TRANSFORMATIONS

In geometry, a transformation is a process by which a set of points is transformed, or changed. These changes can involve location, size, or both. We will be studying the following transformations:

1. Reflections
2. Translations
3. Rotations
4. Dilations

Transformations are sometimes called mappings. We will refer to the initial set of points as the pre-image and the final set of points as the image.

In reflections, translations, and rotations, the image is always congruent to the pre-image. Because of this fact, each of these three transformations is known as a congruence transformation. Another name for congruence transformation is isometry. ("Iso" means same and "metry" means measure.)

In this unit, some transformations will be done within the coordinate plane, and others will be done without reference to any coordinate system. These examples will be intermixed, as the principles of the transformations in either case remain the same.

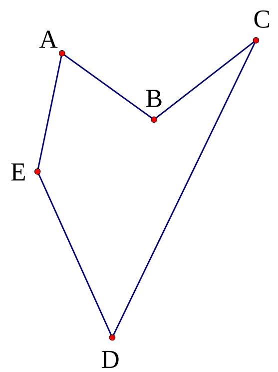
Reflections

Activity

For an introductory exploration of this section, refer to the activity entitled "Reflections with Patty Paper."

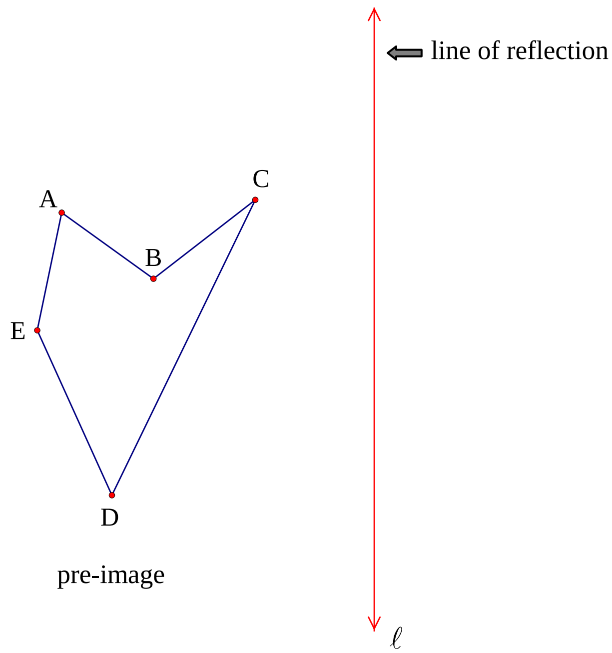
To aid us in our understanding of reflections, think of the ways in which we use the word *reflection* in everyday life, such as a reflection in a mirror, or the reflection of an object in a lake.

Consider pentagon *ABCDE* below. This initial figure is known as the pre-image.

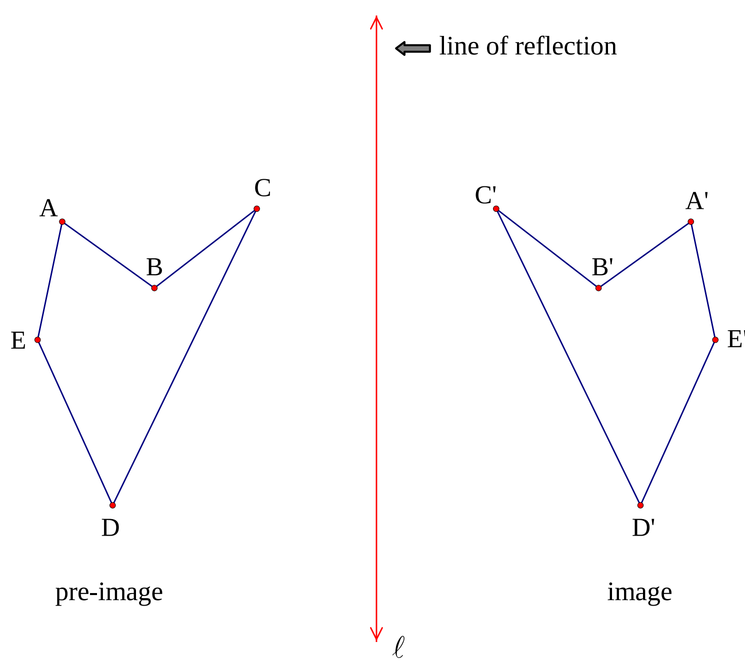


pre-image

We now draw a line of reflection and name the line ℓ.



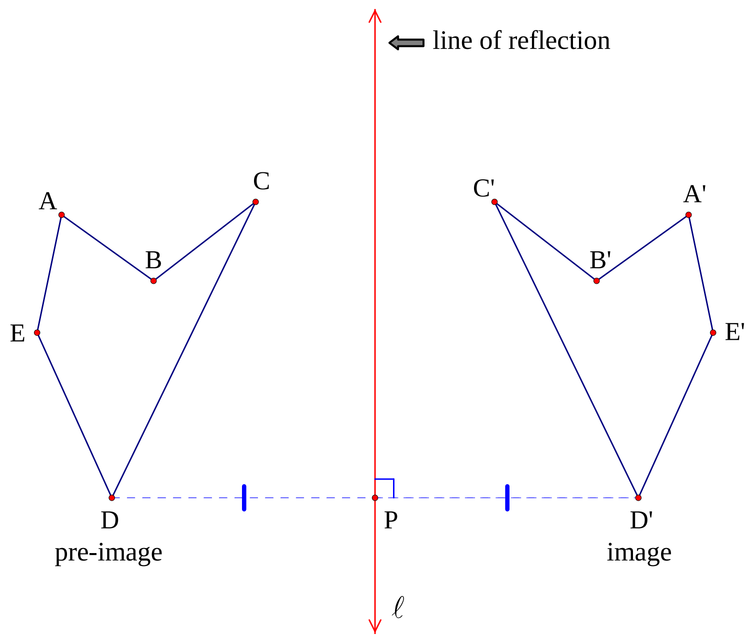
Think of the reflection line as a mirror. We now wish to reflect *ABCDE* about line ℓ, as shown below. Notice that the image of point *A* is denoted as A' (known as "A prime"), the image of point *B* is denoted as B', etc.



We will now look at a feature of the objects known as orientation. Notice that in the pre-image above, we can name the pentagon *ABCDE*, with the vertices being named in a clockwise fashion. On the other hand, if the vertices of the image are named in a a corresponding manner, they are named in a counterclockwise fashion. For this reason, pentagons *ABCDE* and $A'B'C'D'E'$ are said to have reverse orientation.

Compare the corresponding sides and angles of pentagon *ABCDE* with those of $A'B'C'D'E'$. (You can measure them if you wish.) We can easily see that the two pentagons are congruent. A reflection preserves both distance and angle measure; the orientation is simply reversed.

Let us now examine some other properties of reflection. In the figure below, a segment has been drawn between point *D* and its image, D'. The intersection of $\overline{DD'}$ and line ℓ has been labeled as *P*.



Line ℓ is the perpendicular bisector of $\overline{DD'}$. This means that $\ell \perp \overline{DD'}$ and that $\overline{DP} \cong \overline{D'P}$. In a reflection, the reflection line is always the perpendicular bisector of the segment joining a point in the pre-image with its image.

The properties of reflections are summarized in the table below.

Properties of Reflections

If a figure is reflected:

1. The orientation of the image is reversed from the pre-image.
2. The image is congruent to the pre-image. (Therefore, a reflection is a congruence transformation, or isometry.)
3. If a segment is drawn which connects any point in the pre-image with its image, the line of reflection is the perpendicular bisector of that segment.

Below are some ideas of how to create accurate reflections by hand. (Software programs such as *The Geometer's Sketchpad* can also be used to create reflections.)

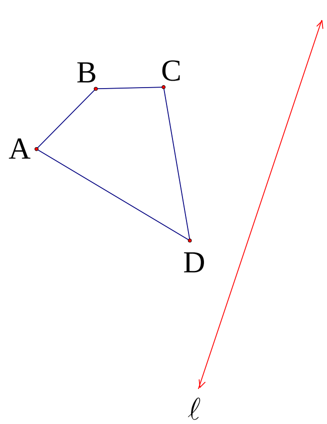
1. Trace the pre-image and the reflection line onto patty paper or thin paper. Then fold the paper along the reflection line, and trace each part of the pre-image onto the other side of the reflection line – this forms the image. Label the image of A as A', B as B', etc.
2. Trace the pre-image and the reflection line onto thin paper. Then fold the paper along the reflection line and place the paper against a window; this should help you to see the pre-image through the paper. Then trace each part of the pre-image onto the other side of the reflection line – this forms the image.
3. Trace or draw the pre-image onto any type of paper. Then place a mirror or a plastic Mira on the reflection line, and 'freehand' the image that you see reflected in the mirror. (The mirror itself gives you an accurate reflection, but you may not be able to draw the reflection as well using this method.)

