of {A,B,C,D}. They are:

{A,B},{A,C}, {A, D}, {B,C}, {B, D} and {C,D}.

Six different 2-member subcommittees can be selected from the 4-member Executive Committee. Each of the possible subcommittees is a combination of the 4 people taken two at a time.

Now let's consider a much bigger problem, we'll draw 5 cards from a standard deck.

(1) Suppose that we draw 5 cards in succession, without replacement, noting the order in which the cards are drawn (i.e. ace of hearts, king of hearts,...,10 of hearts is different

from 10 of hearts, jack of hearts,…, ace of hearts). How many different ways can this be done? That is, how many permutations of 52 cards taken 5 at a time are there?

(2) How many different 5-card hands can be dealt from a standard 52-card deck? Think about the game of "poker". Since the "value" of a 5-card hand does not depend on the order in which the cards are dealt (the hand: ace, king, queen, jack, 10 of hearts is the same as the hand: 10, jack, queen, king, ace of hearts), we are asking for the number of combinations of 52 objects taken 5 at a time.

There is no reasonable way to list all the possible outcomes for either of these questions; a tree diagram is out of question. So we need some way "theoretical" to count the number of permutations and the number of combinations.

First, some notation: Suppose we have a set of n objects and we want to count the number of different ways to select k of them, without replacement (without repetition). If we are going to take order into account (for example, picking a Chair and a Vice-Chair), then we are asking for the number of permutations of n objects taken k at a time.

The *number of permutations of* n *objects taken* k *at a time* is denoted by $P_{n,k}$.

We use the multiplication principle to calculate $P_{n,k}$. There are k steps involved.
<u>Step 1</u>: Pick one of the objects − there are n ways to do this.

<u>Step 2</u>: Pick one of the remaining $n-1$ objects − there are $n-1$ ways to do this.

<u>Step 3</u>: Pick one of the remaining $n-2$ objects − there are $n-2$ ways to do this.

And so on. Notice the pattern here: the Step number vs the number of objects to select from!

•
•
•

<u>Step k</u>: Pick one of the remaining $n-k+1$ objects − there are $n-k+1$ ways to do this.

Therefore, by the multiplication principle:

$$P_{n,k} = n(n-1)(n-2)...(n-k+1)$$

Note that the number of permutations of n objects taken n at a time is:

$$P_{n,n} = n(n-1)(n-2)...3 \cdot 2 \cdot 1$$

This number has a special name. It is called *n factorial* and is denoted by $n!$

Examples 2.2:

$1! = 1$

$2! = 2 \cdot 1 = 2$

$3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

Calculate $(a)\ 5!$ $(b)\ 6!$ $(c)\ \frac{10!}{7!}$ $(d)\ \frac{7!}{4!3!}$

Solutions:

$(a)\ 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

$(b)\ 6! = 6 \cdot 5! = 6(120) = 720$.

$(c)\ \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} = 10 \cdot 9 \cdot 8 = 720$.

$(d)\ \frac{7!}{4!3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4!3!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$.

For notational purposes, it is convenient to define $0! = 1$.

Factorial notation is useful in representing the number of permutations of n objects taken k at a time:

$$P_{n,k} = n(n-1)(n-2)...(n-k+1)$$

$$= n(n-1)(n-2)....(n-k+1) \cdot \frac{(n-k)(n-k-1)...3 \cdot 2 \cdot 1}{(n-k)(n-k-1)...3 \cdot 2 \cdot 1}$$

$$= \frac{n(n-1)(n-2)....(n-k+1)(n-k)(n-k-1)...3 \cdot 2 \cdot 1}{(n-k)(n-k-1)...3 \cdot 2 \cdot 1} = \frac{n!}{(n-k)!}$$

Therefore,

(4) $$P_{n,k} = \frac{n!}{(n-k)!}.$$

Examples 2.3:

1. How many 4-letter “words” can be formed from the first 9 letters of the alphabet if letters can be repeated?

2. How many 4-letter “words” can be formed from the first 9 letters of the alphabet if letters cannot be repeated?

3. Serial numbers for a product are to be made using 3 letters followed by 3 numbers. If the letters are to be selected from the first 9 letters of the alphabet, with no letter repeated, and the numbers are to be selected from 0,1,2,…,9, with no number repeated, how many different serial numbers are there?

Solutions:

1. Step1: Pick the first letter in any one of 9 ways.
   Step 2: Pick the second letter. Since the letters can be repeated, there are 9 ways to do this.
   And so on.

