Quadrilaterals

Definition

A quadrilateral is a four-sided closed figure in a plane that meets the following conditions:

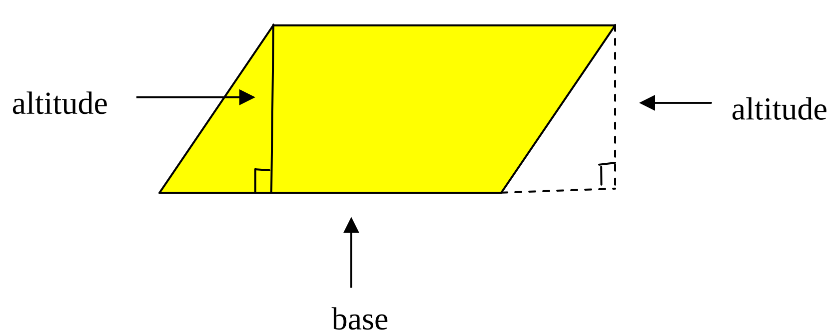
- Each side has its endpoints in common with an endpoint of two adjacent sides.
- Consecutive sides do not lie in a straight line.

In this unit, we will investigate the properties of a special group of quadrilaterals called *parallelograms*. A definition of a parallelogram is given below.

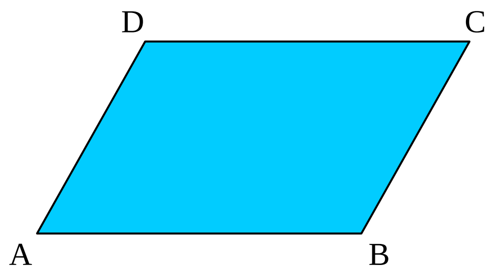
Parallelogram

A quadrilateral with two pairs of opposite sides parallel is a parallelogram.

Any side of a parallelogram can be designated as the base as shown in the figure below. The altitude is the perpendicular segment joining any point on the opposite side with the base or base extended.



A parallelogram is named according to the capital letters assigned to its vertices. Select any vertex to start with and move in a clockwise or counterclockwise direction to select the other three letters to complete its name. A parallelogram with the letters A, B, C, and D assigned to its vertices is shown below.



The symbol ▱ will be used to denote a parallelogram. Using this symbol, the parallelogram above can be named as follows.

▱ ABCD	▱ BADC	▱ DCBA	▱ CBAD
▱ ADCB	▱ BCDA	▱ DABC	▱ CDAB

Properties of a Parallelogram

We will investigate the properties of a parallelogram using the TI-83+ graphing calculator with Cabri Junior™ in the following explorations.

Begin by constructing a parallelogram ABCD on the viewing screen using the procedure outlined below. This is only one of several methods that can be used to construct a parallelogram.

a) Draw and label segment AB.
b) Draw segment AD for an adjacent side to side AB (figure 1).
c) Press F3 and select **Parallel**. To draw a line parallel to $\overline{AB}$ through point D outside the line, move the cursor to point D and press ENTER. Then move the cursor to $\overline{AB}$ and press ENTER. The line parallel to $\overline{AB}$ is shown in figure 1.
d) Press F3 again and select **Parallel.** Select point B and $\overline{AD}$ followed by ENTER. This is the same process as outlined in "c" above. A line will be drawn parallel to $\overline{AD}$ through point B outside the line. Label the last vertex in the parallelogram with the letter C. Refer to figure 2.

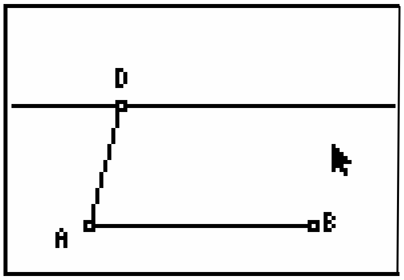


figure 1

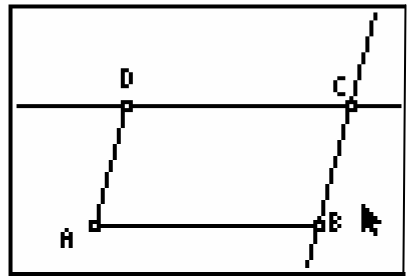


figure 2

Explorations

1. Make a conjecture about the lengths of the opposite sides of a parallelogram.

   Conjecture:____

   Test this conjecture using the parallelogram in figure 2.

   a) Press F5 and select **Measure- D. & length.** Use the cursor to select the endpoints of a segment followed by ENTER after each selection. The measure will appear near the segment with a "hand" attached. Use the arrows on the calculator keypad to move the measure to a desired position on the screen.
   b) How do the measures of the opposite sides of the parallelogram compare with the conjecture made?

   The measures of the sides for parallelogram ABCD are shown in figure 3 below.

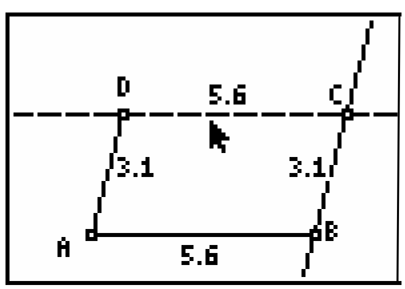


figure 3

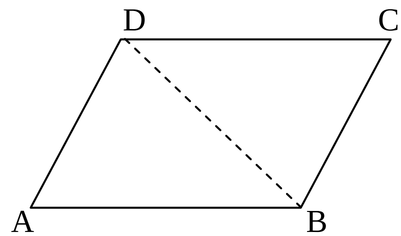
In figure 3, opposite sides of ▱ABCD have the same measure. This leads to the following theorem about parallelograms.

Theorem

Opposite sides of a parallelogram have the same measure and are congruent.

This theorem can be proven using the triangle congruence postulates and theorems from the Congruent Triangles unit.

Given: ▱ABCD
Prove: $\overline{AB} \cong \overline{DC}$
$\overline{AD} \cong \overline{BC}$



It is sometimes necessary to draw an <u>auxiliary line</u> such as the line containing $\overline{DB}$ in the figure above. This <u>auxiliary line</u> can be drawn since two points determine a line. Now it is possible to complete this proof.

Proof:

Statements	Reasons
1. ▱ABCD	1. Given
2. Draw auxiliary line DB	2. Two points determine a line.
3. $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$	3. Definition of a parallelogram
4. $\angle ADB \cong \angle CBD$ $\angle ABD \cong \angle BDC$	4. If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.
5. $\overline{DB} \cong \overline{DB}$	5. Reflexive property
6. $\Delta ABD \cong \Delta CDB$	6. ASA congruence postulate
7. $\overline{AD} \cong \overline{BC}$ $\overline{AB} \cong \overline{DC}$	7. CPCTC

2. Make a conjecture about the measures of the opposite angles of a parallelogram.

Conjecture:____

Test this conjecture using the parallelogram in figure 3. First, Press F5 and select **Hide/show- objects**. Hide the measures of the sides.

a) Press F5 and select **Measure-angle**.
b) Select a point on one of the sides of the angle, the vertex, and a point on the other side of the angle to be measured followed by ENTER after each selection.
c) Move the "hand" to position the angle measure on the figure and press ENTER.
d) Compare the measures of opposite angle pairs. How do these results compare to the conjecture made? Refer to the example in figure 4.

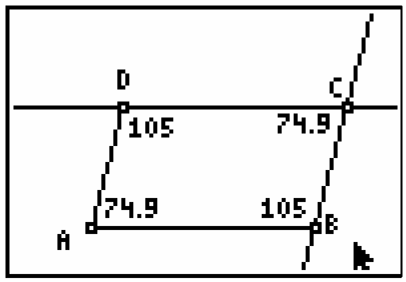


figure 4

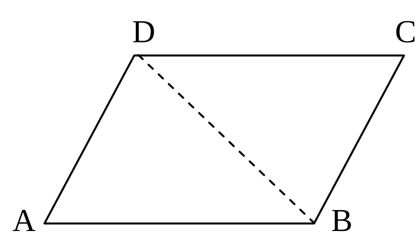
The measures of the opposite angles in ▱ABCD are congruent as shown in figure 4. These results can be stated in the following theorem.

Theorem

The opposite angles of a parallelogram have the same measure and are congruent.

A paragraph-style proof of this theorem is given below.

Given: ▱ABCD
Prove: $\angle A \cong \angle C$
$\angle B \cong \angle D$

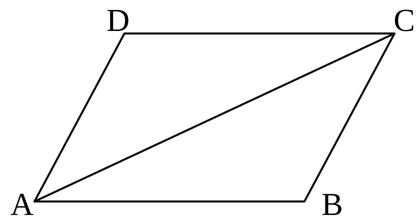


Proof:

Draw auxiliary line DB. We have $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$ (Opposite sides of a parallelogram are congruent.) $\overline{DB} \cong \overline{DB}$ by the reflexive property. It follows that $\Delta ABD \cong \Delta CDB$ by the SSS congruence postulate. By CPCTC, $\angle A \cong \angle C$. To prove $\angle B \cong \angle D$, draw auxiliary line AC and prove $\Delta ADC \cong \Delta CBA$ by the SSS congruence postulate.

Examples

1. Given: ▱ABCD with diagonal $\overline{AC}$
   Complete each statement and give a justification using a definition, postulate or theorem.



   a) $\overline{DC} \parallel$ ______ ____________________________________

   b) $\angle$ADC $\cong$ ______ ____________________________________

   c) $\angle$DCA $\cong$ ______ ____________________________________

   d) $\overline{AD} \cong$ ______ ____________________________________

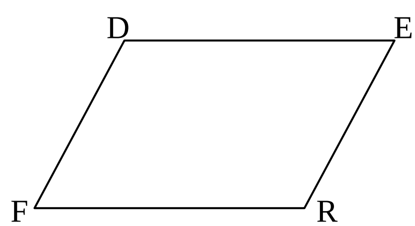
   e) ΔADC $\cong$ ______ ____________________________________

   f) $\angle$BCA $\cong$ ______ ____________________________________

   g) $\overline{DC} \cong$ ______ ____________________________________

   h) $\angle$DCB $\cong$ ______ ____________________________________

2. Given: ▱FRED
   Find the value of x and the indicated measure(s) for each of the following.



   a) DE = (3x - 2) cm
   FR = (2x + 7) cm
   RE = $(\frac{5}{3}x + 4)$ cm

   x = __________
   DE = __________ DF = __________

b) $m\angle F = (8x - 15)$

$m\angle D = (\frac{9}{2}x + 45)$

$m\angle E = (5x + 21)$

$x =$ ________

$m\angle D =$ ________

$m\angle R =$ ________

3. Given: Quadrilateral PQRS with three vertices P(-2,3), Q(4,3), and S(0,8)
   a) Graph these vertices in the coordinate plane.
   b) Determine the coordinates of vertex R so that the quadrilateral is a parallelogram.