Realize that the reflection line need not be vertical, but can be in any position, as shown in the examples below.

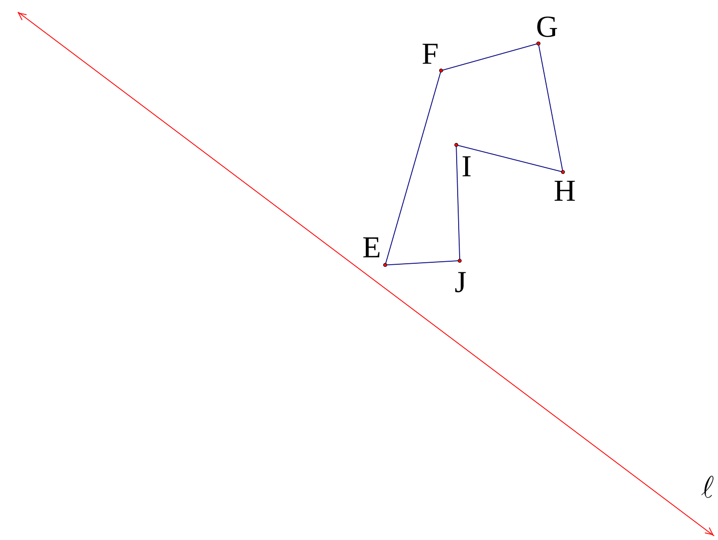
Examples

Reflect the following figures across reflection line ℓ. Label the points of the image with prime notation.

1.

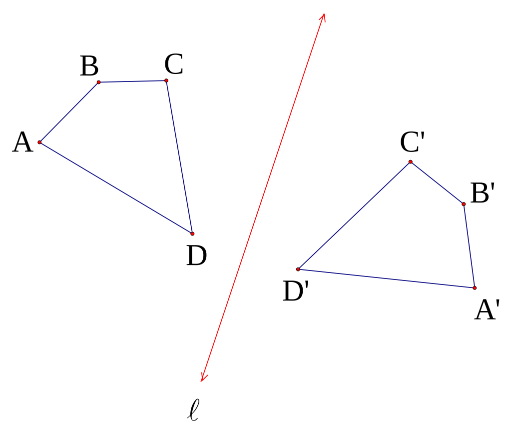


2.

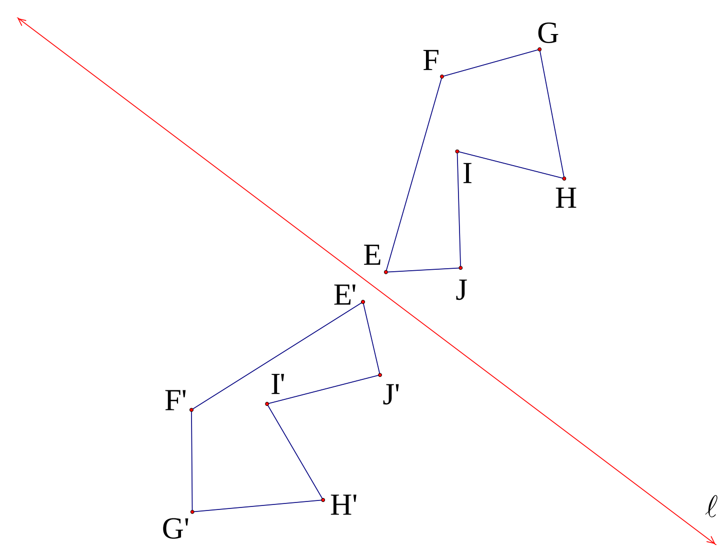


Solutions:

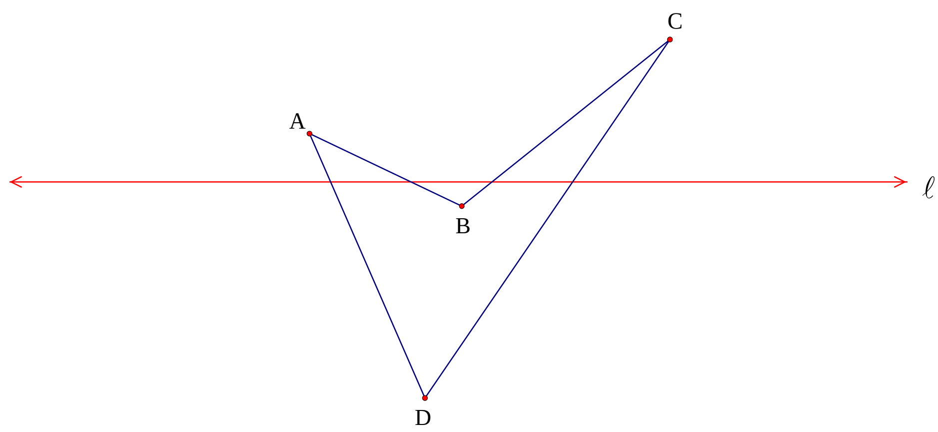
1.



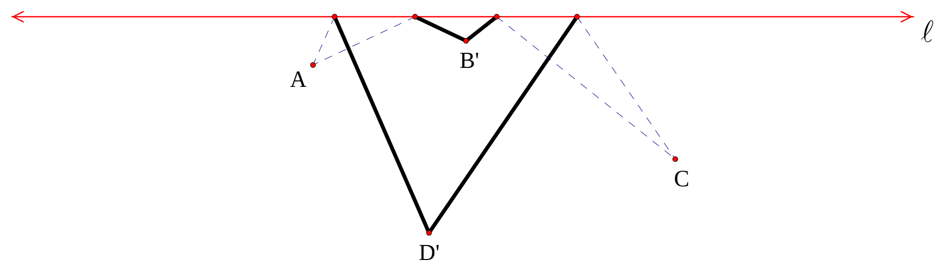
2.



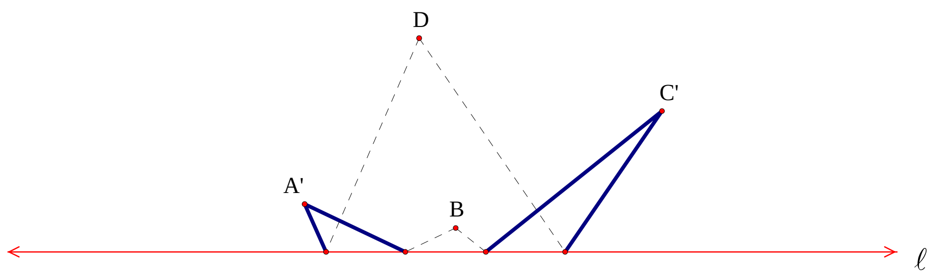
When the pre-image lies on both sides of the reflection line, it takes two steps to draw the reflection. Consider the following example.



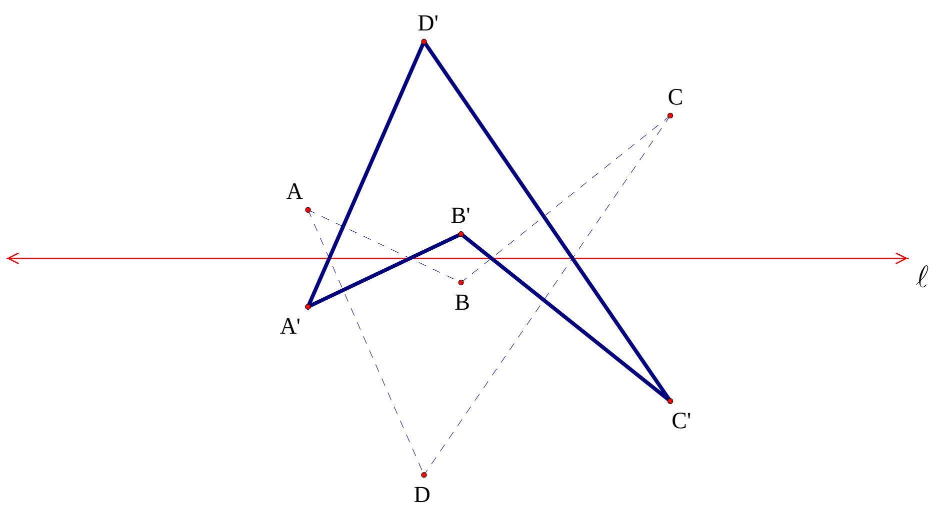
Suppose that we want to perform the reflection of *ABCD* across line ℓ. Trace the pre-image above onto patty paper or thin paper. Then fold along the reflection line, folding the top portion of the pre-image down toward you. <u>On the topmost layer of the paper in front of you, trace what you see from the lower layer of the paper.</u> (If you are using thin paper instead of patty paper, you may need to hold it up to a window or to the light in order to see through to the lower layer of the paper.) The part that you should trace is shown in bold below, and the part from the top layer of the paper (which should not be traced yet) is shown below as dotted segments. Since you are re-tracing the diagram about vertex *B*, label it lightly as B', and since you are re-tracing the diagram about vertex *D*, label it lightly as D'. (Note: The labels for *A* and *C* have not been reversed in the diagram below, but they in reality will be reversed since the top portion of the pre-image has been folded down.)