   Thus, by the multiplication principle, the total number of 4-letter words, repetition allowed, is:

$$9 \cdot 9 \cdot 9 \cdot 9 = 9^4 = 6561\,.$$

2. A 4-letter “word” with no letter repeated is a permutation of the 9 letters taken 4 at a time. Therefore, the total number of 4-letter “words” that can be formed from the first 9 letters, with no letter repeated, is

$$P_{9,4} = \frac{9!}{(9-4)!} = \frac{9!}{5!} = 9 \cdot 8 \cdot 7 \cdot 6 = 3024$$

3. There are two parts to this problem.
   Part 1. Calculate the total number of ways, N_1, to pick the 3 letters.
   Part 2. Calculate the total number of ways, N_2, to pick the 3 numbers.
   Then, by the multiplication principle, the total number of the serial numbers is:

$$N_1 \cdot N_2$$

   In a serial number, the order of the letters and the order of the numbers must be taken into account (for example, *abc*123 is different from *cab*123; *abc*123 is different from *abc*321). Therefore, N_1, the number of the ways to pick the 3 letters, is given by $P_{9,3}$ and N_2, the number of ways to pick the 3 numbers, is given by $P_{10,3}$. Thus, the number of different serial numbers is:

$$N_1 \cdot N_2 = P_{9,3} \cdot P_{10,3} = \frac{9!}{6!} \cdot \frac{10!}{7!} = 9 \cdot 8 \cdot 7 \cdot 10 \cdot 9 \cdot 8 = 362,880 .$$

Now, we turn to combinations. As discussed above, the number of combinations of n objects taken k at a time is simply the number of k-element subsets of the set of n objects, the *order* in which the k objects of a particular subset are listed is immaterial. As illustrated above, when n is small we can simply list the k-element subsets. However, this is not feasible when n is large. So we need a formula.

The number of *combinations of n objects taken k at a time* is denoted by $C_{n,k}$

To motivate a formula for $C_{n,k}$, we take an example with $n = 4$ and $k = 3$.

Example 2.4:

Let $S = \{a, b, c, d\}$. We list the 3-element subsets of S and under each subset we list all the permutations of the letters of that subset. In each case, we will be taking 3 objects, 3 at a time, so there will be $3 \cdot 2 \cdot 1 = 6$ permutations of the letters in each subset.

$\{a,b,c\}$	$\{a,b,d\}$	$\{a,c,d\}$	$\{b,c,d\}$
abc	*abd*	*acd*	*bcd*
acb	*adb*	*adc*	*bdc*
bac	*bad*	*cad*	*cbd*
bca	*bda*	*cda*	*cdb*
cab	*dab*	*dac*	*dbc*
cba	*dba*	*dca*	*dcb*

There are 4, 3-element subsets of S and there are $3!$ ways to permute the elements of each subset. Following this reasoning, we can view the calculation of the number of permutations of the 4 objects taken 3 at a time as a two-step process:

Step 1: Choose a subset − there are 4 ways to do this; there are 4 combinations of 4 objects taken 3 at a time.

Step 2: Calculate the number of permutations of the 3 elements of that subset − there are 3! ways to do this.

Therefore,

$$P_{4,3} = C_{4,3} \cdot 3! \quad \text{and} \quad C_{4,3} = \frac{P_{4,3}}{3!} = \frac{4!}{3!1!}.$$

Exactly the same reasoning applies in general. If we have a set S of n objects and we want to calculate the number of permutations of the n objects taken k at a time, we simply pick a k-element subset of S, there are $C_{n,k}$ ways to do this. Then we calculate the number of ways to permute the k objects in the subset we picked. There are $k!$ ways to do this. Thus, by the multiplication principle,

$$P_{n,k} = C_{n,k} \cdot k!.$$

Solving this equation for $C_{n,k}$ we get

(5) $$C_{n,k} = \frac{P_{n,k}}{k!} = \frac{n!}{k!(n-k)!}.$$

We can now answer the question posed above: How many different 5-card hands can be dealt from a standard 52-card deck? Answer:

$$C_{52,5} = \frac{52!}{47!(5!)} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47!}{47!(5!)} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960$$

Examples 2.5:

Evaluate

$(a)\ \frac{9!}{7!}$ $\qquad (b)\ \frac{11!}{7!4!}$ $\qquad (c)\ P_{52,3}$ $\qquad (d)\ C_{52,3}$

Solutions:

$(a)\ \frac{9!}{7!} = \frac{9 \cdot 8 \cdot 7!}{7!} = 9 \cdot 8 = 72.$

$(b)\ \frac{11!}{7!4!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{7! \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{11 \cdot 10 \cdot 9 \cdot 8}{4 \cdot 3 \cdot 2 \cdot 1} = 330.$

$(c)\ P_{52,3} = \frac{52!}{49!} = \frac{52 \cdot 51 \cdot 50 \cdot 49!}{49!} = 52 \cdot 51 \cdot 50 = 132,600.$

$(d)\ C_{52,3} = \dfrac{52!}{49!3!} = \dfrac{52 \cdot 51 \cdot 50 \cdot 49!}{49! \cdot 6} = 22{,}100$.

