

C HAPTER 1 SETS

I. DEFINITION OF A SET

We begin our study of probability with the discussion of the basic concept of *set*. We assume that there is a common understanding of what is meant by the notion of “a collection of things”, “a set of objects”, or “a group of articles”. Therefore we will not attempt to define this fundamental idea of a “collection of things”, but we will use it to give a definition of the mathematical usage of the term *set*.

What is a set ?

A *set* is a well-defined collection of distinct objects. The objects in the set are called the *elements* or *members* of the set. Capital letters A, B, C, \dots are usually used to denote sets and lowercase letters a, b, c, \dots to denote the elements of a set.

As we shall see by some examples, it is not enough to say that a set is simply a collection of objects. The distinguishing idea is that the collection must be *well-defined*. That is, for a collection S to be set it must always be possible to determine whether or not some given object x is an element of S .

Examples 1.1:

1. The collection of the vowels in the word “probability”.
2. The collection of real numbers that satisfy the equation $x^2 - 9 = 0$.
3. The collection of two-digit positive integers that are divisible by 5.
4. The collection of great football players in the National Football League.
5. The collection of intelligent members of the United States Congress.

The first three collections are sets because in each case it is certainly possible to determine whether or not some given object x is an element of the set. On the other hand, collections 4 and 5 are not sets; for a given NFL football player x some people might consider him to be great, others might not. Similarly, different people would probably come up with different collections of “intelligent” members of Congress.

Collection 5 also suggests the need for a special set, namely the set that has no elements. The set with no elements is called the *empty set* or the *null set*. The empty

set is denoted by the symbol \emptyset . For a concrete example, let B be the set of real numbers x that satisfy the equation

$$x^2 + 1 = 0, \text{ which is the same as } x^2 = -1.$$

Since the square of any real number is always a nonnegative real number, there are no real numbers which satisfy this equation. Thus B is the empty set (B is \emptyset). Other examples which justify the need for the empty set will be given below.

Specifying a set.

There are various ways to denote a set. When the number of the elements in the set is small, or when the elements have an obvious pattern, you can simply *list* the elements, enclosed within braces. This method is called the *roster method*.

Examples 1.2:

1. The set of vowels in the word “probability” is the set $C = \{a, o, i\}$. Note that the letter i appears twice in the word “probability” but we list it once - a set is a collection of *distinct* elements.
2. The set of real numbers that satisfy the equation $x^2 - 9 = 0$ is the set $S = \{-3, 3\}$.
3. The set of two-digit positive integers that are divisible by 5 is the set $K = \{10, 15, 20, 25, \dots, 95\}$. The three dots (called *ellipsis*) indicate that the list of elements continues in the same pattern established by the first few elements.

In each of these examples it is possible to count the number of the elements in the set; C has three elements, S has two elements, K has eighteen elements (verify this). These are examples of *finite sets*. On the other hand, the set of odd positive integers, which can be denoted by $P = \{1, 3, 5, 7, \dots\}$ has infinitely many elements and is an example of an *infinite set*. The set of integers, the set of rational numbers, and the set of real numbers are other examples of infinite sets.

The number of elements in a given set A is called the *cardinality* of A and is denoted by $n(A)$. By definition, $n(\emptyset) = 0$ (the empty set has no elements). If $A \neq \emptyset$, then $n(A)$ is either a positive integer (A is a finite set), or $n(A) = \infty$ (A is an infinite set). You should be able to determine whether a given set is finite or infinite. Our primary interest in this material is in finite sets, but examples involving infinite sets will come up from time to time. Methods for counting the number of elements in a finite set will be considered in the next section.

There are situations where a set has a large number of elements which can not be listed easily, or at all, or where the elements are determined by some characteristic property. For example, it is not possible to list all the elements in the set Q of rational numbers and, although possible, it would be unwieldy to list the elements of

the set T of states in the United States. So we need an alternative to the roster method to denote a set. *Set builder notation* is used to describe a set A whose elements have a defining characteristic property:

$$A = \{x \mid x \text{ has a specific property } P\},$$

read “the set of all x such that x has property P ”.

Examples 1.3:

1. The set of rational numbers Q can be denoted by $Q = \{x \mid x \text{ is a rational number}\}$ or by $Q = \{x \mid x = p/q \text{ where } p \text{ and } q \text{ are integers and } q \neq 0\}$
2. The set T of states in the United States is denoted by $T = \{x \mid x \text{ is a state in the United States}\}$.