We now repeat a similar process from above, but in reverse. Unfold the paper, and fold along the reflection line, folding the bottom part of the pre-image up toward you. <u>On the topmost layer of the paper in front of you, trace what you see from the lower layer of the paper.</u> (If you are using thin paper instead of patty paper, you may need to hold it up to a window or to the light in order to see through to the lower layer of the paper.) The part that you should trace is shown in bold below, and the part from the upper layer of the paper (which has already been traced) is shown below as dotted segments. Since you are re-tracing the diagram about vertex *A*, label it lightly as A', and since you are re-tracing the diagram about vertex *C*, label it lightly as C'. (Note: The labels for *B* and *D* have not been reversed in the diagram below, but they in reality will be reversed since the bottom portion of the pre-image has been folded up.)



Now unfold your paper. Realize that in the steps above, we were tracing onto the back side of the paper. If you used patty paper, you should be able to see all of your tracing lines clearly. If you used thin paper, you should be able to see the tracing showing through from the back side, and should trace over those segments on the front side of the paper. You will also need to re-label the vertices that have prime notation, as the letters were drawn on the back side and will appear reversed. The diagram below shows the final image in bold segments, with the original pre-image in dotted segments.

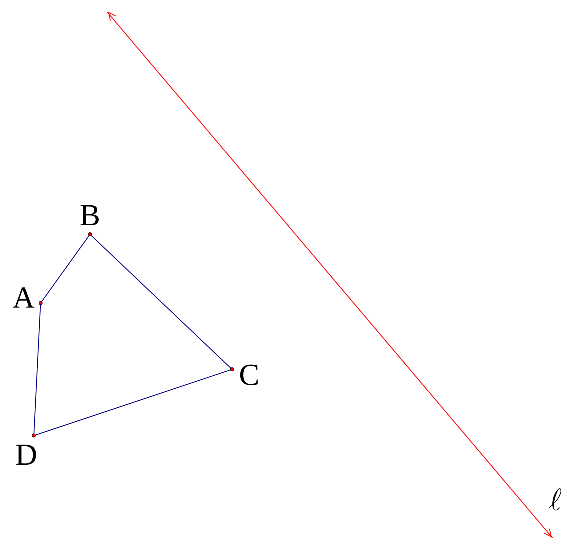


Additional practice exercises for reflections can be found below.

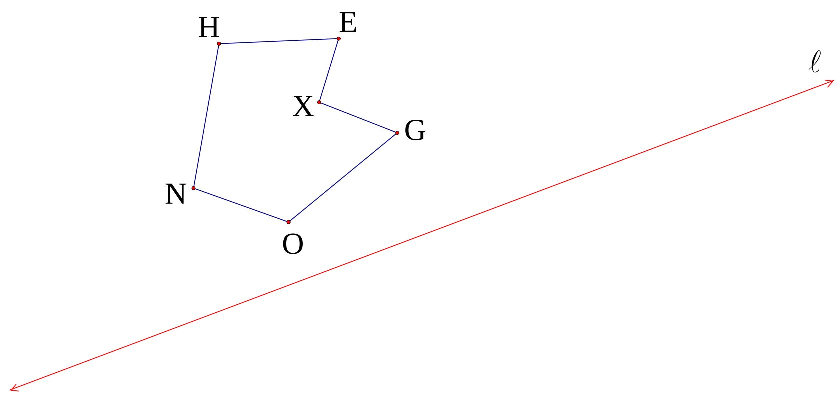
Exercises

Reflect the following figures across reflection line ℓ. Label the points of the image with prime notation.

1.

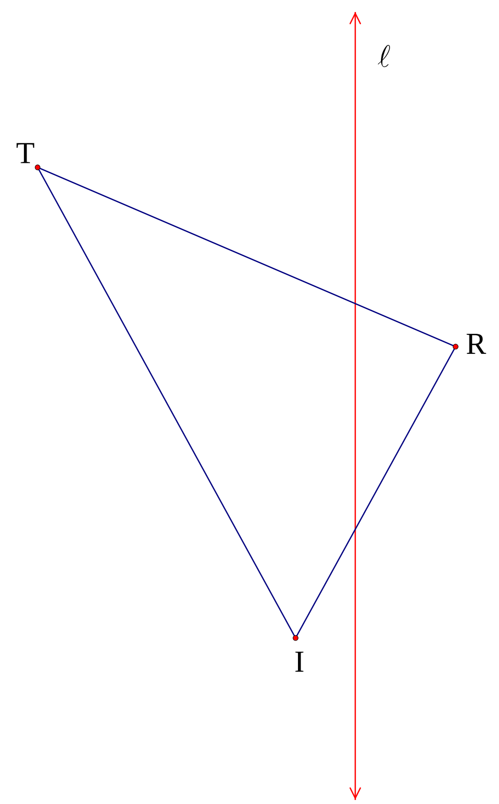


2.

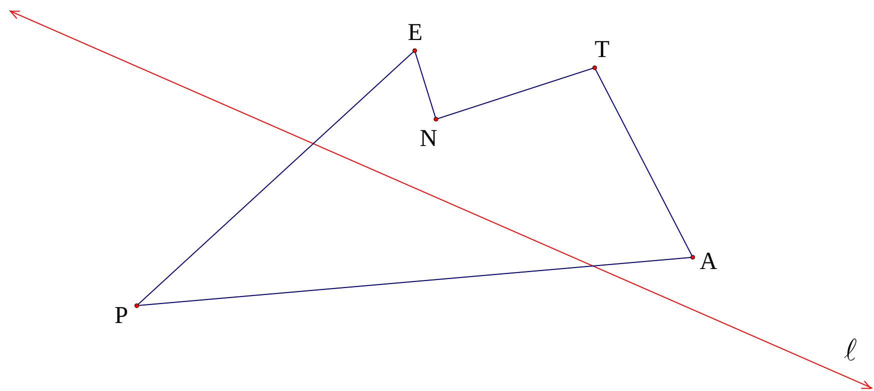


3. Draw a reflection line, and then draw a figure that does not intersect the reflection line. Then reflect the figure across the line of reflection.

4.



5.



6. Draw a reflection line, and then draw a figure that intersects the reflection line. Then reflect the figure across the line of reflection.

We will now perform reflections in the coordinate plane. There is no difference in technique when reflecting figures in the coordinate plane; we must just focus on the reflection line itself and not be distracted by the axes or the grid. (It is possible, of course, that one of the axes is in fact the reflection line, but this is not always the case.)

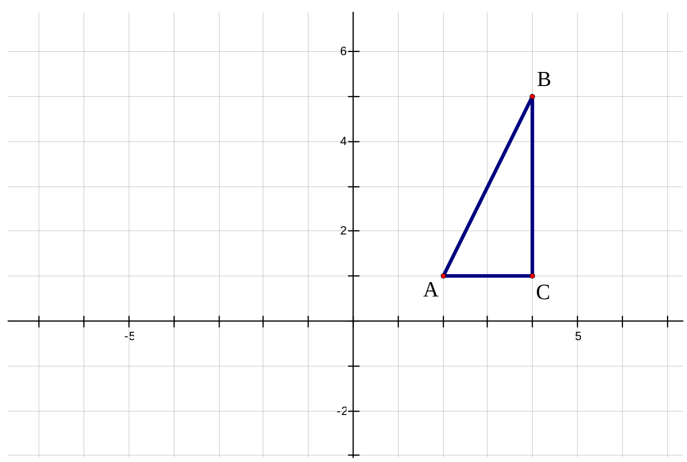
Examples

Given the triangle with vertices $A(2,1)$, $B(4,1)$, and $C(4,5)$,

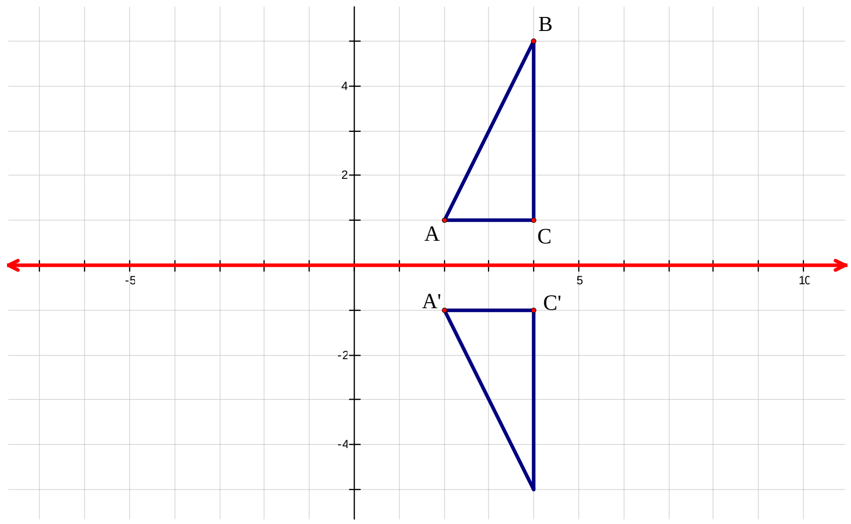
1. Draw a diagram of ΔABC in the coordinate plane.
2. Reflect ΔABC across the following lines:
   a) The *x*-axis
   b) The *y*-axis
   c) The line $x = 7$
   d) The line $y = -2$
   e) The line $y = x - 3$
   f) The line $x = 3$