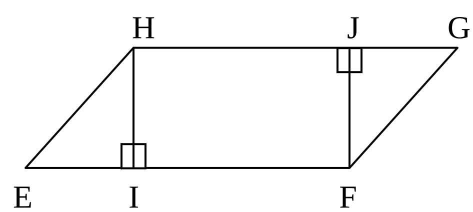
4. Given: Quadrilateral RSTV with vertices R(5,1), S(8,3), T(4,7), and V(1,5)
   a) Graph quadrilateral RSTV in the coordinate plane.
   b) Show quadrilateral RSTV is a parallelogram.

5. Given: ▱EFGH

$\overline{HI} \perp \overline{EF}$

$\overline{JF} \perp \overline{HG}$

Prove: $\overline{HI} \cong \overline{JF}$



Write a paragraph, two-column, or flow-chart proof.

<u>Solutions:</u>

1.	a) $\overline{DC} \parallel \underline{\overline{AB}}$	<u>Definition of a parallelogram.</u>
	b) $\angle ADC \cong$ <u>**∠ABC**</u>	<u>The opposite angles of a ▱ are ≅.</u>
	c) $\angle DCA \cong$ <u>**∠BAC**</u>	<u>If 2 parallel lines are cut by a transversal, then the alternate interior angles are congruent</u>
	d) $\overline{AD} \cong \underline{\overline{BC}}$	<u>The opposite sides of a ▱ are ≅.</u>
	e) $\Delta ADC \cong$ <u>**ΔCBA**</u>	<u>SSS, SAS, or ASA congruence postulates; or AAS congruence theorem</u>

f) $\angle BCA \cong$ **$\angle$DAC** — <u>If 2 parallel lines are cut by a transversal, then the alternate interior angles are congruent.</u>

g) $\overline{DC} \cong \overline{AB}$ — <u>The opposite sides of a ▱ are ≅.</u>

h) $\angle DCB \cong$ **$\angle$BAD** — <u>The opposite angles of a ▱ are ≅.</u>

2. a)

DE = FR	(The opposite sides of a ▱ are ≅.)
3x - 2 = 2x + 7	(Substitution)
x = 9	(Addition/subtraction properties of equality)

DE = 3•9 – 2 = 27 – 2 = 25 cm

DF = RE = $\frac{5}{3}$•9 + 4 = 15 + 4 = 19 cm

b)

m∠F = m∠E	(The opposite angles of a ▱ are ≅.)
8x – 15 = 5x + 21	(Substitution)
3x = 36	(Addition/subtraction properties of equality)
x = 12	(Division/multiplication property of equality)

m∠D = $\frac{9}{2}$•12 + 45 = 54 + 45 = 99

m∠R = m∠D = 99

3. b) R(6,8)

4. b) By the definition of a parallelogram, opposite sides are parallel. Parallel lines have equal slopes. We can show quadrilateral RSTV has opposite sides parallel by comparing the slopes of the lines containing the sides. We will use the slope formula given below to find the slope of each side in the coordinate plane.

Slope Formula

The slope (m) of the line containing the points (x_1, y_1) and (x_2, y_2) is given by the formula:

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

Slope of $\overline{RS} = \dfrac{(3-1)}{(8-5)} = \dfrac{2}{3}$

Slope of $\overline{VT} = \dfrac{(7-5)}{(4-1)} = \dfrac{2}{3}$

Since the slope of $\overline{RS}$ equals the slope of $\overline{VT}$, $\overline{RS} \parallel \overline{VT}$.

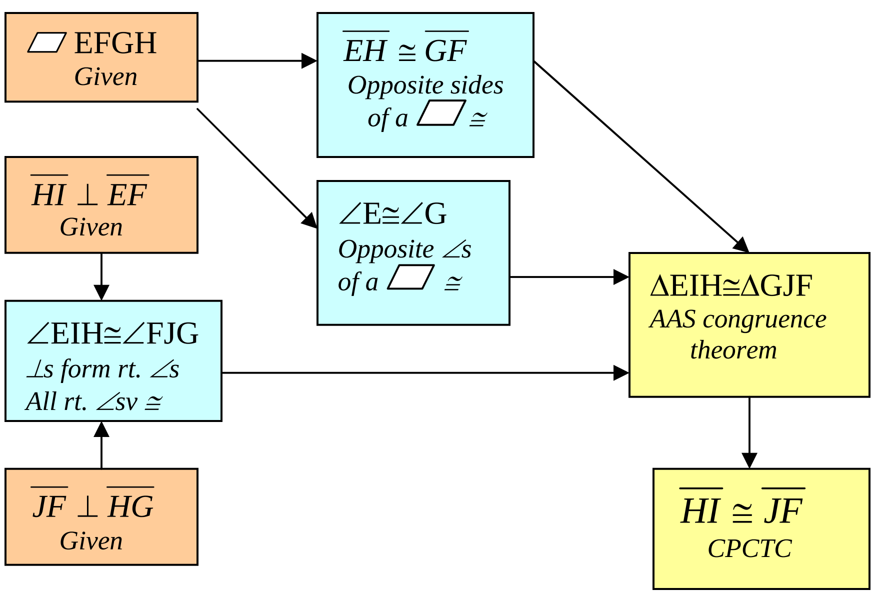
Slope of $\overline{VR} = \dfrac{(5-1)}{(1-5)} = -1$

Slope of $\overline{TS} = \dfrac{(7-3)}{(4-8)} = -1$

Since the slope of $\overline{VR}$ equals the slope of $\overline{TS}$, $\overline{VR} \parallel \overline{TS}$.

We now have the opposite sides of quadrilateral RSTV parallel. This proves that quadrilateral RSTV is a parallelogram by the definition of a parallelogram.

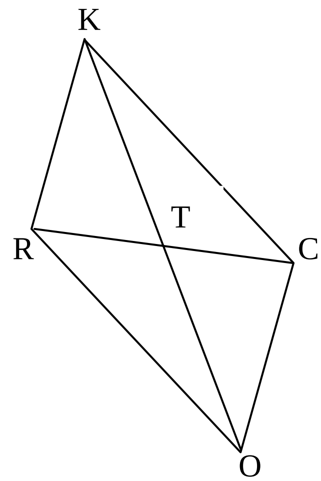
5. A flow-chart proof is given below.



Exercises

1. Given: ▱ROCK with diagonals $\overline{RC}$ and $\overline{KD}$

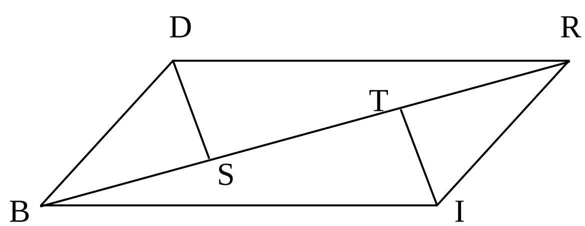
   Justify each statement below with a definition, postulate, or theorem.



   a) $\overline{KC} \cong \overline{OR}$
   b) $\angle RKC \cong \angle ROC$
   c) $\angle CKO \cong \angle ROK$
   d) $\overline{RK} \parallel \overline{OC}$
   e) $\overline{RK} \cong \overline{OC}$

2. Given: ▱BIRD
   $\overline{DS} \perp \overline{BR}$
   $\overline{IT} \perp \overline{BR}$

   Find: x and the indicated measures



   a) DS = (7.5x – 1) cm
   IT = (3.2x + 20.5) cm
   BD = (8.4x + 6.7) cm
   x = ___________
   BD = ___________

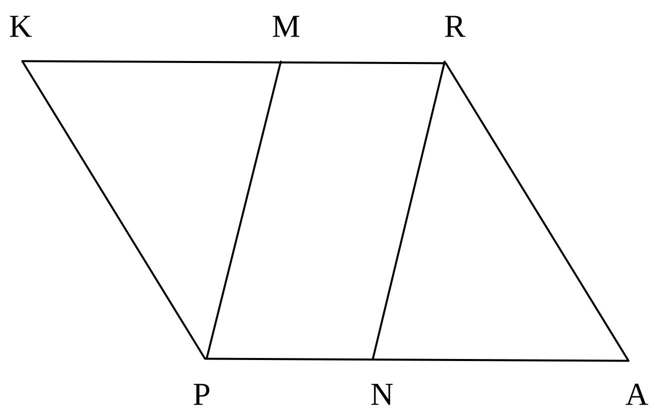
   b) $m\angle DBI = (2x - 5)$
   $m\angle DRI = (\frac{6}{7}x + 19)$
   x = _________
   $m\angle DRI$ = _________

3. Given: Quadrilateral QUAD with vertices Q(-6,2), U(1,-2), and D(-6,7)

   a) Graph these vertices in the coordinate plane.
   b) Determine the coordinates of vertex A so that quadrilateral QUAD is a parallelogram.

4. Given: ▱QUAD in exercise 3
   Find: Perimeter of ▱QUAD

5. Given: ▱PARK
   $\angle KMP \cong \angle RNA$
   Prove: $\overline{MP} \cong \overline{NR}$



We have established the following properties of a parallelogram and will use these properties to investigate other properties.

Some Properties of a Parallelogram

- Opposite sides are parallel. (Definition)
- Opposite sides are congruent. (Theorem)
- Opposite angles are congruent. (Theorem)

Exploration

Use a TI-83+ graphing calculator with Cabri Junior™ to investigate a relationship between the diagonals of a parallelogram.

1. Make a conjecture about the diagonals of a parallelogram.
   Conjecture: ____

   Test this conjecture with the following investigation.

   a) Begin by constructing ▱ABCD or use the parallelogram from a previous investigation. Refer to figure 1 below.

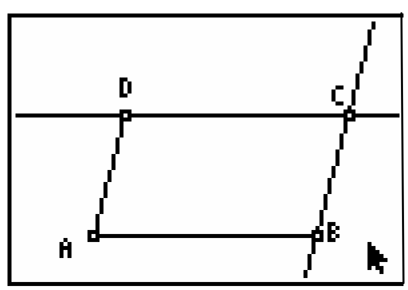


figure 1

   b) Draw diagonals $\overline{DB}$ and $\overline{AC}$. Locate the point of intersection of these diagonals with point M as shown in figure 2 below.