Universal Set.

Connected with a given set there is almost always a surrounding context, a set that contains all the objects of interest. For example, in considering the set C of vowels in the word “probability”, the context is the letters of the English alphabet. We do not have to concern ourselves as to whether the number 3 belongs to C or whether a certain person is in C ; clearly they are not. We only have to worry about whether or not a given letter is in C . Similarly, an encompassing context for the set P of the odd positive integers is the set N of positive integers; we do not have to be concerned as to whether or not the letter a is in P , clearly it is not.

A set U that includes all of the elements under consideration in a particular discussion is called a *universal set*. The purpose of the “universal set” is to focus attention on the objects of interest. The need for it will become apparent in our discussion of set operations below. However, as shown in the following examples, there may be more than one universal set associated with a given set A .

Examples 1.4:

1. $C = \{a, o, i\}$ is the set of vowels in the word “probability”. Since the overall context is the letters of the alphabet, the set $U = \{a, b, c, \dots, x, y, z\}$ is a universal set for C . However, if we are concerned only with vowels, then $U = \{a, e, i, o, u\}$ would also serve as a universal set for C .
2. Suppose we roll a standard, six-sided die and ask whether the result is a prime number. Then we are asking whether the outcome is an element of the set

$D = \{2, 3, 5\}$. An obvious universal set for D is the set of all possible outcomes that can result from rolling the die, namely $U = \{1, 2, 3, 4, 5, 6\}$. On the other hand, the elements of D are positive integers, so we could use the set $N = \{1, 2, 3, \dots\}$ as a universal set for D ; the elements of D are also real numbers so we could use the set $\mathcal{R} = \{x \mid x \text{ is a real number}\}$ as a universal set for D . The set U can be thought of as the “minimal” universal set for D .

II. SET OPERATIONS

Let A be a set and let x be some object. We know that x either is, or is not, an element of A . The statement “ x is an element of A ” (also stated “ x is a member of A ” or “ x is in A ”) is denoted by $x \in A$; the statement “ x is not an element of A ” is denoted by $x \notin A$.

Examples 2.1:

1. Let V be the set of vowels: $V = \{a, e, i, o, u\}$. Then $e \in V$ and $b \notin V$.
2. Let P be the set of prime numbers: $P = \{2, 3, 5, 7, \dots\}$. (Note: P is an infinite set.) Then $17 \in P$ and $39 \notin P$.

Equality.

Given two sets A and B . Our first concern is the question of *equality*; are A and B the same set, or are they different sets. Two sets A and B are *equal*, denoted in the familiar way $A = B$, if and only if they have exactly the same elements. If A and B do not have exactly the same elements then A is not equal to B , denoted by $A \neq B$.

Examples 2.2:

Let $A = \{a, b, c, d\}$, $B = \{a, b, c, d, e, f\}$, $C = \{b, d, a, c\}$. Then

1. $A = C$ (A and C have exactly the same elements, the *order* in which the elements of a set are listed is immaterial)
2. $A \neq B$ (A and B do not have exactly the same elements; $e, f \in B$, but $e, f \notin A$).

Subset.

Equality is not the only relationship between two sets. The sets A and B in the example above are not equal because they do not have exactly the same elements. However, note that each element of A is also an element of B ; the set A is “inside” B . If A and B are sets and each element of A is also an element of B , then A is said to be a *subset* of B , symbolized by $A \subseteq B$.

Example 2.3:

Let $A = \{1,2,3,4,5,6\}$, $B = \{2,3,5\}$, $C = \{1,2,3,5,7\}$. Determine which of the following statements are true.

- (a) $5 \in A$ (b) $5 \subseteq A$ (c) $\{5\} \subseteq A$ (d) $\{5\} \in A$
 (e) $B \subseteq A$ (f) $C \subseteq A$ (g) $A \subseteq A$

Solution:

- (a) True; 5 is an element of A .
 (b) False; 5 cannot be a subset of A since 5 is not a set.
 (c) True; $\{5\}$ is a set and every element of $\{5\}$ is an element of A .
 (d) False; if $\{5\} \in A$, then the set $\{5\}$ must be one of the elements of A ; the element 5 is in A , the set $\{5\}$ is not in A .
 (e) True; every element in B is also in A .
 (f) False; the element 7 is in C but it is not in A .
 (g) True; every element in A is an element in A .