Example 2.6:

A committee consists of 9 men and 6 women.

(*a*) How many ways can the committee elect a Chair, a Vice Chair and a Secretary?

(*b*) How many ways can the committee elect a Chair, a Vice Chair and a Secretary if the by-laws state that the Chair and Secretary must be of opposite gender?

(*c*) How many ways can a 5-member subcommittee be selected?

(*d*) How many 5-member subcommittees have exactly 3 men and 2 women?

(*e*) How many 5-member subcommittees have at least 3 women?

Solutions:

(*a*) Chair, Vice Chair, Secretary: $15 \cdot 14 \cdot 13 = 2730$.

(*b*) Chair, Vice Chair, Secretary; Chair and Secretary of opposite gender:

(*i*) Man as Chair, woman as Secretary:

$$9 \cdot 6 \cdot 13 = 702 .$$

(*ii*) Woman as Chair, man as Secretary:

$$6 \cdot 9 \cdot 13 = 702 .$$

Total: 1404.

(*c*) $C_{15,5} = 3003$

(*d*) Choose 3 men and 2 women: $C_{9,3} \cdot C_{6,2} = 84 \cdot 15 = 1260$.

(e) At least 3 women = exactly 3 women + exactly 4 women + exactly 5 women

$$= C_{6,3} \cdot C_{9,2} + C_{6,4} \cdot C_{9,1} + C_{6,5} = 20(36) + 15(9) + 6 = 861 .$$

Exercises 2.2:

1. A small organization has 15 people.

   a. How many ways can a president, vice-president, and a treasurer be selected?
   b. How many different 5-member committees can be formed?

2. Nine cards are numbered with the digits 1, 2, 3, … , 9. A 3-card hand is dealt, one card at a time. How many hands are possible if:

   a. Order is taken into consideration (1, 2, 3 is different from 3, 2, 1)?
   b. Order is not taken into consideration?

3. How many 4-member committees can be formed from a group of 9 people if:

   a. Both Fred and Sandra must be on the committee?
   b. Either Fred or Sandra, but not both, must be on the committee?
   c. There are no restrictions?

4. Three departments have 12, 15 and 18 members, respectively.

   a. If each department is to select a delegate and an alternate to represent the department at a conference, how many different ways can this be done?
   b. If each department is to send a committee of three to represent the department at a conference, how many different ways can this be done?

5. a. How many 5-card hands can be dealt from a standard deck of 52 cards?
   b. How many 5-card hands will have all hearts?
   c. How many 5-card hands will have all face cards?
   d. How many 5-card hands will have exactly 3 spades and 2 clubs?

6. a. Five people are to be seated in a row for a photograph. How many different arrangements of the people are there?
   b. The same 5 people are to be seated at a round table for dinner. How many different arrangements of the people are there?

7. Eight distinct points are selected on the circumference of a circle.

a. How many chords can be drawn by joining the points in all possible ways?
b. How many triangles can be drawn using the 8 points as vertices?
c. How many quadrilaterals can be drawn using the 8 points as vertices?

8. A committee of 5 people must be selected from 5 men and 8 women.

a. How many ways can this be done?
b. How many ways can this be done if the committee is to have exactly 3 women?
c. How many ways can this be done if the committee is to have at least 3 women?

9. Show that $C_{n,k} = C_{n,n-k}$ for each integer k, $0 \le k \le n$.

III. PASCAL'S TRIANGLE AND THE BINOMIAL THEOREM

The number of subsets of a given set:

Suppose we have a set S with n elements. The set of subsets of a set S is called the *power set* of S and is often denoted by 2^S. The notation is suggestive of the answer to the question: How many subsets does S have? One way to proceed is to look at some examples to see if there is a pattern.

Examples 2.7:

1. $S = \{a\}$, a 1-element set. The subsets are $\varnothing$ and the set itself, $\{a\}$. There are 2 subsets of S.

2. $S = \{a,b\}$. We list the subsets of S, remembering to include the empty set $\varnothing$ and S itself. The subsets of S are:

$$\varnothing,\ \{a\},\ \{b\},\ \{a,b\}.$$

There are 4 subsets of $S = \{a, \mathrm{b}\}$. There are 4 subsets of a set with 2 elements.

3. $S = \{a,b,c\}$. We'll list the subsets of S in a systematic way, the 0-element subset $\varnothing$, the 1-element subsets, the 2-element subsets, and the 3-element subset S:

$$\varnothing,\ \{a\},\ \{b\},\ \{c\},\ \{a,b\},\ \{a,c\},\ \{b,c\},\ \{a,b,c\}.$$

There are 8 subsets of $S = \{a,b,c\}$. There are 8 subsets of a set with 3 elements.