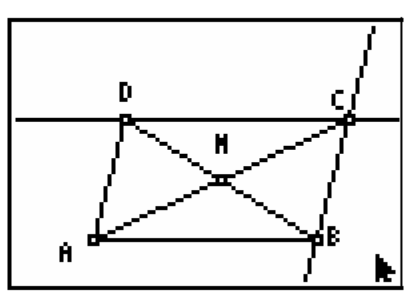


figure 2

c) Measure the lengths of the segments AM, MC, DM, and MB of diagonals $\overline{AC}$ and $\overline{DB}$ as shown in figure 3 below. Compare these results to the conjecture made previously. The figure below is representative of the possible results. Measurements will vary according to the parallelogram drawn.

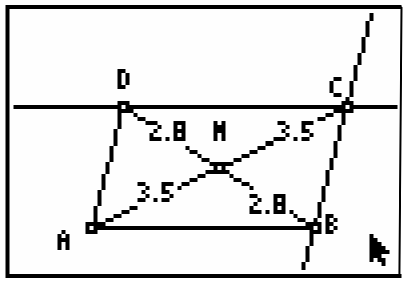


figure 3

These results can be summarized in the following theorem about the diagonals of a parallelogram.

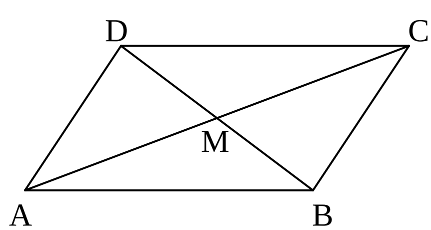
Theorem

The diagonals of a parallelogram bisect each other.

.

This can be proven using other properties of a parallelogram as follows.

Given: ▱ABCD with diagonals $\overline{AC}$ and $\overline{DB}$
Prove: $\overline{AC}$ and $\overline{DB}$ bisect each other



Proof:

By the definition of a parallelogram, we have $\overline{DC} \parallel \overline{AB}$. It follows that $\angle CDB \cong \angle ABD$ since alternate interior angles are congruent when two parallel lines are cut by a transversal. $\angle DMC \cong \angle BMA$ because vertical angles are congruent. $\overline{DC} \cong \overline{DC}$ by the reflexive property. Now we have $\Delta AMB \cong \Delta CMD$ by the AAS theorem. By CPCTC, $\overline{DM} \cong \overline{MB}$ and $\overline{AM} \cong \overline{MC}$. Therefore, $\overline{AC}$ and $\overline{DB}$ bisect each other.

We will investigate another property of parallelograms using the TI-83+ graphing calculator with Cabri Junior™. This property involves the relationship between any two consecutive angles of a parallelogram.

1. Make a conjecture about this relationship.
   Conjecture:____

   Test this conjecture by measuring any two consecutive angles of a parallelogram.

   a) Begin with a parallelogram drawn on the viewing screen of a TI-83+ graphing calculator with Cabri Junior™ or use ▱ABCD from the previous investigation.

   b) Measure two consecutive angles as shown in figure 4 below. These measurements are for the parallelogram drawn.

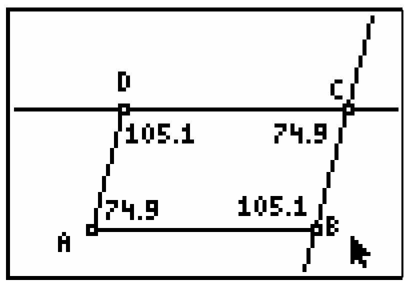


figure 4

These results can be summarized in the following theorem for parallelograms.

Theorem

Any two consecutive angles of a parallelogram are supplementary.

This proof follows from the definition of a parallelogram.

Proof:
In figure 4 above, we have $\angle A$ and $\angle B$ supplementary because $\overline{AD} \parallel \overline{BC}$ by the definition of parallelogram. Using a similar argument, other pairs of consecutive angles in the parallelogram can be shown supplementary.

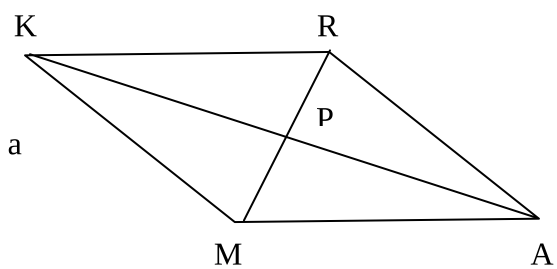
We can summarize the properties of a parallelogram as follows.

Properties of a Parallelogram

1. Opposite sides are parallel.
2. Opposite sides are congruent.
3. Opposite angles are congruent.
4. Diagonals bisect each other.
5. Any two consecutive angles are supplementary.

Examples

1. Given: ▱MARK with diagonals $\overline{AK}$ and $\overline{MR}$



Complete each statement and write a justification for each answer.

a) $\overline{KP} \cong$ ________ ______________________________

b) $\overline{KR} \parallel$ ________ ______________________________

c) $\overline{AP} \cong$ ________ ______________________________

d) $\overline{KM} \cong$ ________ ______________________________

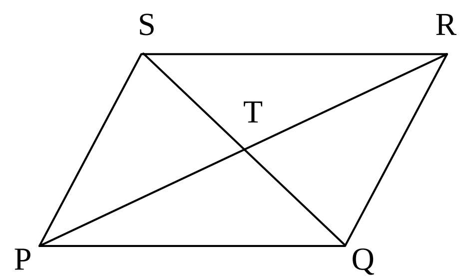
e) $\angle K + \angle M =$ ________ ° ______________________________

f) $\overline{KM} \parallel$ ________ ______________________________

g) $\angle KMA \cong$ ________ ______________________________

2. Given: ▱PQRS with diagonals $\overline{PR}$ and $\overline{SQ}$

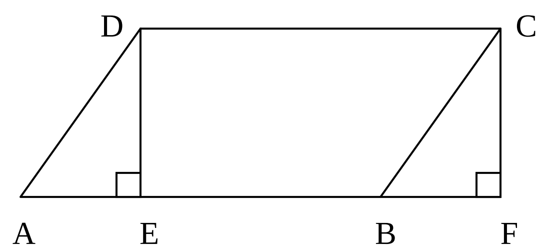
Find x, y, and the indicated measures.



PT = (2x + y) cm
TR = 7 cm
ST = 5 cm
TQ = (x + y) cm
PQ = (x + y + 1) cm
x = __________
y = __________
PQ= __________

3. Given: ▱ABCD
$\overline{DE} \perp \overline{AB}$
$\overline{CF} \perp \overline{AF}$

Prove: $\overline{DE} \cong \overline{CF}$



Complete the following proof with the appropriate statements and reasons.

Proof:

Statements	Reasons
1. ▱ABCD, $\overline{DE} \perp \overline{AB}$, $\overline{CF} \perp \overline{AF}$	1. ____________________
2. $\overline{AD} \cong \overline{BC}$	2. ____________________
3. ____________________	3. Opposite angles of a ▱ ≅
4. ____________________	4. ⊥s form right ∠s
5. ∠DEA ≅ ∠CFB	5. ____________________
6. ____________________	6. AAS congruence theorem
7. $\overline{DE} \cong \overline{CF}$	7. ____________________

Solutions:

1.	a) $\overline{KP} \cong \underline{\overline{PA}}$	The diagonals of a ▱ bisect each other.
	b) $\overline{KR} \parallel \underline{\overline{AM}}$	The opposite sides of a ▱ are parallel.
	c) $\overline{AP} \cong \underline{\overline{KP}}$	The diagonals of a ▱ bisect each other.
	d) $\overline{KM} \cong \underline{\overline{AR}}$	The opposite sides of a ▱ are parallel.
	e) $\angle K + \angle M =$ $\underline{\mathbf{180°}}$	Consecutive angles of a parallelogram are supplementary; supplementary $\angle$s = 180°.
	f) $\overline{KM} \parallel \underline{\overline{AR}}$	The opposite sides of a ▱ are parallel.
	g) $\angle KMA \cong \underline{\mathbf{\angle KRA}}$	The opposite angles of a ▱ are congruent.

2.	PT = TR	(The diagonals of a ▱ bisect each other.)
	2x + y = 7	(Substitution)
	x + y = 5	(ST = TQ and substitution)
	x = 2	(Subtraction property of equality)
	2 + y = 5	(Substitution)
	y = 3	(Subtraction property of equality)
	PQ = (x + y + 1)	(Given)
	PQ = (2 + 3 + 1) = 6 cm	(Substitution)

3. *Proof:*

Statements	Reasons
1. ▱ABCD $\overline{DE} \perp \overline{AB}$, $\overline{CF} \perp \overline{AF}$	1. **Given**
2. $\overline{AD} \cong \overline{BC}$	2. **Opposite sides of a ▱ ≅**
3. **∠A ≅ ∠DCB**	3. Opposite angles of a ▱ ≅
4. **∠DEA and ∠CFB are rt. ∠s**	4. ⊥s form right ∠s
5. ∠DEA ≅ ∠CFB	5. **All right ∠s are congruent.**
6. **ΔAED ≅ ΔBFC**	6. AAS congruence theorem
7. $\overline{DE} \cong \overline{CF}$	7. **CPCTC**

Exercises

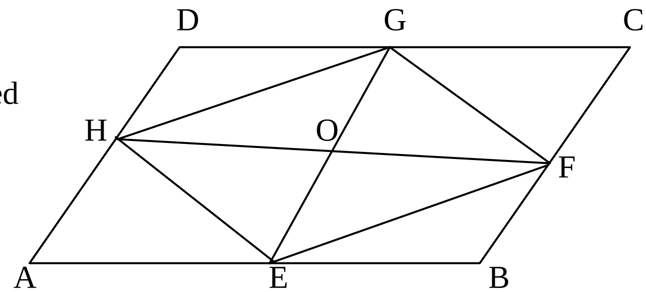
1. Given: Quadrilateral WADE with vertices W(-6,3), A(0,-3), and D(3,0)

   a) Graph the three vertices in the coordinate plane.

   b) Locate the fourth vertex to make the quadrilateral a parallelogram.

   c) Find the value of *x* and *y* in the coordinates of the fourth vertex E(x-2, y+1) so that quadrilateral WADE is a parallelogram.