As illustrated in the example above, a set is a subset of itself; for any set A , $A \subseteq A$. What about the empty set, \emptyset ? The convention is that the empty set is a subset of every set. One way to justify this is by asking: Are there any elements in \emptyset which are not in a given set A ? Since there are no elements in \emptyset , the answer is “no”. Thus, $\emptyset \subseteq A$. A set B is said to be a *proper subset* of A if B is a subset of A and $B \neq A$, $B \neq \emptyset$.

Example 2.4: Let $A = \{a, b, c\}$. The proper subsets of A are:

$\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$.

Union.

Given two sets A and B . The *union* of A and B , denoted by $A \cup B$, is the set of elements that are either in A or in B . In symbols,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

The use of the word “or” here is inclusive; that is “ x is either in A or in B ” means either x is in A , or x is in B , or x is in both A and B .

Intersection.

Given two sets A and B . The *intersection* of A and B , denoted by $A \cap B$, is the set of elements that are in both A and B . In symbols,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Complement.

The last set operation we will define is the *complement* of a set. Here we will have to use the notion of a universal set introduced above. Let A be a set and U be a universal set for A . The *complement* of A , denoted by A^c , is the set of elements in U that are not in A . That is,

$$A^c = \{x \in U \mid x \notin A\}$$

Examples 2.5: Let U (universal set) = $\{1, 2, 3, 4, \dots, 20\}$. Let A, B and C be subsets of U with $A = \{1, 3, 5, \dots, 19\}$, $B = \{x \mid x \text{ is even and } 4 < x < 12\}$, and $C = \{x \mid x \text{ is an odd prime number}\}$. Find, in roster form, each of the following sets:

- (a) $A \cup B$ (b) $A \cap B$ (c) A^c (d) $C \cap B^c$ (e) $A^c \cap C$

Solution:

$B = \{6, 8, 10\}$ and $C = \{3, 5, 7, 11, 13, 17, 19\}$.

- (a) $A \cup B = \{1, 3, 5, 6, 7, 8, 9, 10, 11, 13, 15, 17, 19\}$.
 (b) $C \subseteq A$; $A \cap C = \{3, 5, 7, 11, 13, 17, 19\} = C$.
 (c) $C^c = \{1, 2, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20\}$.
 (d) $B^c = \{1, 2, 3, 4, 12, 13, \dots, 20\}$; $C \cup B^c = \{3, 13, 17, 19\}$.
 (e) $A^c = \{2, 4, 6, \dots, 20\}$; $A^c \cap C = \emptyset$.

As illustrated in the example above, the intersection of two sets could be empty (another reason why we need to have the empty set). The sets A and B are said to be *disjoint* if there are no elements that are in both A and B ; that is, if $A \cap B = \emptyset$.

Venn Diagrams.

A picture is worth a thousand words. A *Venn diagram* provides a way to represent sets and set operations pictorially. Let A be a set with universal set U . The Venn diagram representing U and A is

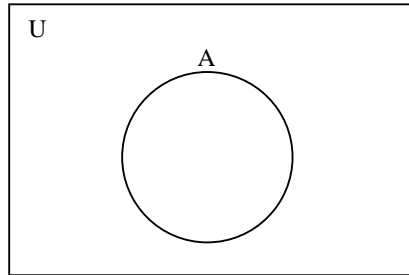


Figure 1

The region inside the rectangle corresponds to the elements in the universal set U ; the region inside the circle are the elements of the set A .

Now let A and B be sets contained within the same universal set U . The following figures represent possible relationships between A and B :

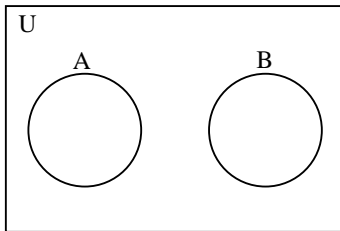


Figure 2

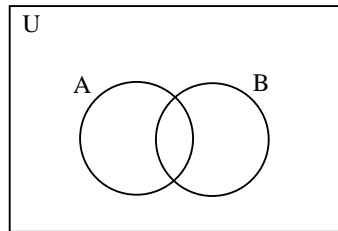


Figure 3

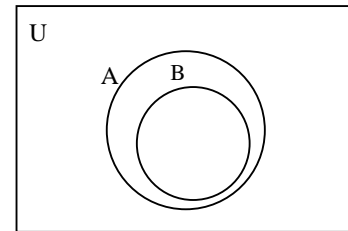


Figure 4

Figure 2 displays disjoint sets A and B ; the sets A and B in Figure 3 are not disjoint; and Figure 4 illustrates the subset relationship. Here B is a subset of A , $B \subseteq A$.