<u>Solutions:</u>

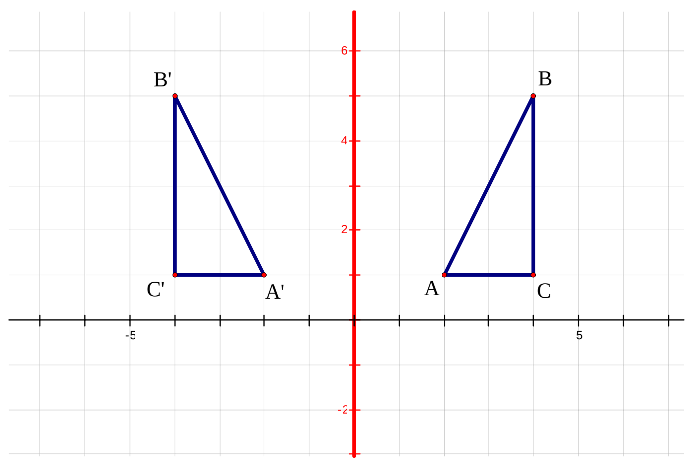
1.



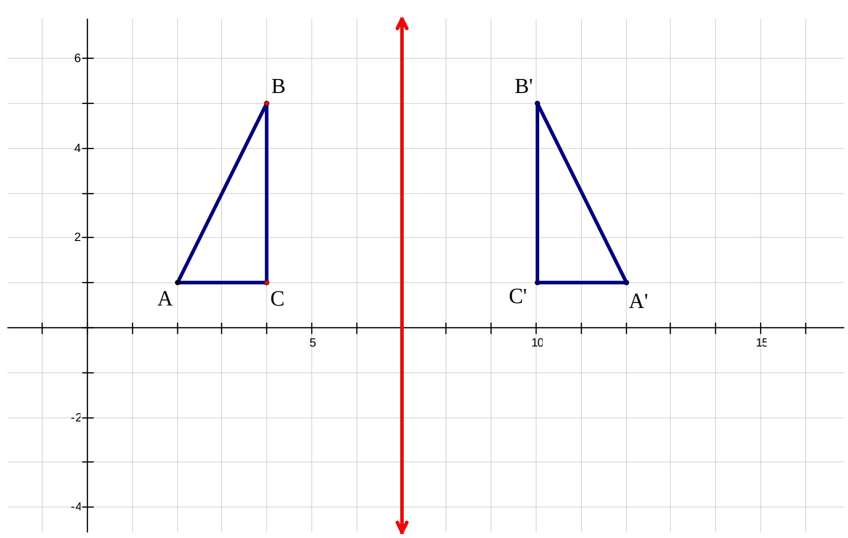
2. a)



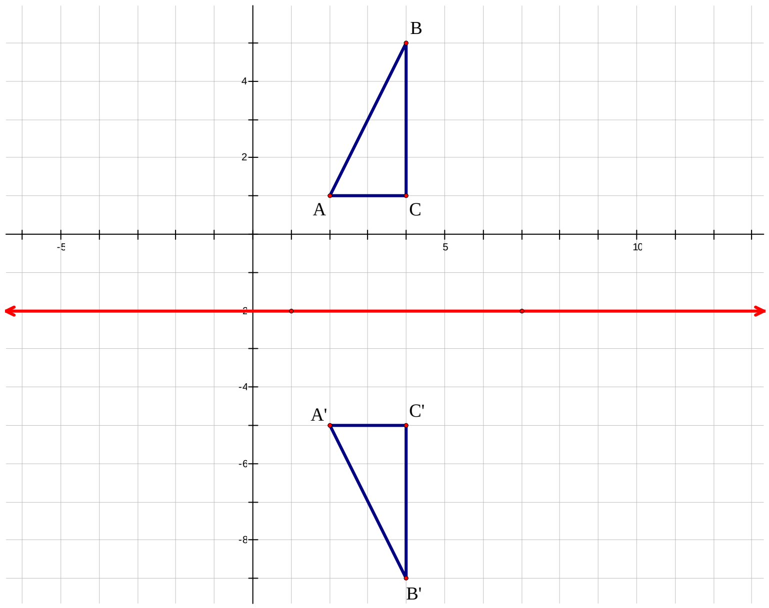
b)



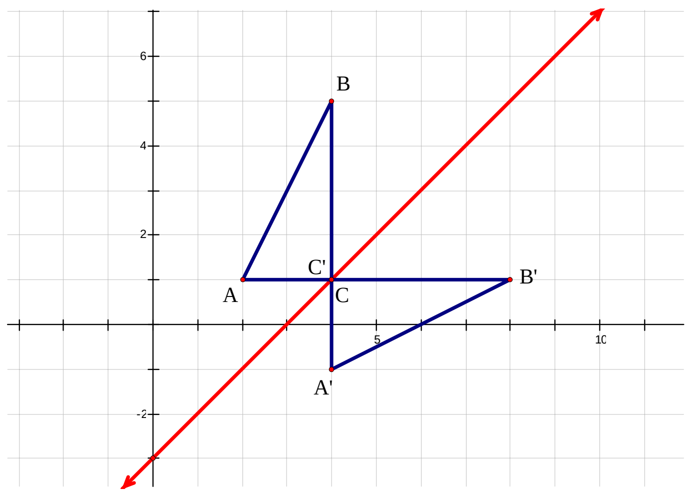
c)



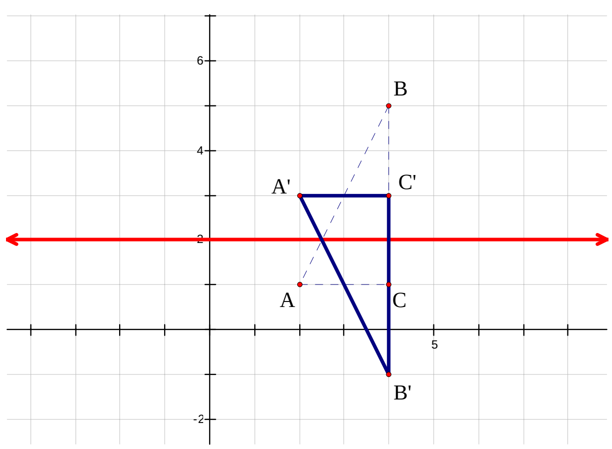
d)



e)



f)



The segments of the pre-image are dashed to avoid confusion between the pre-image and the image.

Exercises

Given the triangle with vertices $A(-6,-1)$, $B(-4,-3)$, and $C(-2,4)$,

1. Draw a diagram of ΔABC in the coordinate plane.
2. Reflect ΔABC across the following lines:
   a) The *x*-axis
   b) The *y*-axis
   c) The line $x=-1$
   d) The line $y=4$
   e) The line $y=x-1$
   f) The line $y=1$

Translations

A transformation can be thought of as a slide that involves no rotation. Some properties of translations are listed below.

<u>Properties of Translations</u>

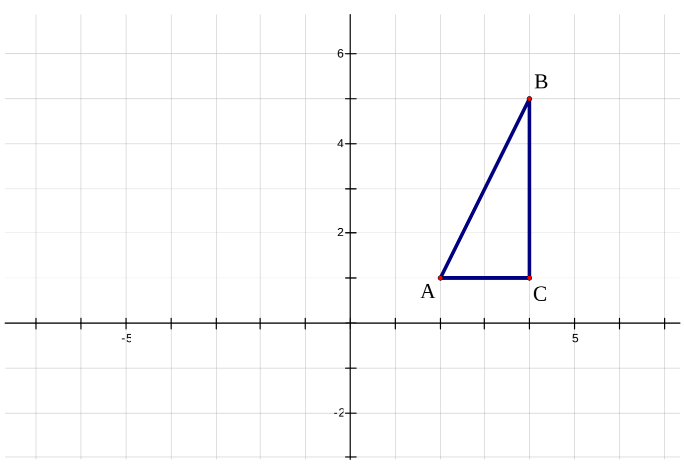
If a figure is translated:

1. The orientation of the image is the same as that of the pre-image.
2. The image is congruent to the pre-image. (Therefore, a translation is a congruence transformation, or isometry.)