2. Given: ▱EFGH
▱ABCD
Find x, y, and the indicated measures.



a) $m\angle HEF = (7x + 61)$
$m\angle HGF = (8x + 53)$
$m\angle HEA = (5x - 17)$

x = ________
$m\angle FEB$ = ________

b) GO = $(3x + 2y)$ in
OE = 9 in
OH = $(x + 2y)$ in
OF = 11 in

x = ________
y = ________

c) DC = $(x^2 + 5x - 2)$ ft
AB = $(x^2 + 3x + 10)$ ft
BC = $(5x + 26)$ ft

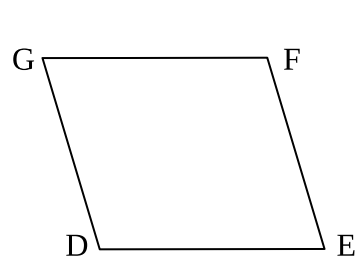
x = ________
perimeter of ▱ABCD = __________

d) $m\angle A = (\frac{7}{2}x + 30)$
$m\angle C = 65$

x = ________
$m\angle D$ = ________

3. Given: ▱DEFG

Prove: $\angle D + \angle E + \angle F + \angle G = 360°$



Let's consider the following properties of a parallelogram stated in the "if-then" format.

a) If a quadrilateral is a parallelogram, then the opposite sides are parallel.
b) If a quadrilateral is a parallelogram, then the opposite sides are congruent.
c) If a quadrilateral is a parallelogram, then the opposite angles are congruent.
d) If a quadrilateral is a parallelogram, then consecutive angles are supplementary.
e) If a quadrilateral is a parallelogram, then the diagonals bisect each other.

Each of these statements is called a <u>conditional statement</u>. The "if" part of the statement is called the <u>hypothesis</u> and the "then" part is called the <u>conclusion</u>. When the hypothesis and conclusion are reversed, the <u>converse</u> of the statement is formed as follows.

a) If the opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.
b) If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.
c) If the opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.
d) If consecutive angles of a quadrilateral are supplementary, then the quadrilateral is a parallelogram.
e) If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Each of these converse statements is a theorem that will be proven in the following examples or exercises. While each of these converse statements is true, the converse of a statement may not always be true. Such is the case in the following statement and its converse.

Statement: *If it is 23° outside, then we have cold weather.*
Converse: *If we have cold weather, then it is 23° outside.*

A statement that is true could have a converse that is false. If there is one counterexample for a given statement, then it is not always true and cannot be considered a true statement.

In the following example, both the statement and its converse are true.

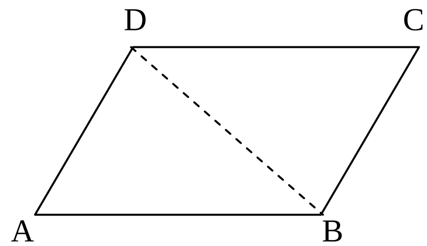
Statement: *If a quadrilateral is a parallelogram, then one pair of sides is both parallel and congruent.*
Converse: *If a quadrilateral has one pair of sides both parallel and congruent, then the quadrilateral is a parallelogram.*

We will prove the converse statement and establish another way to show that a quadrilateral is a parallelogram in example 1.

Examples

1. Given: Quadrilateral ABCD
   $\overline{DC} \cong \overline{AB}$
   $\overline{DC} \parallel \overline{AB}$

   Prove: Quadrilateral ABCD is a ▱



Proof:
Complete the following two-column proof with the appropriate statement or reason.

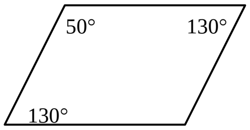
Statements	Reasons
1. Quadrilateral ABCD $\overline{DC} \cong \overline{AB}$ $\overline{DC} \parallel \overline{AB}$	1. ____________________
2. $\angle CDB \cong \angle ABD$	2. ____________________
3. ____________________	3. Reflexive property
4. $\Delta ABD \cong \Delta CDB$	4. ____________________
5. $\overline{AD} \cong \overline{BC}$	5. ____________________
6. Quadrilateral ABCD is a parallelogram	6. ____________________

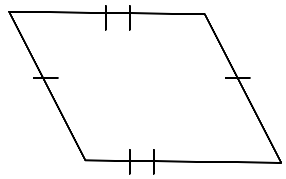
We now have six ways to prove a quadrilateral is a parallelogram. These are summarized in the following table. Use these theorems and the definition of a parallelogram to show that a quadrilateral is a parallelogram.

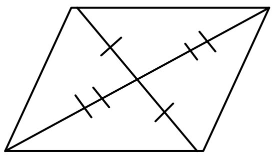
Ways to Prove a Quadrilateral is a Parallelogram

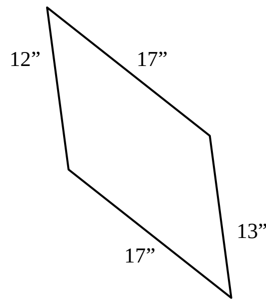
1. Show opposite sides of a quadrilateral are parallel. (Definition)
2. Show opposite sides of a quadrilateral are congruent. (Theorem)
3. Show opposite angles of a quadrilateral are congruent. (Theorem)
4. Show consecutive angles of a quadrilateral are supplementary. (Theorem)
5. Show the diagonals of a quadrilateral bisect each other. (Theorem)
6. Show one pair of sides of a quadrilateral are both congruent and parallel. (Theorem)

2. Given: Quadrilaterals as shown
   Determine which of the quadrilaterals below is a parallelogram
   Justify each answer with a definition or theorem.

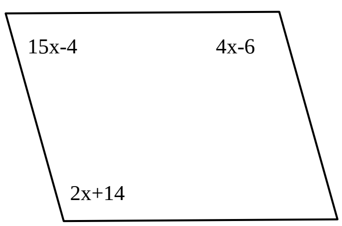
   a) 

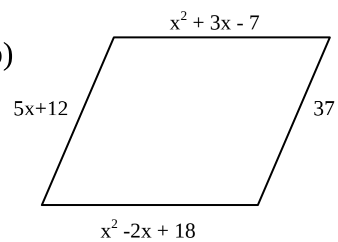
   b) 

   c) 

   d) 

3. Given: Quadrilaterals as shown below
   Find: *x* so that each quadrilateral is a parallelogram

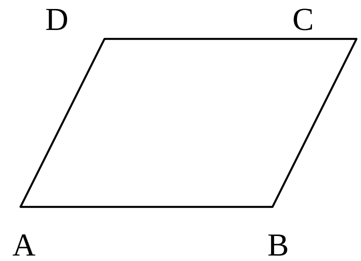
   a) 

   b) 

4. Prove the theorem: *If the consecutive angles of a quadrilateral are supplementary, then the quadrilateral is a parallelogram.*

   In this proof, we must provide the “given” and “prove”. The theorem has been given as a <u>conditional statement</u> in the “*if-then*” format. The “*if*” part of this statement provides us with the “*given*” and the “*then*” part gives us what we must “*prove*”. In this proof, we are given *consecutive angles of a quadrilateral are supplementary* and we need to prove *the quadrilateral is a parallelogram.*

Given: Quadrilateral ABCD
∠A and ∠B are supplementary
∠A and ∠D are supplementary
Prove: Quadrilateral ABCD is a ▱



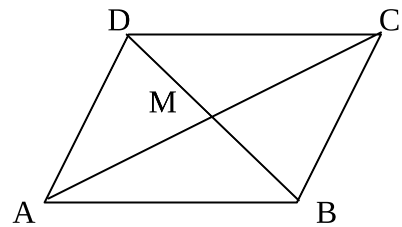
Complete the proof below with the appropriate statements and reasons.
Proof:

Statements	Reasons
1. Quadrilateral ABCD ∠A and ∠B are supplementary ∠A and ∠D are supplementary	1. ____________________
2. $\overline{AD} \parallel \overline{BC}$	2. ____________________
3. ____________________	3. If two lines are cut by a transversal so that the interior angles on the same side are supplementary, then the lines are parallel.
4. Quadrilateral ABCD is a ▱	4. ____________________

5. Prove the theorem: *If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.*

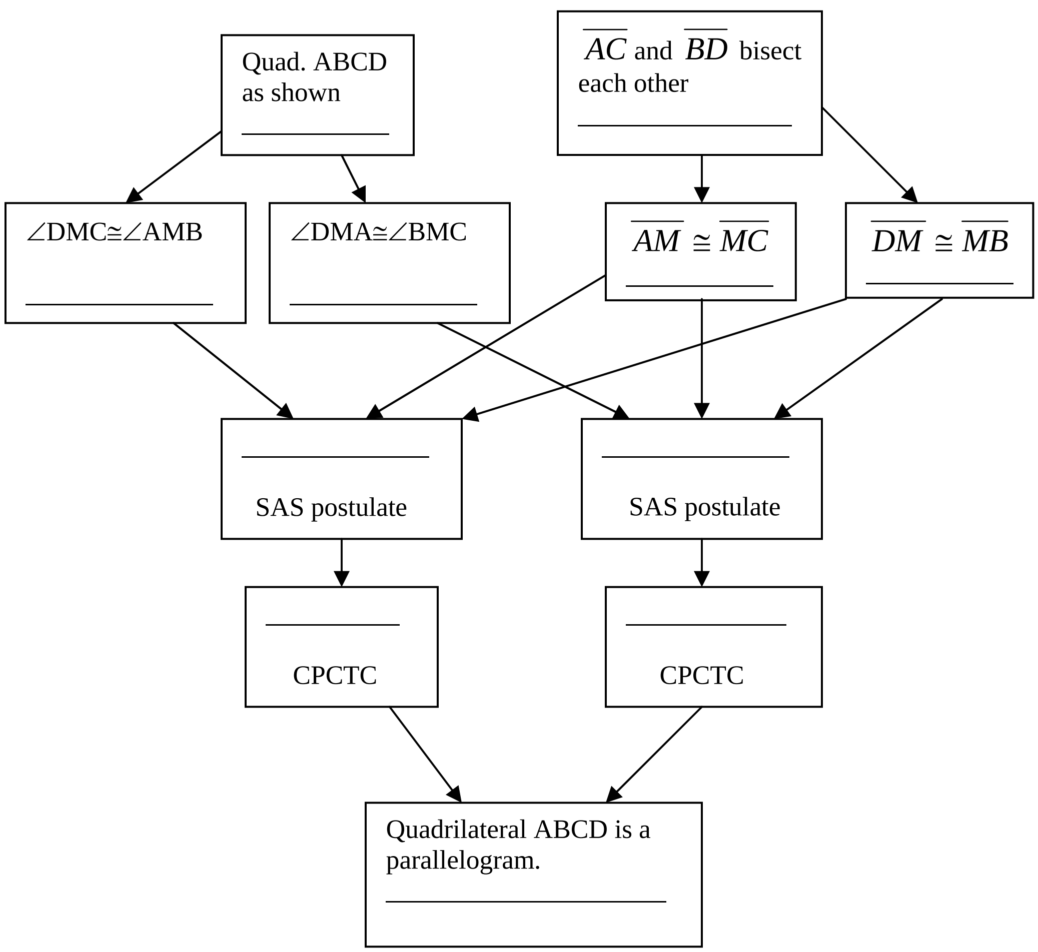
Given: Quadrilateral ABCD as shown
$\overline{AC}$ and $\overline{DB}$ bisect each other

Prove: Quadrilateral ABCD is a ▱



Complete the flow-chart proof below with the appropriate statements or reasons.