4. $S = \{a,b,c,d\}$. It would be a little tedious to list all the subsets of S, so we'll simply try to count them.

There is 1 0-element subset, namely the empty set $\varnothing$.

There are 4 1-element subsets.

The number of 2-element subsets is the number of combinations of 4 objects taken 2 at a time:

$$C_{4,2} = \frac{4!}{2!\cdot 2!} = \frac{24}{4} = 6\,.$$

The number of 3-element subsets is the number of combinations of 4 objects taken 3 at a time:

$$C_{4,3} = \frac{4!}{3!\cdot 1!} = \frac{24}{6} = 4\,.$$

There is 1 4-element subset, namely the set itself.

Thus, the number of subsets of a 4-element set is: $1+4+6+4+1=16$.

Note that $2^1 = 2,\ \ 2^2 = 4,\ \ 2^3 = 8,\ \ 2^4 = 16$. From this, we might be led to conclude that the number of subsets of a 5-element set is $2^5 = 32$, and, in general, the number of subsets of an n-element set is 2^n. These conclusions are, indeed, correct.

To count the number of subsets of an n-element set we simply add up the number of 0-element subsets ($1 = C_{n,0}$), the number of 1-element subsets ($n = C_{n,1}$), the number of 2-element subsets ($C_{n,2}$), and so on.

The number of subsets of a set with n elements is given by:

$$C_{n,0} + C_{n,1} + C_{n,2} + C_{n,3} + \ldots + C_{n,n-1} + C_{n,n}\,.$$

So now the question is: Why do these numbers add up to 2^n? To answer this question we go to what appears to be a completely different topic:

The Binomial Theorem and Pascal's Triangle:

First, we consider the powers of $(a+b)$. Recall from algebra that

$$(a+b)^0 = 1,$$

$$(a+b)^1 = a+b,$$

$$(a+b)^2 = a^2 + 2ab + b^2.$$

You can verify that, by multiplying out and arranging the terms so that the powers on a decrease and the powers on b increase,

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

and

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Note that $(a+b)^1$ has 2 terms, $(a+b)^2$ has 3 terms, and so on. Note also that in each expansion, the sum of the powers on a and b in each term add up to the power on $(a+b)$. Assuming that these patterns continue, we will obtain the expansion of $(a+b)^5$.

The expansion of $(a+b)^5$ will have 6 terms and will look like

$$(a+b)^5 = a^5 + (?)a^4b + (?)a^3b^2 + (?)a^2b^3 + (?)ab^4 + b^5.$$

So the question reduces to finding the coefficients of the terms in the middle. To find these we write down the coefficients in the first four expansions:

1
1 2 1
1 3 3 1
1 4 6 4 1

This triangular pattern begins what is known as *Pascal's Triangle*. To find the next row, note that in each row the first and last numbers are 1 and each of the other numbers can be obtained by adding the two numbers that are diagonally placed in the row above:

third row: $3 = 1+2, \quad 3 = 2+1$

fourth row: $4 = 1+3, \quad 6 = 3+3, \quad 4 = 3+1.$

From this pattern, the fifth row will be:

1 5 10 10 5 1

and the expansion of $(a+b)^5$ is:

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 .$$

As you can verify, either by multiplication or by using a calculator, this result is correct.

The first 6 rows of Pascal's triangle are:

$$\begin{array}{c} 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \end{array}$$

and

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 .$$

We could go on to find the expansion of $(a+b)^n$ for any positive integer n.

To see the connection between the binomial theorem, Pascal's triangle, and the number of subsets of a set with n elements, we go back to our counting of the number of subsets of a set with n elements.

For a 0-element set, i.e. $S = \varnothing$, number of subsets − 1.

For a 1-element set: number of subsets with 0 elements − $1 = C_{1,0}$
number of subsets with 1 element − $1 = C_{1,1}$

For a 2-element set: number of subsets with 0 elements − $1 = C_{2,0}$
number of subsets with 1 element − $2 = C_{2,1}$
number of subsets with 2 elements − $1 = C_{2,2}$

For a 3-element set: number of subsets with 0 elements − $1 = C_{3,0}$
number of subsets with 1 element − $3 = C_{3,1}$
number of subsets with 2 elements − $3 = C_{3,2}$
number of subsets with 3 elements − $1 = C_{3,3}$

For a 4-element set: number of subsets with 0 elements − $1 = C_{4,0}$
number of subsets with 1 element − $4 = C_{4,1}$
number of subsets with 2 elements − $6 = C_{4,2}$
number of subsets with 3 elements − $4 = C_{4,3}$
number of subsets with 4 elements − $1 = C_{4,4}$

And so on. The rows of Pascal's triangle!