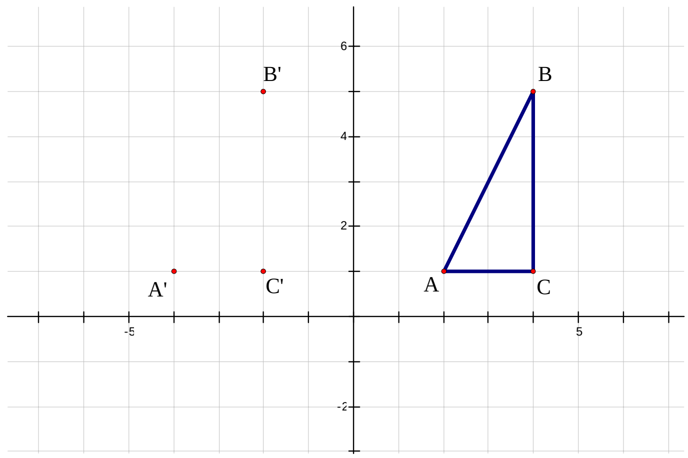
Translations are best introduced in the coordinate plane.

Example

Consider ΔABC below with vertices $A(2,1)$, $B(4,1)$, and $C(4,5)$.

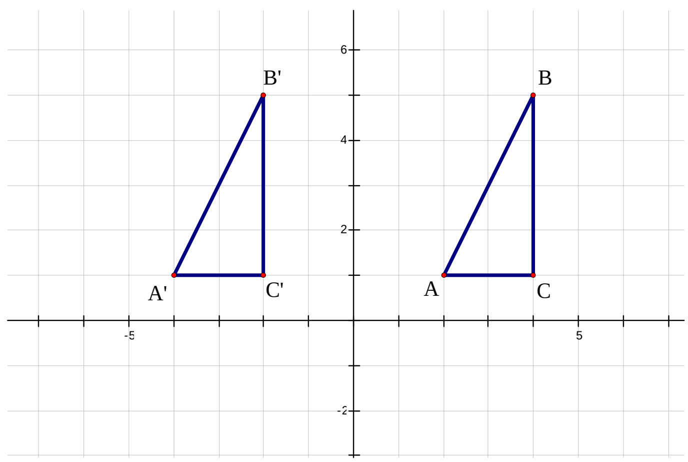


Suppose that we want to translate ΔABC six units to the left. We first move the vertices A, B, and C six units to the left and label them as A', B', and C', as shown below.



Next, we draw segments to complete the image, $\Delta A'B'C'$.

Final Solution:

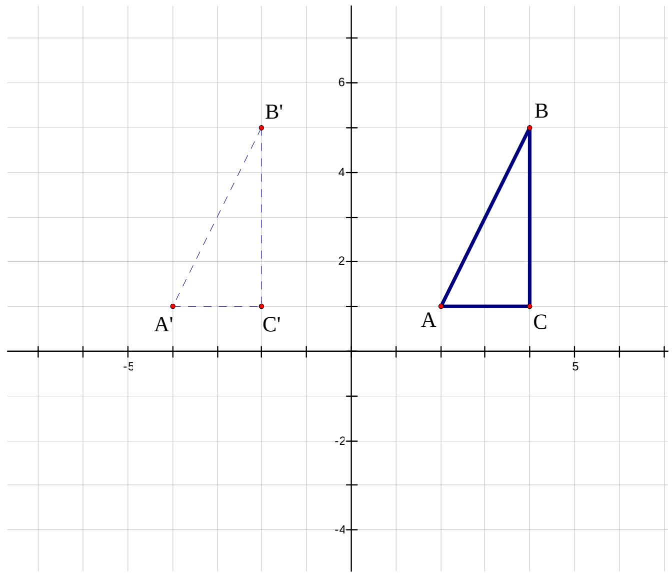


Example

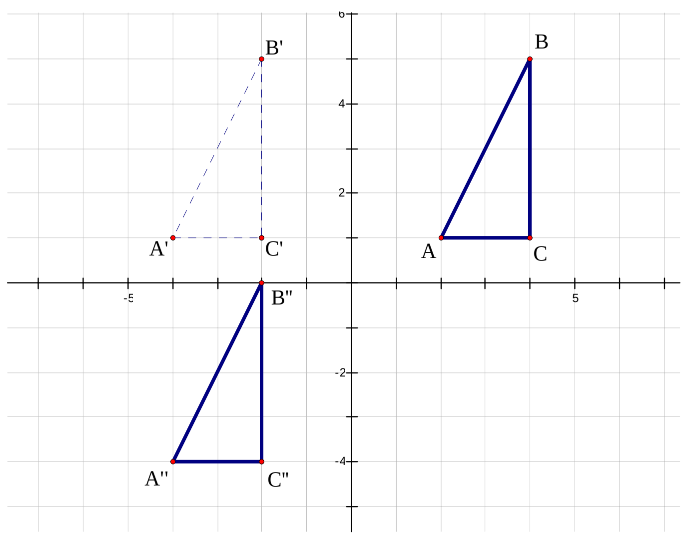
Let us again consider ΔABC with vertices $A(2,1)$, $B(4,1)$, and $C(4,5)$. Suppose that we instead want to translate ΔABC six units to the left and five units down.

We first translate ΔABC six units to the left as explained in the previous example, and we label the vertices A', B', and C'. Realize, however, that this image is not our final solution, as we still need to translate five units down.

Intermediate Step:



We now translate the intermediate figure five units down and label the appropriate vertices as A'', B'', and C'' to obtain our final image $\Delta A''B''C''$. (Notice the "double prime" notation, since two transformations were applied to the pre-image.)



Note: The above transformation could have been completed without the intermediate step, by simply moving each vertex left six units and down five units in one step. In that case, the final solution would be labeled with single prime notation, since only one image would be drawn. When two or more transformations are applied consecutively, this is known as a <u>composition</u> of transformations.

Exercises

Given the triangle with vertices $A(-6,-1)$, $B(-4,-3)$, and $C(-2,4)$,

1. Draw a diagram of ΔABC in the coordinate plane.

2. a) Translate ΔABC three units up.
   b) Translate ΔABC five units to the right.

3. a) Translate ΔABC four units down and two units to the left.
   b) Translate ΔABC two units to the left and four units down.
   c) Are the final images in parts (a) and (b) identical? (Are they not only congruent, but in the same location?)

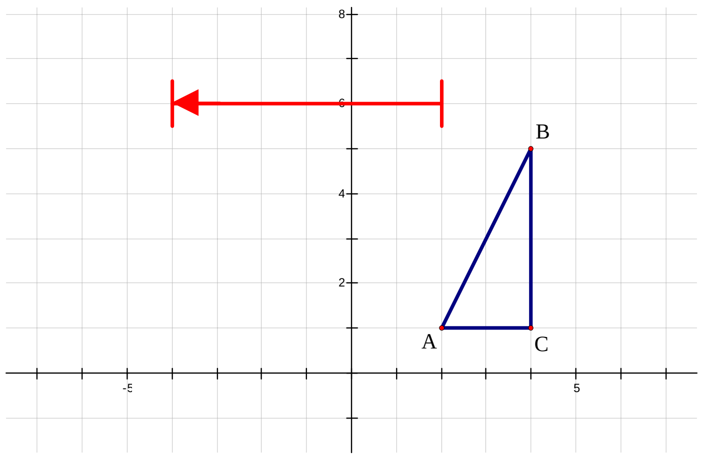
4. a) Translate ΔABC three units down and then reflect the intermediate image across the line $x=1$.
   b) Reflect ΔABC across the line $x=1$ and then translate the intermediate image three units down.
   c) Are the final images in parts (a) and (b) identical? (Are they not only congruent, but in the same location?)
   d) Consider the translation of any object down three units. Does this change the horizontal position of the object, or the vertical position?
   e) Consider the reflection of any object across the line $x=1$. Does this change the horizontal position of the object, or the vertical position?
   f) Make a conjecture about part (c) based on your answers to (d) and (e).

5. a) Translate ΔABC three units down and then reflect the intermediate image across the line $y=1$.
   b) Reflect ΔABC across the line $y=1$ and then translate the intermediate image three units down.
   c) Are the final images in parts (a) and (b) identical? (Are they not only congruent, but in the same location?)
   d) Consider the translation of any object down three units. Does this change the horizontal position of the object, or the vertical position?
   e) Consider the reflection of any object across the line $y=1$. Does this change the horizontal position of the object, or the vertical position?
   f) Make a conjecture about part (c) based on your answers to (d) and (e).