Proof:



Solutions:

1.

Statements	Reasons
1. Quadrilateral ABCD $\overline{DC} \cong \overline{AB}$ $\overline{DC} \parallel \overline{AB}$	1. **Given**
2. $\angle$CDB $\cong$ $\angle$ABD	2. **If two lines are parallel and cut by a transversal, the alternate interior angles are congruent.**
3. $\overline{DB} \cong \overline{DB}$	3. Reflexive property
4. ΔABD $\cong$ ΔCDB	4. **SAS congruence postulate**
5. $\overline{AD} \cong \overline{BC}$	5. **CPCTC**
6. Quadrilateral ABCD is a parallelogram	6. **If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.**

2. a) The quadrilateral is a parallelogram.
   If the consecutive angles of a parallelogram are supplementary, then the quadrilateral is a parallelogram.

   b) The quadrilateral is a parallelogram.
   If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

   c) The quadrilateral is a parallelogram.
   If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

   d) The quadrilateral is not a parallelogram.
   The opposite sides are not congruent.

3. a) Show consecutive angles supplementary.

$$(2x + 14) + (15x - 4) = 180$$
$$17x + 10 = 180$$
$$17x = 170$$
$$x = 10$$

$(2x + 14) = 2 \bullet 10 + 14 = 20 + 14 = 34$
$(15x - 4) = 15 \bullet 10 - 4 = 150 - 4 = 146$
$(4x - 6) = 4 \bullet 10 - 6 = 40 - 6 = 34$
$34 + 146 = 180$

When $x = 10$, consecutive angles are supplementary.

b) Show opposite sides congruent.

$$x^2 + 3x - 7 = x^2 - 2x + 18$$
$$3x - 7 = -2x + 18$$
$$5x = 25$$
$$x = 5$$

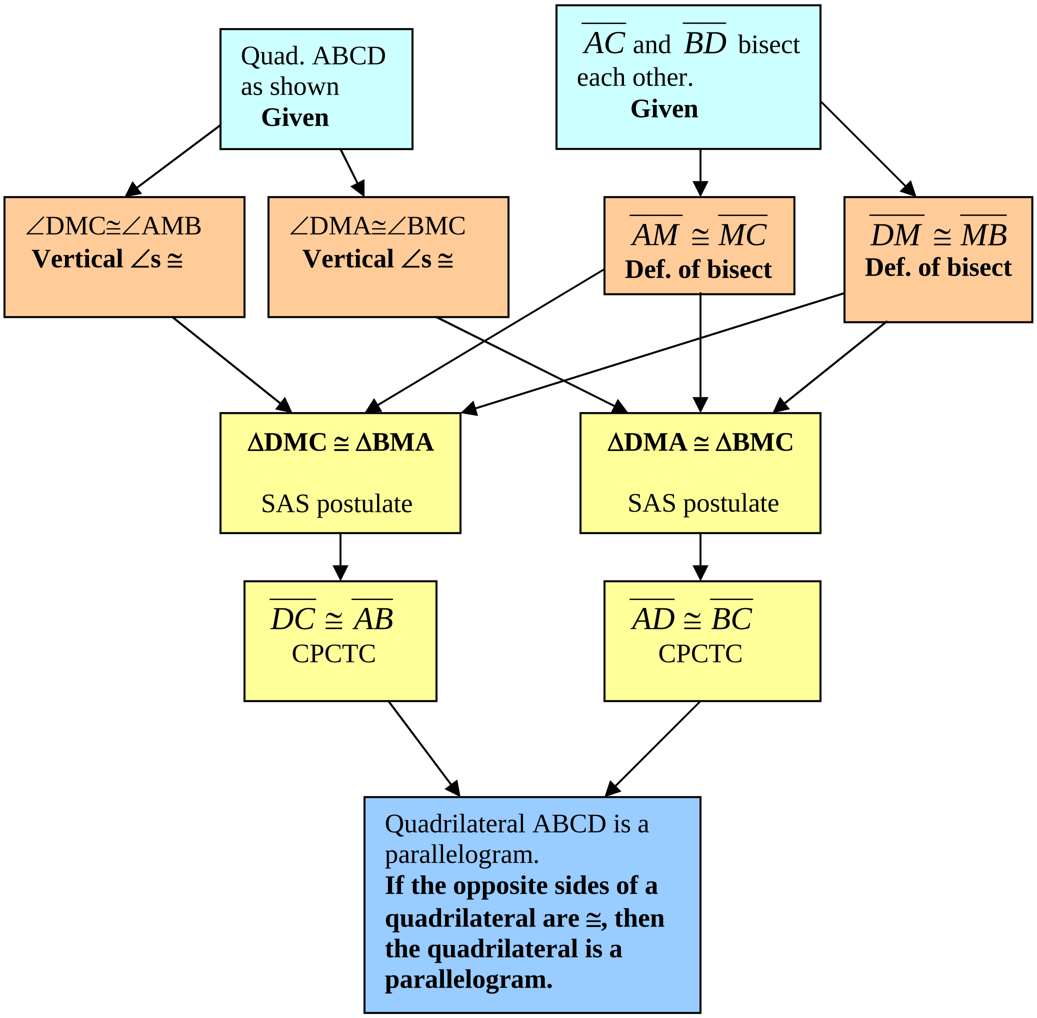
$x^2 + 3x - 7 = 5^2 + 3 \bullet 5 - 7 = 25 + 15 - 7 = 33$
$x^2 - 2x + 18 = 5^2 - 2 \bullet 5 + 18 = 25 - 10 + 18 = 33$

$5x + 12 = 5 \bullet 5 + 12 = 25 + 12 = 37$

4.

Statements	Reasons
1. Quadrilateral ABCD $\angle A$ and $\angle B$ are supplementary $\angle A$ and $\angle D$ are supplementary	1. **Given**
2. $\overline{AD} \parallel \overline{BC}$	2. **If two lines are cut by a transversal so that the interior angles on the same side are supplementary, then the lines are parallel.**
3. $\overline{AB} \parallel \overline{DC}$	3. If two lines are cut by a transversal so that the interior angles on the same side are supplementary, then the lines are parallel.
4. Quadrilateral ABCD is a ▱	4. **If the opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.**

5. *Proof:*

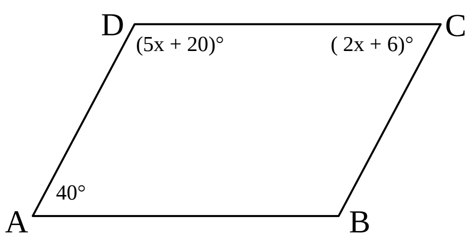


Quad. ABCD as shown
Given
$\overline{AC}$ and $\overline{BD}$ bisect each other.
Given
$\angle DMC \cong \angle AMB$
Vertical $\angle$s $\cong$
$\angle DMA \cong \angle BMC$
Vertical $\angle$s $\cong$
$\overline{AM} \cong \overline{MC}$
Def. of bisect
$\overline{DM} \cong \overline{MB}$
Def. of bisect
ΔDMC $\cong$ ΔBMA
SAS postulate
ΔDMA $\cong$ ΔBMC
SAS postulate
$\overline{DC} \cong \overline{AB}$
CPCTC
$\overline{AD} \cong \overline{BC}$
CPCTC
Quadrilateral ABCD is a parallelogram.
If the opposite sides of a quadrilateral are $\cong$, then the quadrilateral is a parallelogram.

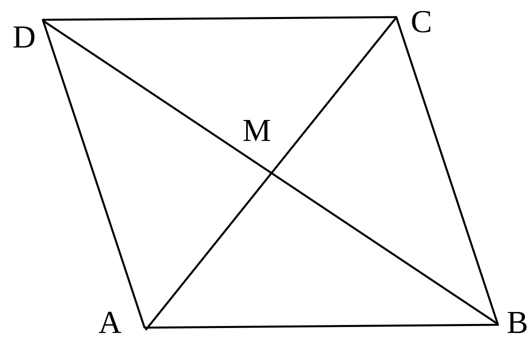
Exercises

1. Given: Quadrilateral ABCD
   Find: *x* so that quadrilateral ABCD is a parallelogram

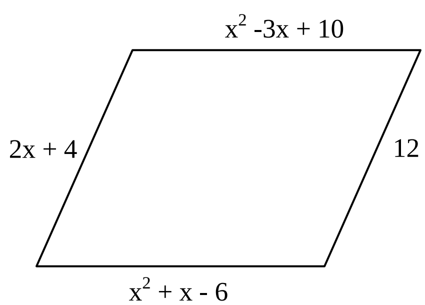
   a)



   b) DM = (2x + 1) cm
      MB = 11 cm
      AM = (x + 4) cm
      MC = (2x – 1) cm



   c)

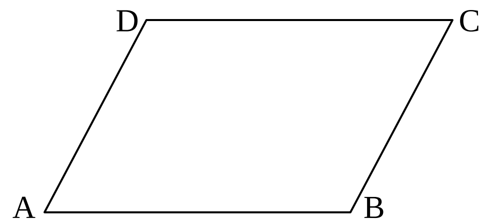


2. The vertices of quadrilateral RSTV are R(2,-4), S(8,-4), T(2,2) and V(-4,2).

   a) Graph quadrilateral RSTV in the coordinate plane.
   b) Show quadrilateral RSTV is a parallelogram.

3. Prove the theorem: *If the opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.*

Given: Quadrilateral ABCD
$\angle A \cong \angle C$
$\angle B \cong \angle D$
Prove: Quadrilateral ABCD is a ▱



Complete the proof below with the appropriate statements and reasons.
Proof:

Statements	Reasons
1. Quadrilateral ABCD $\angle A \cong \angle C$ $\angle B \cong \angle D$	1. ______________________
2. $\angle A + \angle C + \angle B + \angle D = 360°$	2. ______________________
3. $\angle A + \angle A + \angle B + \angle B = 360°$	3. ______________________
4. $2\angle A + 2\angle B = 360°$	4. ______________________
5. $\angle A + \angle B = 180°$	5. ______________________
6. ______________________	6. Definition of supplementary $\angle$s
7. ______________________	7. If 2 lines are cut by a transversal so that the interior angles on the same side of the transversal are supplementary, then the lines are parallel.