Venn diagrams provide a convenient way to illustrate the set operations: complement, union and intersection. Figure 5 shows A^c , the set of elements that are not in A .

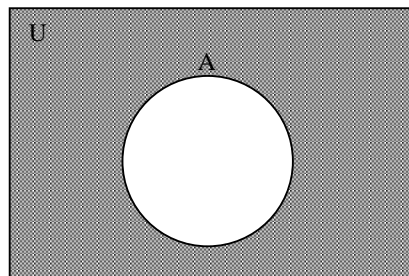


Figure 5

$A \cup B$ is the set of elements that are either in A or in B or in both A and B ; $A \cap B$ is the set of elements that are in both A and B . These cases are shown in Figures 6 and 7.

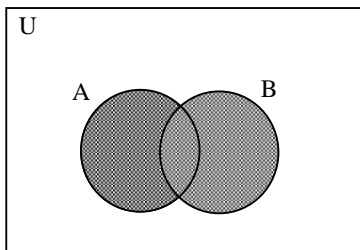
 $A \cup B$

Figure 6

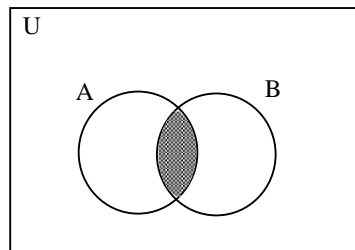
 $A \cap B$

Figure 7

Examples 2.6:

For each of the following, use a copy of the Venn diagram in Figure 3 to shade the indicated region.

- (a) $A \cup B^c$ (b) $A^c \cap B$ (c) $(A \cup B)^c$ (d) $(A \cap B)^c$

A Venn diagram for two sets describes the four disjoint regions labeled R_1, R_2, R_3 and R_4 in Figure 8.

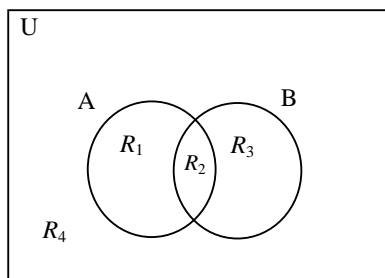


Figure 8

R_1 consists of the elements that are in A but not in B . Thus, $R_1 = A \cap B^c$.

R_2 consists of the elements that are in both A and B . Thus, $R_2 = A \cap B$.

R_3 consists of the elements that are in B but not in A . Thus, $R_3 = A^c \cap B$.

R_4 consists of the elements that are not in A and not in B . Thus, $R_4 = A^c \cap B^c$.

Venn diagrams can also be used to illustrate the various unions, intersections and complements of three sets. Let A , B and C be sets contained in some universal set U . A Venn diagram for the three sets is shown in Figure 9.

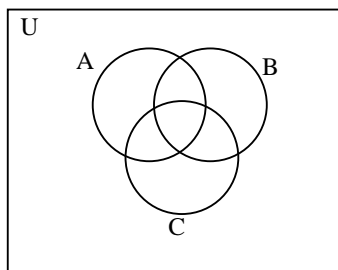


Figure 9

Notice that a Venn diagram representing three sets describes eight distinct, disjoint regions as shown in Figure 10.

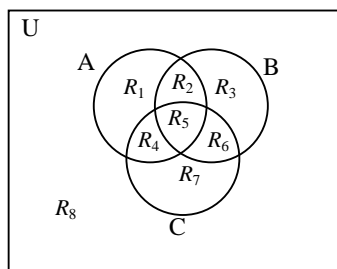


Figure 10

R_1 consists of the elements that are in A but not in B and not in C . In set notation, $R_1 = A \cap B^c \cap C^c$.

R_2 consists of the elements that are in A and B but not in C . Thus, $R_2 = A \cap B \cap C^c$. And so on.