We will now use some of the previous examples from the coordinate plane to introduce the method of translating figures that are not within any coordinate system. Recall our first translation example, where ΔABC had vertices $A(2,1)$, $B(4,1)$, and $C(4,5)$ -- and we wanted to translate ΔABC six units to the left.

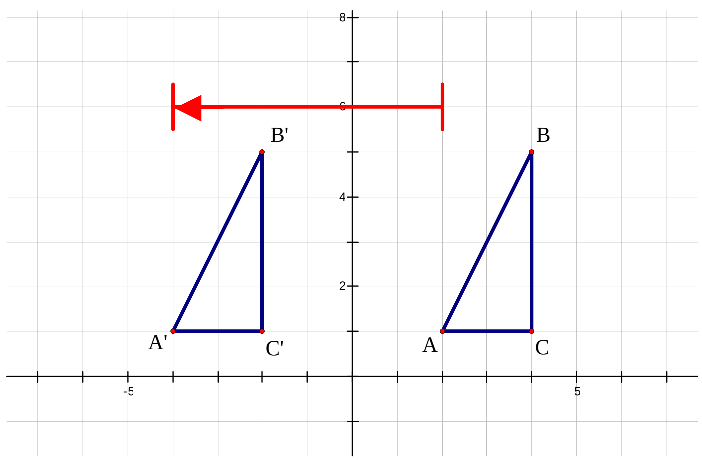
The crucial piece of information needed to perform this translation was the phrase "six units to the left." We needed to know both the amount and the direction in which ΔABC was to be translated. The information "six units to the left" is known as a <u>translation vector</u>. A <u>vector</u> specifies both magnitude (i.e. size – in this case "six units") and direction (in this case, "to the left"). A vector can easily be drawn to indicate both the size and direction of the translation, as shown in the figure below.

<u>Translation vector indicating a translation of "six units to the left"</u>



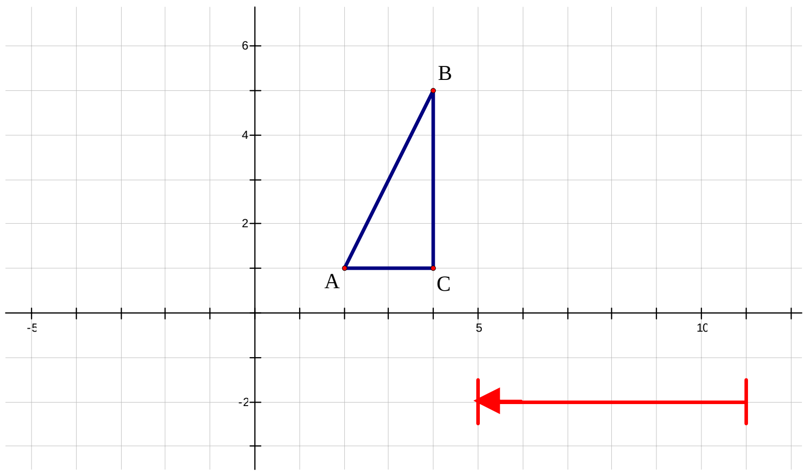
The length of the vector above clearly indicates that the vector is six units long, and the position of the arrow shows that the figure is to be translated directly to the left. The solution to the translation is shown below.

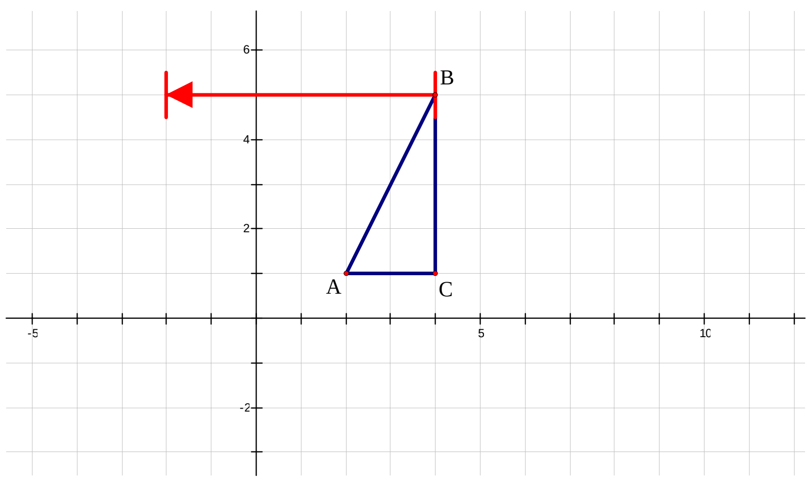
<u>Solution to the above translation problem:</u>



It is important to note that *the translation vector is not unique*. Two vectors are considered to be equal if their magnitude and direction are the same. The following diagrams are other examples of vectors indicating that ΔABC should be translated six units to the left, and all of the following examples yield the exact same image as in the solution found above.

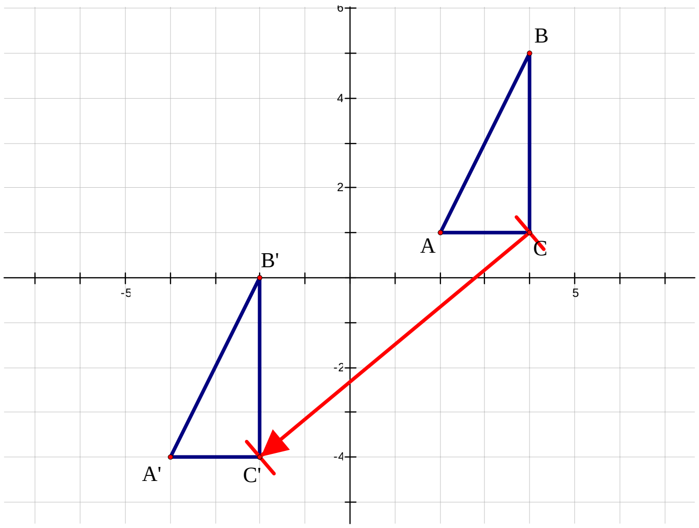
<u>Equal translation vectors which yield the same image as the solution above:</u>





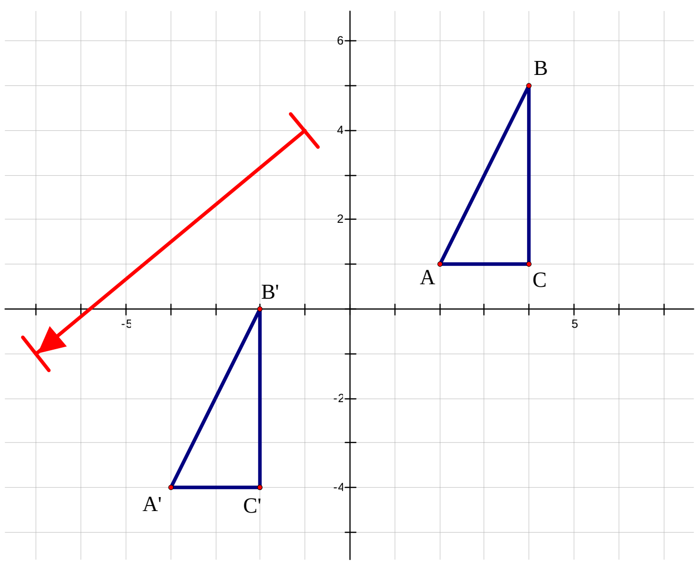
(Note: Many people like for the translation vector to be 'connected' to the pre-image as in the figure above, but this is not at all necessary.)

In our second translation example, we used the same triangle as above, and translated it six units to the left and five units down. A translation vector has been drawn to represent this translation, as shown below.



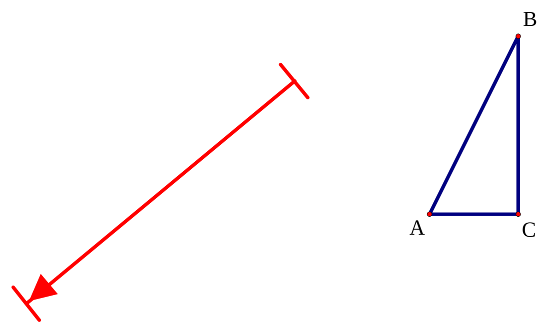
Notice how the translation vector was drawn between a point on the pre-image and its corresponding image point. Although this translation vector is not unique (and the two figures do not have to be 'connected' by the translation vector), this is certainly the easiest way to draw a translation vector when the pre-image and image are already given.

Another translation vector is drawn below which yield the same image. (Notice that in order to get from the starting point of the translation vector to its ending point, you need to move six units to the left and five units down.).



Now that we have a better understanding of translation vectors, we can better explore how to perform translations which lie outside of the coordinate plane. Let us take, for example, the previous diagram – but without the coordinate plane (and without the solution showing!)