In a similar way, it can be shown that $\overline{AB}$ is parallel to $\overline{DC}$. Quadrilateral ABCD can be shown to be a parallelogram using the definition "*If the opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram*".

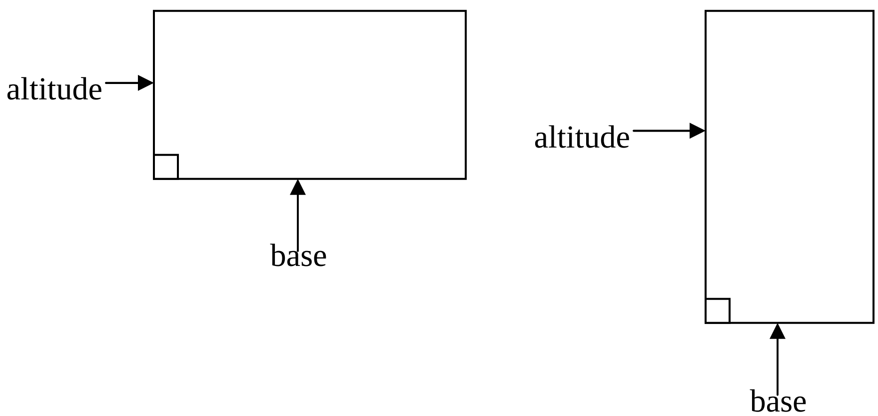
4. Prove the theorem: *If the opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.*

5. Prove the theorem: *If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.*

Rectangles

Definition
A rectangle is a parallelogram with one right angle.

Any side of a rectangle can be designated as a base and an adjacent side will be the altitude as shown in the figures below.

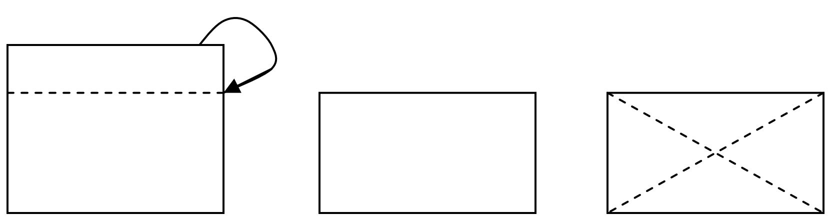


Since a rectangle is a parallelogram, it has all the properties of a parallelogram. But it also has a special property involving its diagonals. Investigate this property in the following exploration.

Exploration

1. Make a conjecture about the diagonals of a rectangle below.
   Conjecture:____

2. Test this conjecture using patty paper. Fold down one edge of a piece of patty paper so that the fold is parallel to the opposite side as shown below. Cut off the excess along the fold to form a rectangle. Then fold the diagonals. Use the edge of another piece of patty paper to compare the lengths of the diagonals.



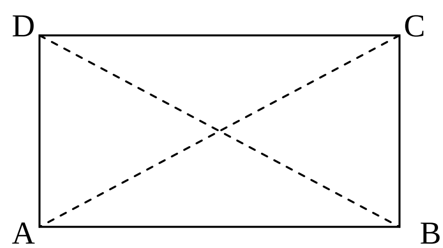
The results of this exploration lead us to the following theorem.

Theorem

The diagonals of a rectangle are congruent.

The proof of this theorem is as follows:

Given: Rectangle ABCD with diagonals $\overline{AC}$ and $\overline{BD}$
Right angle DAB
Prove: $\overline{AC} \cong \overline{BD}$



Proof:

We have $\overline{AD} \cong \overline{BC}$ since a rectangle is a parallelogram and the opposite sides of a parallelogram are congruent. $\angle DAB$ and $\angle CBA$ are supplementary because consecutive angles of a parallelogram are supplementary. We can now write $m\angle DAB + m\angle CBA = 180$ by definition of supplementary angles. Using substitution, the equation becomes $90 + m\angle CBA = 180$. Solving this equation , $m\angle CBA = 90$. We have $\angle DAB \cong \angle CBA$ since all right angles are congruent. $\overline{AB} \cong \overline{AB}$ by the reflexive property. Now $\Delta DAB \cong \Delta CBA$ by the SAS congruence postulate. $\overline{AC} \cong \overline{BD}$ by CPCTC.

The converse of this theorem can be stated as follows. The proof of this theorem will be left as an exercise.

Theorem

If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

This theorem can be used to prove a parallelogram is a rectangle.

We now have the following properties of a rectangle:

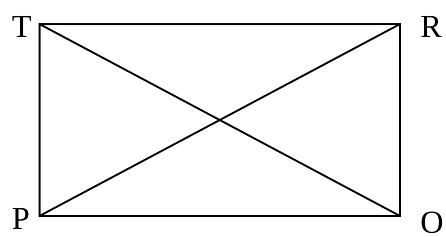
Properties of a Rectangle

1. Opposite sides of a rectangle are parallel.
2. Opposite sides of a rectangle are congruent.
3. Opposite angles of a rectangle are congruent.
4. Consecutive angles of a rectangle are supplementary.
5. The diagonals of a rectangle bisect each other.
6. The diagonals of a rectangle are congruent.

These properties will be applied in the following examples.

Examples

1. Given: Rectangle PORT with diagonals $\overline{OT}$ and $\overline{PR}$
   Find: *x* and the indicated measures



a) OT = (7x – 13) in
   PR = (4x + 11) in

   x = __________
   OT = __________

b) m∠TPR = 6x + 15
   m∠RPO = 5x + 9

   x = ________
   m∠TPO = ________

2. Given: ▱ANGL with vertices A(-5,1), N(3,-7), G(7,-3), and L(-1,5)

   Show ▱ANGL is a rectangle

3. Given: $\square$PARL with vertices P(-3,-1), A(4,-1), R(4,3), L(x+2,y-3)

   Find the value of *x* and *y* so that $\square$ PARL is a rectangle.

Solutions:

1. a) OT = PR (The diagonals of a rectangle are congruent.)
   7x – 13 = 4x + 11 (Substitution)
   3x = 24 (Addition/subtraction property of equality)
   x = 8 (Multiplication/division property of equality)

   OT = 7•8 – 13 = 56 – 13 = 43 in

   b) m∠TPR + m∠RPO = 90 (A rectangle has 1 right ∠; complementary angles have a sum of 90°.)
   6x + 15 + 5x + 9 = 90 (Substitution)
   11x + 24 = 90
   11x = 66
   x = 6
   m∠TPR = 6•6 + 15 = 36 + 15 = 51
   m∠RPO = 5•6 + 9 = 30 + 9 = 39

   ∠TPR + ∠RPO = ∠TPO (Angle addition property)
   51 + 39 = 90 (Substitution)
   m∠TPO = 90

2. Since we are given $\square$ANGL, we need to show that this figure has 1 right angle for it to be a rectangle. Right angles are formed by perpendiculars. By showing that two sides are perpendicular, we can establish a right angle. We will use the slopes of two adjacent sides to do this. The following property of perpendicular lines will be needed.

Property of Perpendicular Lines in the Coordinate Plane Two lines are perpendicular if and only if their slopes have a product of -1 and neither line is vertical.

Let's show $\overline{AL} \perp \overline{AN}$. First, we will find the slope of the line containing each of these sides and show their product is -1. Neither line is vertical.

$m_{AL} = \dfrac{(5-1)}{(-1+5)} = \dfrac{4}{4} = 1$ $\qquad$ $m_{AN} = \dfrac{(-7-1)}{(3+5)} = \dfrac{-8}{8} = -1$

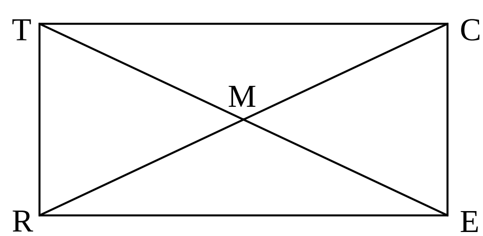
$m_{AL} \bullet m_{AN} = 1(-1) = -1$

3. In order for ▱PARL to be a rectangle, vertex L must have the coordinates (-3,3). Write the following equations and solve for *x* and *y*.

$$x + 2 = -3 \qquad y - 3 = 3$$
$$x = -5 \qquad y = 6$$

Exercises

1. Given: Rectangle RECT with diagonals $\overline{RC}$ and $\overline{ET}$
   Find: *x*, *y*, and the measures indicated



a) RE = (3x – y) cm
   TC = 7 cm
   TR = (2x-y) cm
   x = _________
   y = _________
   RC = _________
   TE = _________

b) m∠RTC = 5x + 20
   x = ________

c) m∠TRC = 3x + 10
   m∠CRE = ______
   x = ______
   m∠TCR = ______

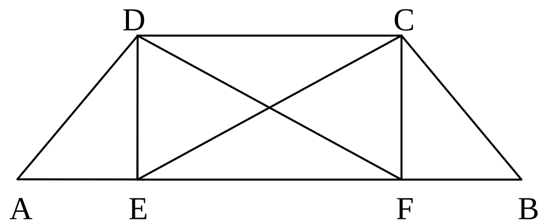
d) TM = (x + 7) ft
   ME = (2x – 1) ft
   x = _________
   RC = _________

2. Given: ▱PDQT with vertices P(-2,5), D(3,5), Q(3,9), and T(x+1,y-2)
   Find: *x* and *y* so that the coordinates of vertex T make ▱PDQT a rectangle

3. Given: ▱ABCD with vertices A(-5,1), B(3,-7), C(7,-3), and D(-1,5)
   Show ▱ABCD is a rectangle.

4. Given: Rectangle DEFC
   AE = FB
   ∠ADF ≅ ∠BCE

   Prove: ∠DFE ≅ ∠CEF



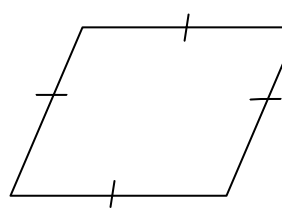
Rhombi

<u>Rhombi</u> is the plural of rhombus, another type of parallelogram defined as follows.

Definition

A rhombus is a parallelogram with two adjacent sides congruent.

Since a rhombus is a parallelogram, opposite sides are congruent. Using the definition above, all sides are congruent to each other. A rhombus can be described as an <u>equilateral parallelogram</u> as shown below.