Exercises:

1. In each of the following, state whether or not the given collection is a set. If it is, specify the set using the roster method.
 - a. Distinct letters in the word “mathematics”.
 - b. Handsome men in Texas.
 - c. Counting numbers strictly between 10 and 20.
 - d. Good students enrolled at Texas A&M University.
 - e. The odd counting numbers.
 - f. The positive multiples of 3.

2. Write the following sets using set builder notation.
 - a. $\{1, 3, 5, 7, \dots\}$.
 - b. $\{0, \pm 1, \pm 2, \dots\}$
 - c. $\{\text{Sunday, Monday}, \dots, \text{Saturday}\}$
 - d. $\{a, e, i, o, u\}$
 - e. $\{\text{Washington, Adams, Jefferson}, \dots, \text{Clinton, Bush}\}$
 - f. $\{101, 102, 103, 104, \dots\}$

In Problems 3 and 4, let U (universal set) = $\{0, 1, 2, \dots, 10\}$, $A = \{0, 2, 4, 6, 8\}$, $B = \{1, 3, 5, 7, 10\}$ and $C = \{2, 3, 5, 6, 9, 10\}$.

3. Indicate T – “true” or F – “false” for each of the following:
 - a. $2 \in A$
 - b. $2 \in B$
 - c. $3 \notin B$
 - d. $\{6\} \subseteq A$
 - e. $\{1, 2, 3, 4\} \not\subseteq C$
 - f. $\emptyset \in B$
 - g. $A = \{8, 0, 6, 2, 4\}$
 - h. $B \neq \{1, 3, 5, 7, 9\}$
 - i. $\{2, 4, 6\} \subseteq A$
4. Use the roster method to specify each of the following sets:
 - a. $A \cup B$
 - b. $A \cap C$
 - c. B^c
 - d. $B \cap C^c$
 - e. $(A \cap B)^c$
 - f. $A^c \cup B^c$
5. Let A and B be sets in a universal set U . Use copies of the Venn diagram in Figure 3 to shade each of the following regions:
 - a. A^c
 - b. $A \cap B^c$
 - c. $A^c \cap B$
 - d. $A \cap A^c$
 - e. $(A \cup B)^c$
 - f. $A^c \cup B^c$
6. Let A , B and C be sets in a universal set U . Use copies of the Venn diagram in Figure 9 to shade each of the following regions:
 - a. $A \cup B \cup C$
 - b. $A \cap (B \cup C)$
 - c. $(A \cap B) \cup C$
 - d. $A^c \cup (B \cap C)$
 - e. $(A \cup C) \cap B^c$
 - f. $B^c \cap (A \cap C)$
7. Let A and B be sets in a universal set U . What can you conclude if:
 - a. $A \cup B = A$?
 - b. $A \cap B = A$?
8. Use copies of the Venn diagram in Figure 3 to show that:
 - a. $(A \cup B)^c = A^c \cap B^c$.
 - b. $(A \cap B)^c = A^c \cup B^c$.

These results are known as *DeMorgan's Laws* after the 19th century mathematician Augustus DeMorgan.

Answers to the Exercises – Chapter 1

1.
 - a. $\{m, a, t, h, e, i, c, s\}$
 - b. Not a set.
 - c. $\{11, 12, 13, \dots, 19\}$
 - d. Not a set.
 - e. $\{1, 3, 5, 7, \dots\}$
 - f. $\{m, i, s, p\}$
 - g. $\{3, 6, 9, 12, \dots\}$

2.
 - a. $S = \{x \mid x \text{ is an odd positive integer}\} = \{x \mid x \text{ is an odd counting number}\}$
 - b. $S = P\{x \mid x \text{ is an integer}\}$
 - c. $S = \{x \mid x \text{ is a day of the week}\}$
 - d. $S = \{x \mid x \text{ is a vowel}\}$
 - e. $S = \{x \mid x \text{ is a President of the United States}\}$
 - f. $S = \{x \mid x \text{ is a positive integer and } x > 100\}$

3.

a. T	b. F	c. F
d. T	e. T	f. F
g. T	h. T	i. T

4.

a. $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 10\}$	b. $(2, 6)$
c. $\{0, 2, 4, 6, 8, 9\}$	d. $\{1, 7\}$
e. U	f. U

7.
 - a. If $A \cup B = A$, then $B \subseteq A$.
 - b. If $A \cap B = A$, then $A \subseteq B$.