Example
Translate ΔABC using the translation vector below:

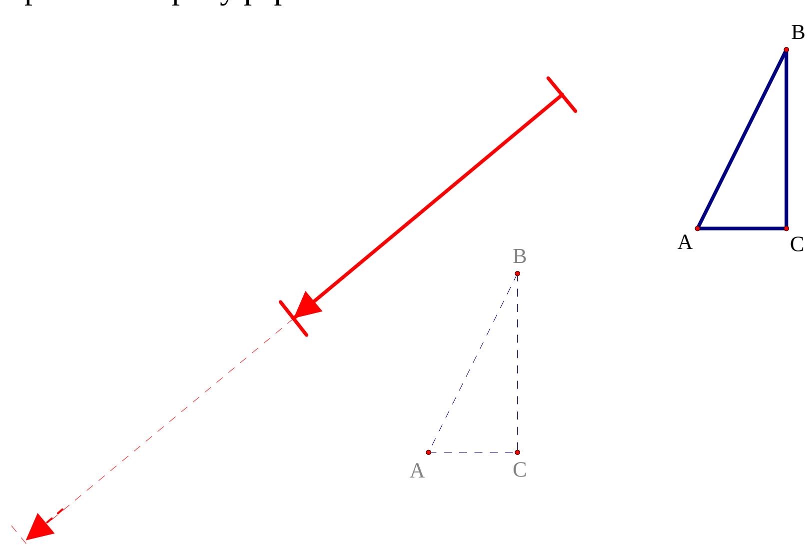


As with reflections outside the coordinate plane, we will be using patty paper (or thin paper) to perform the translation. First, place a piece of patty paper on top of the original diagram. Trace the pre-image as well as the translation vector.

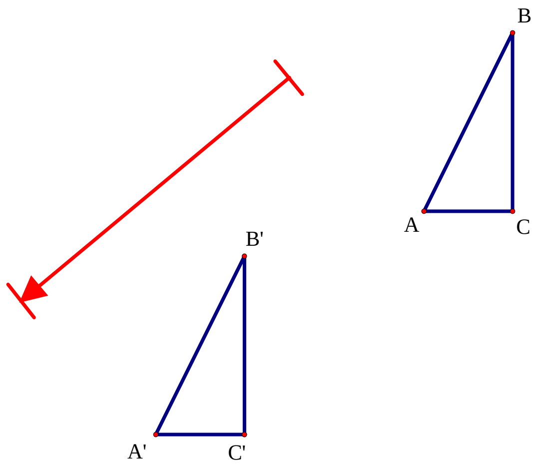
We now have two options as to how to perform the translation, as described below.

> <u>Option 1:</u>
> Keeping the patty paper on top of the initial diagram, slide the ***bottom piece of paper (not patty paper) in the direction of the translation vector*** and for the entire distance of the translation vector.
>
> <u>Option 2:</u>
> Keeping the patty paper on top of the initial diagram, ***slide the top piece of paper (the patty paper) in the OPPOSITE direction of the translation vector*** and for the entire distance of the translation vector.

A diagram is shown below which represents what you will see using either of the two methods described above. The dotted figures represent the lower piece of paper and the solid figures represent the patty paper.



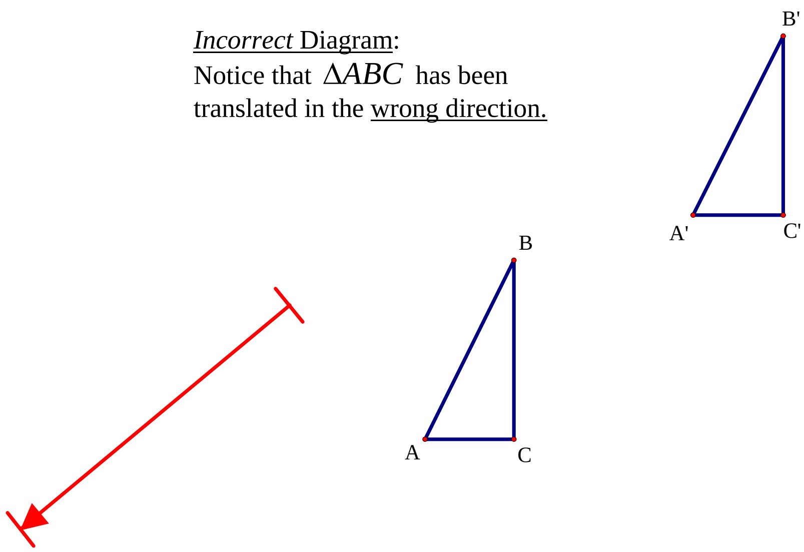
Now trace the triangle that you see from the bottom paper on to the patty paper. *We would not recommend re-tracing the image of the translation vector, as it may cause confusion.* As you re-trace ΔABC, label the vertices of the image as A', B', and C'. Then remove the patty paper from the bottom sheet of paper. A diagram of the patty paper, which represents the final solution, is shown below.



We would recommend that you experiment with the two methods above, decide which method you like best, and then stick with one method. A common error in performing these types of translations is to move the paper in the wrong direction, which places the image in the wrong position.

If you move the paper in the wrong direction, you will obtain the diagram below. (After performing a translation, you should always check the location of the image in relation to the pre-image to make sure that you did not slide the paper in the wrong direction.)

Incorrect Diagram:
Notice that ΔABC has been translated in the wrong direction.



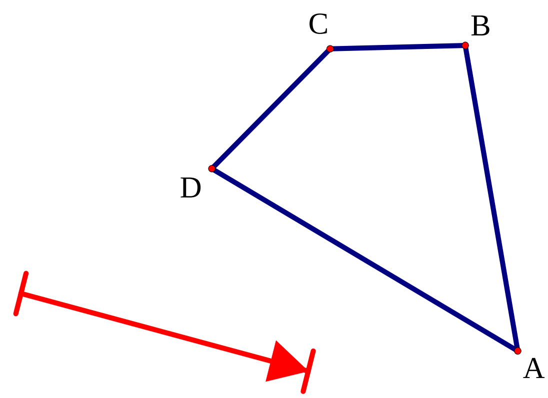
Another common error is to treat the translation vector as a reflection line. A reflection line has two arrows, while a translation vector has one arrow indicating the direction of the translation as well as ‘caps’ on both ends to show the magnitude of the translation.

Examples

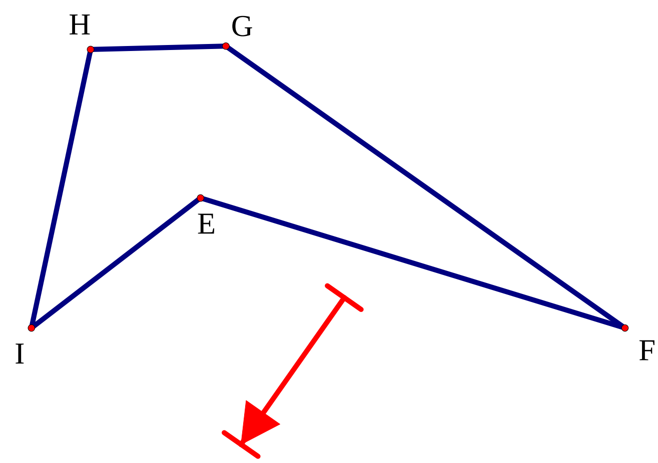
Answer the following.

1. Translate the figures below as indicated by the translation vectors.

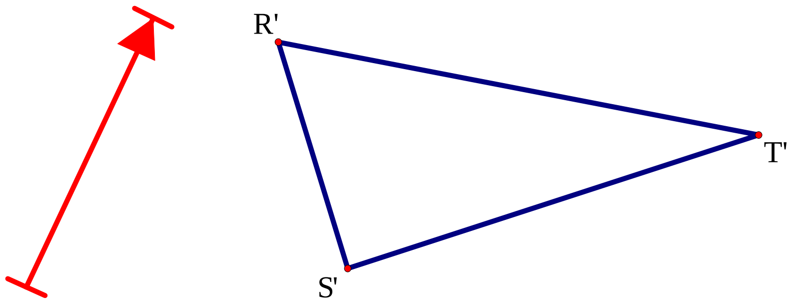
a)



b)

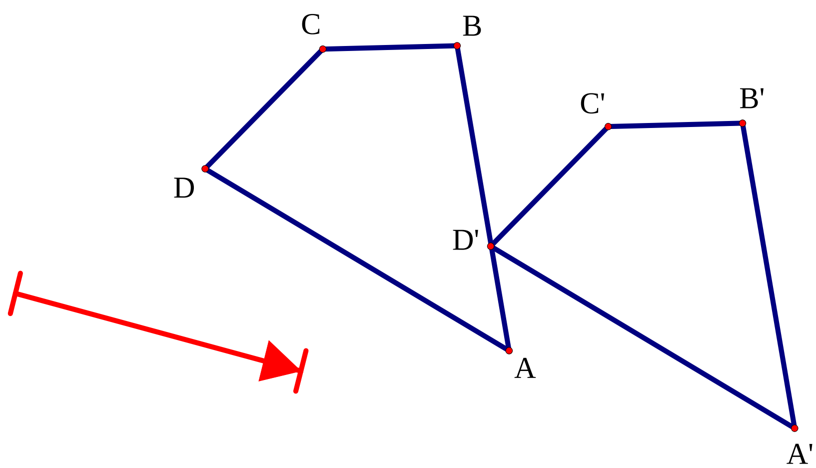


2. ΔRST has been translated using the following translation vector, resulting in the image below. Draw ΔRST .