We now conclude that

$$(a+b)^n = C_{n,0}a^n + C_{n,1}a^{n-1}b + C_{n,2}a^{n-2}b^2 + \ldots + C_{n,n-1}ab^{n-1} + C_{n,n}b^n .$$

This result is the important **binomial theorem**.

Finally, if we let $a = b = 1$ in the binomial theorem, we get

$$(1+1)^n = 2^n = C_{n,0}1^n + C_{n,1}(1^{n-1})1 + C_{n,2}(1^{n-2})1^2 + \ldots + C_{n,n-1}(1)1^{n-1} + C_{n,n}1^n$$

$$= C_{n,0} + C_{n,1} + C_{n,2} + \ldots + C_{n,n-1} + C_{n,n}$$

Therefore,

$$2^n = C_{n,0} + C_{n,1} + C_{n,2} + \ldots + C_{n,n-1} + C_{n,n} ;$$

the number of subsets of a set with n elements is 2^n.

Examples 2.8:

1. Let $A = \{a, b, c, d, e, f\}$.
   a. How many subsets does A have?
   b. How many proper subsets does A have?

2. Use Pascal's triangle to find the expansion of:
   a. $(p+q)^7$ b. $(m-1)^5$ c. $(x+2y)^4$

3. Use the binomial theorem to find:
   a. The fifth term in the expansion of $(a+b)^{11}$.
   b. The fourth term in the expansion of $(x-2)^8$.
   c. The coefficient of x^3y^4 in the expansion of $(x+2y)^7$.

Solutions:

1. a. A has 6 elements. Therefore, A has $2^6 = 64$ subsets.
   b. The improper subsets of A are $\emptyset$ and A itself. Therefore A has 62 proper subsets.
2. a. The first six rows of Pascal's triangle are given above. Therefore, the seventh row is:

   1 7 21 35 35 21 7 1

   and

$$(p+q)^7 = p^7 + 7p^6q + 21p^5q^2 + 35p^4q^3 + 35p^3q^4 + 21p^2q^5 + 7pq^6 + q^7.$$

   b. The fifth row of Pascal's triangle is 1 5 10 10 5 1. Therefore,

$$\begin{aligned}(m-1)^5 &= m^5 + 5m^4(-1) + 10m^3(-1)^2 + 10m^2(-1)^3 + 5m(-1)^4 + (-1)^5 \\ &= m^5 - 5m^4 + 10m^3 - 10m^2 + 5m - 1.\end{aligned}$$

   c. The fourth row of Pascal's triangle is 1 4 6 4 1. Therefore,

$$\begin{aligned}(x+2y)^4 &= x^4 + 4x^3(2y) + 6x^2(2y)^2 + 4x(2y)^3 + (2y)^4 \\ &= x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4.\end{aligned}$$

4. a. The fifth term in the binomial expansion of $(a+b)^{11}$ is:

$$C_{11,4}a^7b^4 = \frac{11!}{4!7!}a^7b^4 = 330a^7b^4.$$

   b. The fourth term in the expansion of $(x-2)^8$ is:

$$C_{8,3}x^5(-2)^3 = \frac{8!}{3!5!}(-8)x^5 = -448x^5.$$

   c. The x^3y^4 term in the expansion of $(x+2y)^7$ is $C_{7,4}x^3(2y)^4$. Therefore, the coefficient of x^3y^4 is:

$$C_{7,4}(2)^4 = \frac{7!}{4!3!}16 = 35 \cdot 16 = 560.$$

Exercises 2.3:

1. Write out the first 8 rows of Pascal's triangle.

2. Use Pascal's triangle to expand:

   a. $(x+2)^5$ b. $(x-2y)^4$

3. Use the binomial theorem to expand:

   a. $(2a-b)^5$ b. $(x-1)^6$

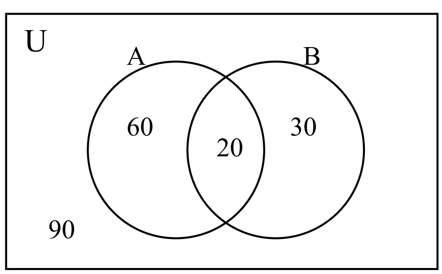
4. a. Find the 9th term in the expansion of $(p+q)^{13}$.
   b. Find the 5th term in the expansion of $(x-1)^{16}$.
   c. Find the 3rd term in the expansion of $(x-3)^5$.