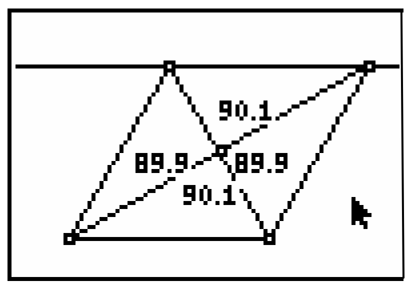
Since a rhombus is a parallelogram, it has all the properties of a parallelogram. In addition, a rhombus has some unique properties of its own. The following explorations will reveal those properties.

Explorations

1. We know that the diagonals of a rhombus bisect each other since a rhombus has the properties of a parallelogram. But, is there another relationship between the diagonals that is unique to a rhombus? Make a conjecture below.

   Conjecture: ____

2. Test this conjecture using a TI-83+ graphing calculator with Cabri Junior™.
   a) Construct a rhombus and its diagonals on the viewing screen.
   b) Measure each of the angles and round to the nearest degree.

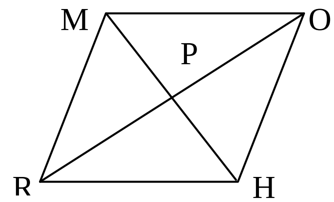


   c) How do the results compare with the conjecture made?

These results lead us to the following theorem about the diagonals of a rhombus.

Theorem
The diagonals of a rhombus are perpendicular.

The proof of this theorem is as follows.



Given: Rhombus RHOM with diagonals $\overline{MH}$ and $\overline{OR}$
Prove: $\overline{MH} \perp \overline{OR}$

Proof:

1. Rhombus RHOM with diagonals $\overline{MH}$ and $\overline{OR}$	1. Given
2. $\overline{RM} \cong \overline{MO}$	2. Definition of a rhombus
3. $\overline{RP} \cong \overline{PO}$	3. The diagonals of a rhombus bisect each other.
4. $\overline{MP} \cong \overline{MP}$	4. Reflexive property
5. ΔRMP $\cong$ ΔOMP	5. SSS congruence postulate
6. $\angle$RPM $\cong$ $\angle$OPM	6. CPCTC
7. $\angle$RPM and $\angle$OPM are supplementary	7. Linear pairs are supplementary.
m$\angle$RPM + m$\angle$OPM = 180	$\therefore$ Definition of supplementary $\angle$s
9. m$\angle$RPM + m$\angle$RPM = 180 2 m$\angle$RPM = 180	9. Substitution
10. m$\angle$RPM = 90	10. x/÷ property of equality
11. $\angle$RPM is a right angle	11. A right $\angle$ has a measure of 90°
12. $\therefore$ $\overline{MH} \perp \overline{OR}$	12. $\perp$s form right $\angle$s

Note: The symbol $\therefore$ means “therefore” and can be used in the last statement /conclusion.

Another property of a rhombus also involves the diagonals and will be proven as an exercise.

Theorem

Either diagonal of a rhombus divides a pair of opposite angles into two congruent angles.

This theorem can be stated as "*Either diagonal of a rhombus bisects a pair of opposite angles.*"

We can summarize the properties of a rhombus as follows:

Properties of a Rhombus

1. Opposite sides of a rhombus are parallel.
2. Opposite sides of a rhombus are congruent.
3. Opposite angles of a rhombus are congruent.
4. Consecutive angles of a rhombus are supplementary.
5. The diagonals of a rhombus bisect each other.
6. The diagonals of a rhombus are perpendicular.
7. Either diagonal of a rhombus bisects a pair of opposite angles.

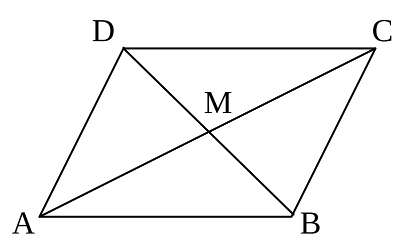
To show that a parallelogram is a rhombus, the converse of the following theorem can be used.

Theorem: *If a parallelogram is a rhombus, then its diagonals are perpendicular.*

Converse: *If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.*

The proof of the converse is given below.

Given: $\square$ABCD
$\overline{AC} \perp \overline{DB}$
Prove: $\square$ABCD is a rhombus



Proof:

$\angle$CMD and $\angle$CMB are right angles because $\overline{AC} \perp \overline{DB}$. $\angle$CMD $\cong$ $\angle$CMB since all right angles are congruent. The diagonals of a parallelogram bisect each other and $\overline{DM} \cong \overline{MB}$. $\overline{MC} \cong \overline{MC}$ by the reflexive property. Now we have ΔDMC $\cong$ ΔBMC by the SAS congruence postulate. By CPCTC, $\overline{DC} \cong \overline{BC}$. Parallelogram ABCD is a rhombus by the definition of a rhombus.

Squares

A special kind of rhombus is a square defined as follows:

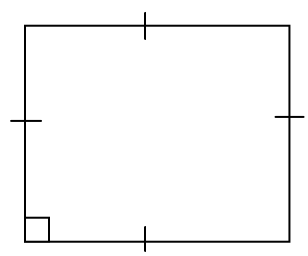
Definition

A square can be defined as a rhombus with one right angle.

Since a square is also a rectangle, it can be defined as follows:

Definition

A square can be defined as a rectangle with two adjacent sides congruent.



Since a square is both a rhombus and a rectangle, it has the properties of both as follows.

Properties of a Square

1. Opposite sides of a square are parallel.
2. Opposite sides of a square are congruent.
3. Opposite angles of a square are congruent.
4. Consecutive angles of a square are supplementary.
5. The diagonals of a square bisect each other.
6. The diagonals of a square are perpendicular.
7. Either diagonal of a square bisects a pair of opposite angles.

To prove a parallelogram is a square, one of the following theorems can be used.

Theorems

1. If the diagonals of a parallelogram are congruent and perpendicular, then the parallelogram is a square.

2. If the parallelogram has one right angle and two adjacent sides congruent, then the parallelogram is a square.

These theorems are converses of previously stated theorems for squares.

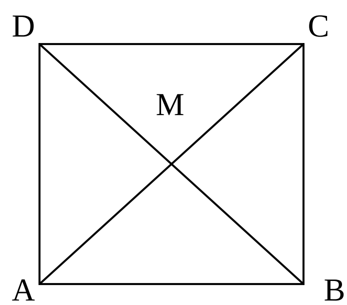
Theorem: *If a parallelogram is a square, then its diagonals are congruent and perpendicular.*
Converse: *If the diagonals of a parallelogram are congruent and perpendicular, then the parallelogram is a square.*

Theorem: *If a parallelogram is a square, then the parallelogram has one right angle and two adjacent sides congruent.*
Converse: *If a parallelogram has one right angle and two adjacent sides congruent, then the parallelogram is a square.*

We will apply the properties of squares and rhombi in the following examples.

Examples

1. Given: Square ABCD with diagonals $\overline{DB}$ and $\overline{AC}$
   Find: x, y, and the indicated measures



a) m$\angle$DMA = 2x + 10
m$\angle$CAB = x + 5
x = ________
m$\angle$CAD =_________

b) AB = (7x – 3) cm
DC = (5x + 9) cm
x = __________
AD = __________

2. The vertices of ▱ABCD are A(-2,-1), B(2,-3), C(4,1), and D(0,3).
   a) Graph the parallelogram in the coordinate plane.
   b) Identify the parallelogram as a rectangle, square, or rhombus.
   c) Justify this answer.

<u>Solutions:</u>

1. a) $\overline{DB} \perp \overline{AC}$ (The diagonals of a square are $\perp$.)
m$\angle$DMA = 90 ($\perp$s form right angles.)
2x + 10 = 90 (Substitution)
2x = 80 (Addition/subtraction property of equality)
x = 40 (Division/multiplication property of equality)

m$\angle$CAD + m$\angle$CAB = m$\angle$DAB (Angle addition property)
m$\angle$CAD = m$\angle$CAB (A diagonal bisects a pair of opposite angles in a square.)
m$\angle$DAB = 90 (Definition of a square)
m$\angle$CAD + m$\angle$CAD = 90 (Substitution)
2 m$\angle$CAD = 90
m$\angle$CAD = 45 (Division/multiplication property of equality)

b) AB = DC (Opposite sides of a square $\cong$ and have = measures.)
7x-3=5x+9 (Substitution)
2x = 12
x = 6 (Division/multiplication property of equality)
AD = AB (Definition of a square)

AD = 7•6 – 3 = 42-3 = 39 cm

2. We will first find the slopes of two adjacent sides to check for a right angle.

$m_{AB} = \dfrac{(-1+3)}{(-2-2)} = \dfrac{2}{-4} = -\dfrac{1}{2}$ $\qquad m_{AD} = \dfrac{(1+3)}{(2+0)} = \dfrac{2}{1}$

$m_{AB} \bullet m_{AD} = -\dfrac{1}{2} \bullet \dfrac{2}{1} = -1 \qquad \Rightarrow \qquad \overline{AB} \perp \overline{AD}$

Next, we will find the lengths of two adjacent sides and compare.

$AB = \sqrt{(-1+3)^2 + (-2-2)^2} = \sqrt{2^2 + (-4)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$

$AD = \sqrt{(1+3)^2 + (2+0)^2} = \sqrt{4^2 + 2^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$

We now have two adjacent sides congruent and one right angle. A square has two adjacent sides congruent and one right angle.

Exercises

1. Complete the table below by placing a √ in the appropriate boxes to indicate the properties of the given quadrilaterals.

Property	**Parallelogram**	**Rectangle**	**Square**	**Rhombus**
Opposite sides are parallel.				
Opposite sides are congruent.				
Opposite angles are congruent.				
Consecutive angles are supplementary.				
Diagonals are congruent.				
Diagonals bisect each other.				
Diagonals are perpendicular.				
Either diagonal bisects a pair of opposite angles.				

2. Determine which of the following statements is *always*, *sometimes*, or *never* true and state a justification for each choice.

a) A square is a rectangle.
b) A rectangle is a square.
c) A rhombus is a square.
d) A rectangle has one pair of opposite sides parallel and congruent.
e) A diagonal bisects a pair of opposite angles in a rectangle.
f) Diagonal $\overline{BD}$ is drawn in square ABCD. $\angle$CAB and $\angle$DAC are complementary.
g) Diagonal $\overline{AC}$ is drawn in rhombus ABCD. ΔADC is an isosceles triangle.
h) Diagonal $\overline{BD}$ is drawn in rhombus ABCD. m$\angle$DAC = 45.
i) If a quadrilateral is a parallelogram, then it is a rhombus.
j) If a quadrilateral is a square, then it is a rhombus.
k) The diagonals of a rectangle are perpendicular.
l) The diagonals of a square intersect to form four non-overlapping congruent triangles.

m) If one pair of consecutive angles in a quadrilateral are supplementary, then the quadrilateral is a parallelogram.
n) A rhombus is equiangular (all angles are congruent).
o) A diagonal drawn in a parallelogram divides it into two congruent triangles.