5. Find the coefficient of:
   a. a^5b^6 in the expansion of $(a-b)^{11}$.
   b. x^{12} in the expansion of $(x-1)^{16}$.
   c. s^5 in the expansion of $(s-2)^{10}$.

Answers to the Exercises – Chapter 2

Exercises 2.1.1

1.

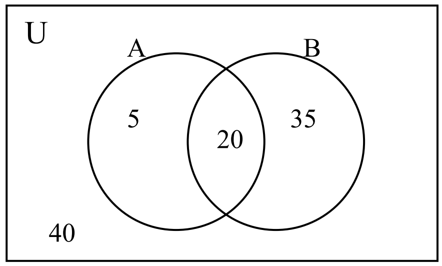


a. $n(A\cup B)=110$
b. $n(A^c\cap B)=120-90=30$

c. $n(A^c \cap B^c) = n[(A \cup B)^c] = 200 - 110 = 90$ (see Chapter 1, exercise 8)

d. $n[(A \cup B)^c] = n(U) - n(A \cup B) = 200 - 110 = 90$ (complement principle)

2.



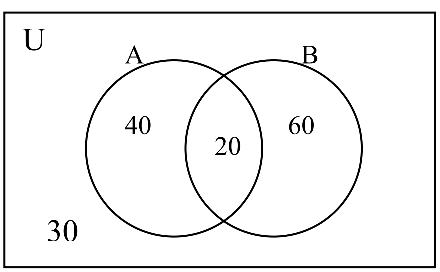
a. $n(A \cap B) = n(A) + n(B) - n(A \cup B) = 25 + 55 - 60 = 20$ (addition principle)

b. $n(A \cap B^c) = 5$

c. $n(A^c \cap B^c) = n[(A \cup B)^c] = n(U) - n(A \cup B) = 40$

d. $n[(A \cup B)^c] = n(U) - n(A \cup B) = 40$ (complement principle)

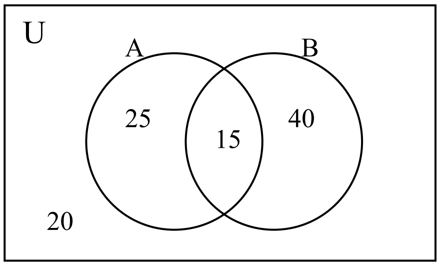
3. First construct a Venn diagram with the information given, which is $n(A) = 60$, $n(B) = 80$, $n(U) = 150$, and $n(A \cap B) = 20$.



$\cap$	***A***	A^c	**Totals**
B	20	60	80
B^c	40	30	70
Totals	60	90	**150**

We are given $n(A \cap B) = 20$, so fill in that cell first. Next find $n(A \cup B) = 120$ using the addition principle. Find $n[(A \cup B)^c] = 30$ using a DeMorgan law. Now we can find the remaining unknowns using the Venn diagram: $n(A \cap B^c) = 60 - 20 = 40$, $n(B \cap A^c) = 80 - 20 = 60$, $n(A^c \cap B^c) = n[(A \cup B)^c] = n(U) - n(A \cup B) = 150 - 120 = 30$

4.

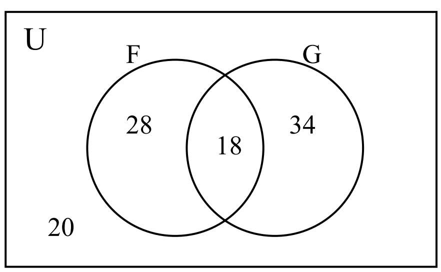


∩	***A***	$\boldsymbol{A^c}$	**Totals**
B	15	40	55
$\boldsymbol{B^c}$	25	20	45
Totals	40	60	**100**

Proceed as in previous problem:

$n(A^c \cap B^c) = n[(A \cup B)^c] = n(U) - n(A \cup B) = 100 - 80 = 20$,
$n(B^c) = n(U) - n(B) = 100 - 55 = 45$ so it follows that $n(B^c \cap A) = 45 - 20 = 25$ and that
$n(A \cap B) = n(A) + n(B) - n(A \cup B) = 40 + 55 - 80 = 15$, and $n(B \cap A^c) = 55 - 15 = 40$
Finally, $n(A^c) = n(U) - n(A) = 100 - 40 = 60$

5.