3. The vertices of ▱ABCD are A(0,0), B(5,-4), C(9,1), and D(4,5).
   a) Graph the parallelogram in the coordinate plane.
   b) Identify the parallelogram as a rectangle, square, or rhombus.
   c) Justify the identification.

4. Prove the theorem: *Either diagonal of a rhombus bisects a pair of opposite angles.* (Optional)

5. Prove the theorem: *If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.* (Optional)

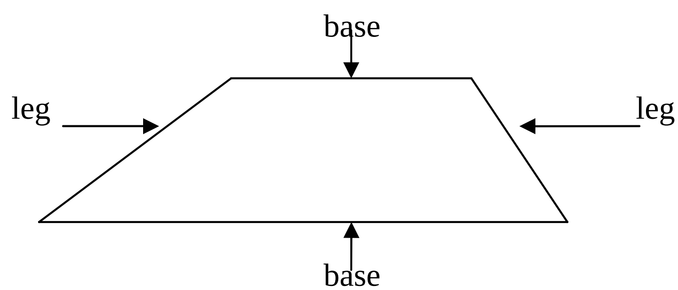
Trapezoids

Definition

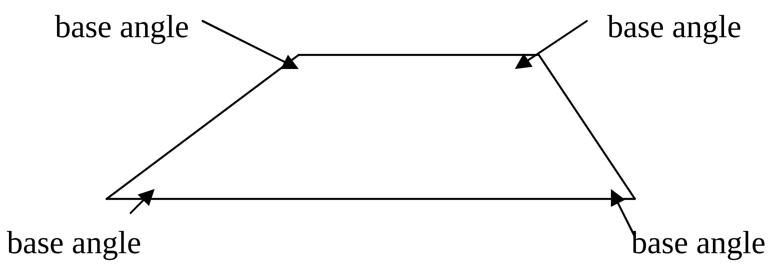
A trapezoid is a quadrilateral with one pair of sides parallel.

A trapezoid can also be defined as a quadrilateral with at least two sides parallel. For our purposes, we will use the definition above.

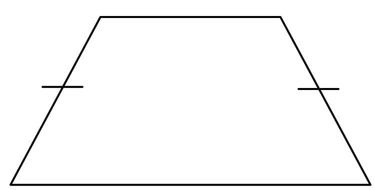
The two sides that are parallel are called <u>bases</u>. The two sides that are not parallel are called <u>legs</u> as shown below.



A trapezoid also has two pairs of <u>base angles</u>. These base angles are shown in the figure below.



A trapezoid with congruent legs is an <u>isosceles trapezoid</u>. The figure below is an example of an isosceles trapezoid.



We will consider two theorems relating to an isosceles trapezoid as follows.

Isosceles Trapezoid Theorem
If a trapezoid is isosceles, then the base angles are congruent.

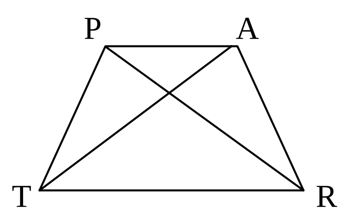
Isosceles Trapezoid Theorem
If a trapezoid is isosceles, then the diagonals are congruent.

These theorems will be applied in the following examples.

Examples

1. Given: Isosceles trapezoid TRAP
   Prove: $\angle TPR \cong \angle RAT$

   Complete the proof with the appropriate statements and reasons.

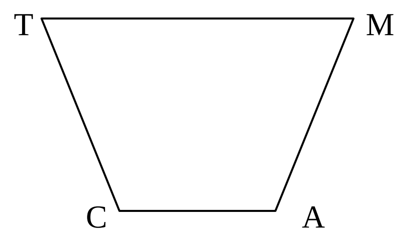


Proof:

Statements	Reasons
1. Isosceles trapezoid TRAP	1. ____________
2. $\overline{TP} \cong \overline{AR}$	2. ____________
3. $\angle PTR \cong \angle ART$	3. ____________
4. ____________	4. Reflexive property
5. ____________	5. ____________
6. $\therefore \angle TPR \cong \angle RAT$	6. ____________

2. Given: Isosceles trapezoid CAMT
   $m\angle T = 8x - 10$
   $m\angle M = 5x + 20$

   Find: x = ________
   $m\angle T$ = ________
   $m\angle C$ = ________



3. Given: Trapezoid WORK with vertices W(-2,-1), O(6,-1), R(4,3)

   a) Graph trapezoid WORK in the coordinate plane.
   b) Find the coordinates of vertex K so that trapezoid WORK is isosceles.

<u>Solutions:</u>

1.

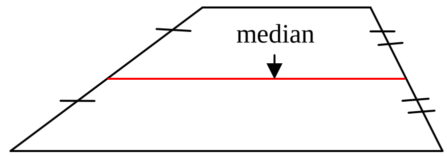
Statements	Reasons
1. Isosceles trapezoid TRAP	1. <u>**Given**</u>
2. $\overline{TP} \cong \overline{AR}$	2. <u>**Definition of an isosceles trapezoid**</u>
3. $\angle PTR \cong \angle ART$	3. <u>**The base angles of an isosceles trapezoid are congruent.**</u>
4. <u>$\overline{TR} \cong \overline{TR}$</u>	4. Reflexive property
5. <u>**ΔPTR ≅ ΔART**</u>	5. <u>**SAS congruence postulate**</u>
6. $\therefore \angle TPR \cong \angle RAT$	6. <u>**CPCTC**</u>

2. $m\angle T = \angle M$ (Base angles of an isosceles trapezoid ≅)
   $8x - 10 = 5x + 20$ (Substitution)
   $3x = 30$ (Addition/subtraction properties of equality)
   $x = 10$ (Division/multiplication properties of equality)

   $m\angle T = 8 \bullet 10 - 10 = 80-10 = 70$
   $m\angle C = 180 - 70 = 110$

3. b) Vertex K has the coordinates (0,3).

The segment joining the midpoints of the two legs of a trapezoid is the <u>median</u> as shown below.



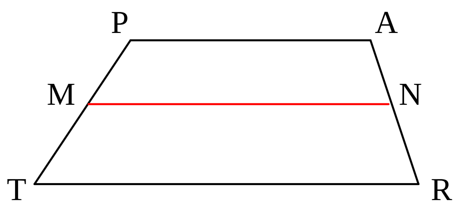
The median of a trapezoid has a special relationship to the bases. This relationship is given as a theorem.

> **Theorem**
>
> The median of a trapezoid is parallel to the two bases and has a length equal to one-half the sum of the lengths of the two bases.

In the figure below, we have trapezoid TRAP with median $\overline{MN}$. Using this theorem, we can write the following statements:

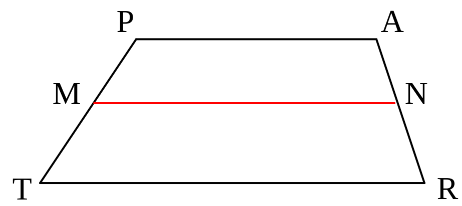
$$\overline{MN} \parallel \overline{AP} \text{ and } \overline{MN} \parallel \overline{TR}$$

$$\text{MN} = \frac{1}{2}(\text{AP} + \text{TR})$$



We will apply the previous theorem in the following examples.

1. Given: Trapezoid TRAP
   Median $\overline{MN}$
   TR = (3x – 4) cm
   AP = (x + 2) cm
   MN = 15 cm



   Find: *x* and TR
   x = __________
   TR = __________

2. Given: Trapezoid TRAP in example 1
   Median $\overline{MN}$
   m∠PMN = (10x – 27)
   m∠PTR = (3x + 29)

   Find: *x* and m∠PTR

<u>Solutions:</u>

1. $MN = \frac{1}{2}(AP + TR)$ (The median of a trapezoid has a length equal to one-half the sum of the lengths of the two bases.)

   $15 = \frac{1}{2}(x + 2 + 3x - 4))$ (Substitution)

   $30 = (x + 2 + 3x - 4)$ (Multiplication property of equality)

   $30 = 4x - 2$

   $32 = 4x$

   $8 = x$

   TR = 3•8 – 4 = 24 – 4 = 20 cm

2. $\overline{MN} \parallel \overline{TR}$ (The median of a trapezoid is parallel to both bases.)

   m∠PMN = m∠PTR (If two parallel lines are cut by a transversal, then the corresponding angles are congruent.)

   $10x - 27 = 3x + 29$ (Substitution)

   $7x = 56$

   $x = 8$

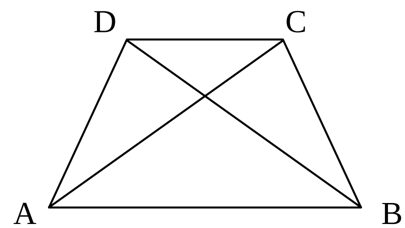
   m∠PTR = 3•8 + 29 = 24 + 29 = 53

Exercises

1. Prove the theorem: *The diagonals of an isosceles trapezoid are congruent.*

   Given: Isosceles trapezoid ABCD with diagonals $\overline{AC}$ and $\overline{BD}$

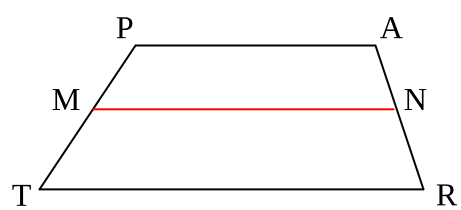
   Prove: $\overline{AC} \cong \overline{BD}$



   Complete the proof below with the appropriate statements and reasons.
   Proof:

1. ____________________	1. Given
2. ____________________	2. Definition of an isosceles trapezoid
3. $\angle DAB \cong \angle CBA$	3. ____________________
4. ____________________	4. Reflexive property
5. $\Delta DAB \cong \Delta CBA$	5. ____________________
6. $\therefore$____________________	6. SAS congruence postulate

2. Given: Trapezoid TRAP
   Median $\overline{MN}$
   MN = (7x – 4) cm
   TR = (5x + 12) cm
   AP = (3x + 10) cm



   Find: *x* and the indicated measure
   x = ___________
   MN = ___________

3. Given: Trapezoid ABCD with vertices A(-2,1), B(3,1), C(5,5), D(x-1, y+3)

   a) Graph the given coordinates of trapezoid ABCD
   b) Find the value of *x* and *y* in the coordinates of vertex D so that trapezoid ABCD is isosceles.