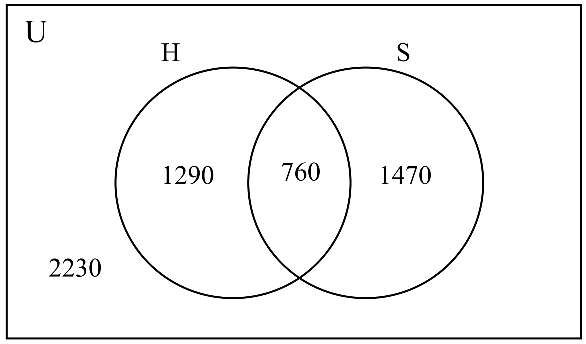
a $n(F \cup G) = n(U) - n(F \cup G)^c = 100 - 20 = 80$ (complement principle)

b. $n(F \cap G) = n(F) + n(G) - n(F \cup G) = 46 + 52 - 80 = 18$ (addition principle)

c. $n(F \cap G^c) =$ 28

d. $n(G \cap F^c) =$ 34

6.



a. $n(H \cup S) = n(U) - n(H \cup S)^c = 5000 - 2230 = 2770$ (complement principle)

b. First we need to find $n(H \cap S) = 2050 + 1480 - 2770 = 760$. (addition principle)

Then we can find $n(H \cap S^c) = 2050 - 760 = 1290$

c. $n(S \cap H^c) = 2230 - 760 = 1470$

Exercises 2.1.2

1. There are 6 x 3 x 5 x 2 = 180 variations of the car.

2. There are 4 x 7 x 3 = 84 different outfits.

3.

a) Repetitions permitted: There are
10 x 10 x 10 x 10 x 10 = 10^5 = 100,000 different plates.

b) Repetitions not permitted: There are
10 x 9 x 8 x 7 x 6 = 30,240 different plates.

c) Repetitions not permitted, first digit cannot be 0: There are
9 x 9 x 8 x 7 x 6 = 27,216 different plates.

4.

a) Repetitions permitted, first letter must be *K* or *W*:
2 x 26 x 26 x 26 = 35,152 different call letters.

b) Repetitions not permitted, first letter must be *K* or *W*: There are
2 x 25 x 24 x 23 = 27,600 different call letters.

c) There are 2 choices for the first letter, there are 5 choices for the vowel and the vowel can be placed in any one of three "slots. After the first letter and the vowel have been chosen and placed, there are 20 choices for the third letter and 19 choices for the fourth letter: There are
2 x 5 x 3 x 20 x 19 = 11,400 different call letters.

5.

a) 18 x 17 x 16 x 15 = 73,440

b) (10 x 9)(8 x 7) = 5040

6.

a) 5 x 5 x 4 x 3 = 300 different numbers.

b) 5 x 6 x 6 x 3 = 540 different numbers.

c) There are 5 choices for the first number and 6 choices for the second number. Since the four-digit number must be divisible by 4, the last two digits must be a number that is divisible by 4. There are 8 choices for this (see the module on Numbers). Therefore there are 5 x 6 x 8 = 240 numbers divisible by 4.

7. There are two choices for the side of the table. There are 5 x 4 x 3 x 2 x 1 = 120 seating arrangements on each side**.** Therefore there are 2 x 120 x 120 = 28,800 different seating arrangements.

8.

a) There are 4 positions for the pair *AB*, and *AB* can sit in either order. Thus, there are 8 choices for arranging *A* and *B.* After *A* and *B* are seated, there are 3 choices for the third seat, 2 choices for the fourth seat, and 1 choice for the last seat. There are 8 x 3 x 2 x 1 = 48 different seating arrangements.

b) We consider all of the positions *B* might occupy:

(i) If *B* sits in the first seat, then there are 4 choices for *C,* 3 choices for the third seat. 2 for the fourth and 1 for the last seat. Thus there are 4 x 3 x 2 x 1 = 24 possibilities.

(ii) If *B* sits in the second seat, then there are 3 choices for *C* and the remaining 3, 2 and 1 for the others; this gives 3x 3 x 2 x 1 = 18 possibilities.

(iii) If *B* sits in the third seat, we get 2x 3 x 2 x 1 = 12 possibilities.

(iv) If *B* sits in the fourth seat, there are 1 x 3 x 2 x 1 = 6 possibilities.

Obviously, *B* cannot sit in the fifth seat so the total number of arrangements is 24 + 18 + 12 + 6 = 60.

c) First consider how many ways the five letters can be arranged regardless of position; there are 5 x 4 x 3 x 2 x 1 = 120 arrangements. Now subtract the cases where *D* and *E* are next to one another. By the result in (a) there are 48 arrangements in which D and E are next to each other. Therefore there are 120 – 48 = 72 arrangements in which *D* and *E* are not next to each other.

Exercises 2.2

1.

a) Order is important: There are $P_{15,3}$ = 15 x 14 x 13 = 27 30 different ways to fill the positions.

b) Order is not important: There are $\boldsymbol{C_{15,5}}$ **=**
$\frac{15!}{(5!)(10!)} = \frac{15\times14\times13\times12\times11}{5\times4\times3\times2\times1} = \frac{360360}{120} =$ 3003 combinations.

2.

a) Order is important: $P_{9,3} = \frac{9!}{6!} = 9\times8\times7 = 504$ possible hands.

b) Order is not important: $\boldsymbol{C_{9,3}} = \frac{9!}{(3!)(6!)} = \frac{9\times8\times7}{3\times2\times1} = \frac{504}{6} =$ 84 possible hands.

3.

a) There are $\boldsymbol{C_{7,2}} = \frac{7!}{(2!)(5!)} = 21$ different committees.

b) Suppose Fred is on the committee. This leaves 3 vacancies to be filled from 7 people (since we cannot count Sandra); $\boldsymbol{C_{7,3}}$ **= 35.** The same holds true if Sandra is on the committee (we cannot count Fred). Thus, there are 35 + 35 = 70 different committees.

c) $C_{9,4}$ = 126 different committees.

4.

a) Order is important: The total number of ways a delegate and alternate can be selected from the 3 departments is (12 x 11)(15 x 14)(18 x 17) = 8,482,320.

b) $C_{12,3}$ x $C_{15,3}$ x $C_{18,3}$ = 81,681,600.

5.

a) $C_{52,5}$ = 2,598,960 possible hands.

b) There are 13 hearts, so there are $C_{13,5}$ = 1287 possible hands.

c) A face card is a Jack, Queen, or King, and there are 4 suits, making a total of 12 face cards, so there are $C_{12,5}$ = 792 possible hands.

d) There are $C_{13,3}$ = 286 ways to choose 3 spades, and $C_{13,2}$ = 78 ways to choose 2 clubs, so there are $286 \times 78 =$ 22,308 possible hands.

6.

a) There are $P_{5,5}$ = 5 x 4 x 3 x 2 x 1 different arrangements.

b) There are 4 x 3 x 2 x 1 = 24 different arrangements.

7. Order is not important. For example, **t**he line segment from *A* to *B* and the line segment from *B* to A produce the same chord; the ordering of the points is not important.

a) $C_{8,2}$ = 28 chords.

b) $C_{8,3}$ = 56 triangles.

c) $C_{8,4}$ = 70 quadrilaterals.

8.

a) $C_{13,5}$ = 1287 ways.

b) $\boldsymbol{C_{8,3}}$ **=** 56, $\boldsymbol{C_{5,2}}$ **=** 10 ways. The number of possible committees is 56 x 10 = 560.

c) $C_{8,3}\, C_{5,2} + C_{8,4}\, C_{5,1} + C_{8,5}\, C_{5,0}$ = 560 + 350 + 56 = 966 ways.

9. $\boldsymbol{C_{n,k}} = \dfrac{n!}{k!(n-k)!} = \dfrac{n!}{(n-k)!(n-(n-k))!} = \boldsymbol{C_{n,n-k}}$

Exercises 2.3

1.

$$\begin{array}{c} 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\ 1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\ 1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \end{array}$$

2.

a) $$\begin{aligned}(x+2)^5 &= x^5 + 5(2)x^4 + 10(2)^2x^3 + 10(2)^3x^2 + 5(2)^4x + (2)^5 \\ &= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32\end{aligned}$$

b) $$\begin{aligned}(x-2y) &= x^4 + 4(-2y)x^3 + 6(-2y)^2x^2 + 4(-2y)^3x + (-2y)^4 \\ &= x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^5\end{aligned}$$

3. Use the formula for $(a+b)^n$.

a)

$$\begin{aligned}(2a-b)^5 &= C_{5,0}(2a)^5 + C_{5,1}(2a)^4(-b) + C_{5,2}(2a)^3(-b)^2 + C_{5,3}(2a)^2(-b)^3 + C_{5,4}(2a)(-b)^4 + C_{5,5}(-b)^5 \\ &= 32a^5 - 80a^4b + 80a^3b^2 - 20a^2b^3 + 10ab^4 - b^5\end{aligned}$$

b)

$$\begin{aligned}(x-1)^6 &= C_{6,0}x^6 + C_{6,1}x^5(-1) + C_{6,2}x^4(-1)^2 + C_{6,3}x^3(-1)^3 + C_{6,4}x^2(-1)^4 + C_{6,5}x(-1)^5 + (-1)^6 \\ &= x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1\end{aligned}$$

4.

a) $C_{13,8}p^5q^8 = 1287p^5q^8$

b) $C_{16,4}(x)^{12}(-1)^4 = 1820x^{12}$

c) $C_{5,2}x^3(-3)^2 = 10x^3(9) = 90x^3$

5.

a) $C_{11,6}(-1)^6 = 462$

b) $C_{16,4}(-1)^4 = 1820$

c) $C_{10,5}(-2)^5 = 252(-32) = -8064$