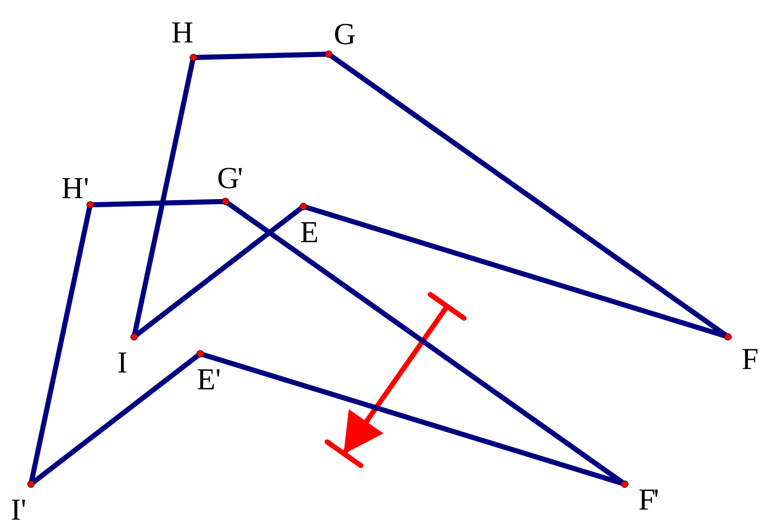


Solutions:

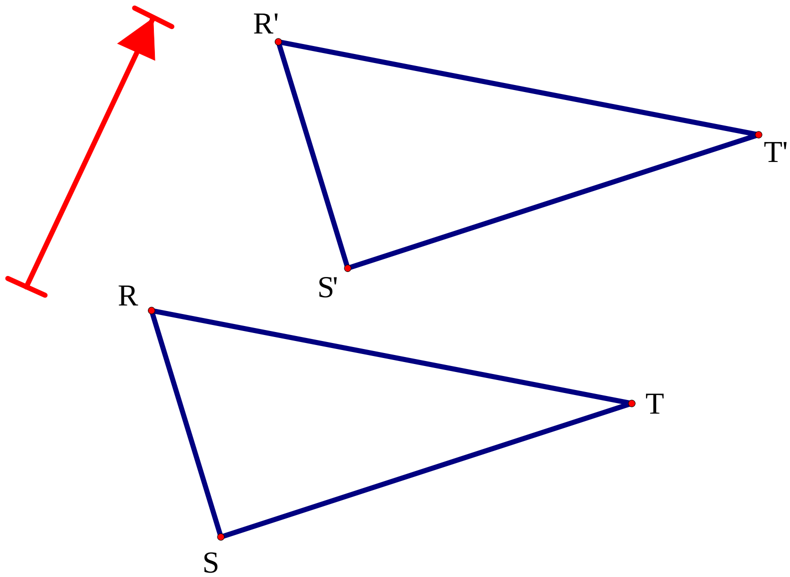
1. a)



b)



2. In this problem, the image, $\Delta R'S'T'$ is given and we need to find the pre-image, ΔRST . We need to work backwards to find the pre-image, and therefore must move the paper in the *opposite* direction of our normal method of translation. The solution is found below.

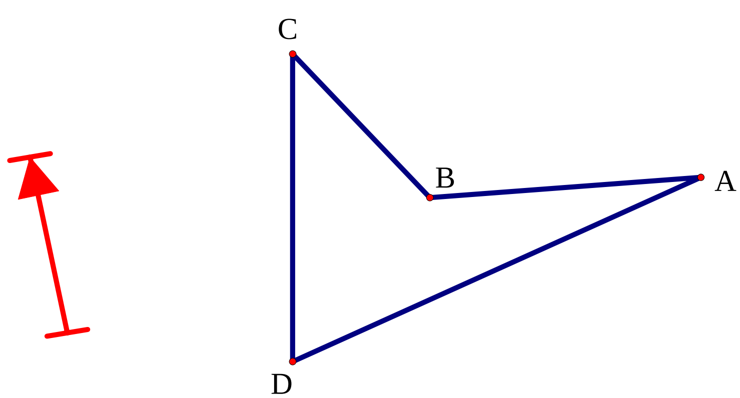


Activity
When a figure is reflected over a line, and then the intermediate image is reflected over another line which is parallel to the first, a translation is obtained. For further exploration of this topic, refer to the activity entitled "Translations with Patty Paper."

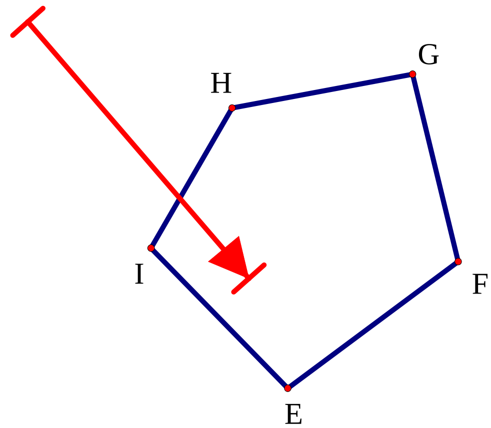
Exercises
Answer the following.

1. Translate the figures below as indicated by the translation vectors.

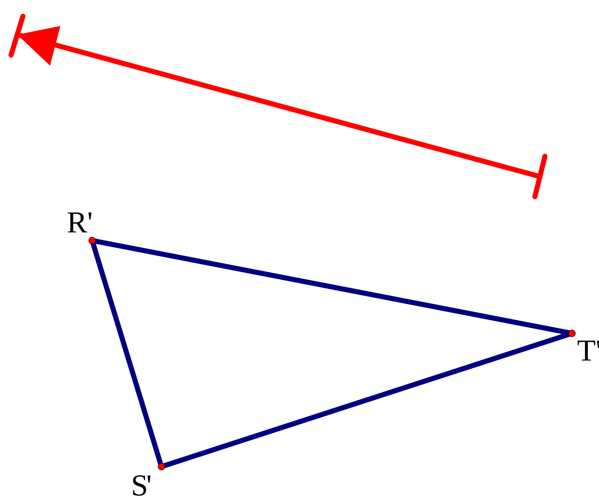
a)



b)



2. ΔRST has been translated using the following translation vector, resulting in the image below. Draw ΔRST.



3. Draw two parallel lines and label them as ℓ and m. (You may want to trace two parallel lines from notebook paper.) Then draw ΔTRA (a triangle of your choice). Reflect ΔTRA over line ℓ and label the image as $\Delta T'R'A'$. Then reflect $\Delta T'R'A'$ over line m and label the final image as $\Delta T''R''A''$. Compare ΔTRA with $\Delta T''R''A''$. What do you notice?

Rotations

A rotation is a transformation whereby an figure is turned in a circular arc around a fixed center point. The center point is known as the center of rotation.

Some properties of rotations are listed below.

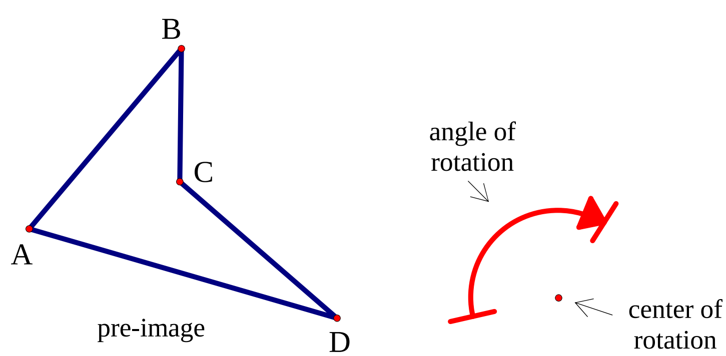
Properties of Rotations

If a figure is rotated:

1. The orientation of the image is the same as that of the pre-image.
2. The image is congruent to the pre-image. (Therefore, a rotation is a congruence transformation, or isometry.)

We will begin our discussion of rotations without reference to the coordinate plane.

Example
Consider the diagram below. We wish to rotate quadrilateral *ABCD* around the given center of rotation, by the given angle of rotation.



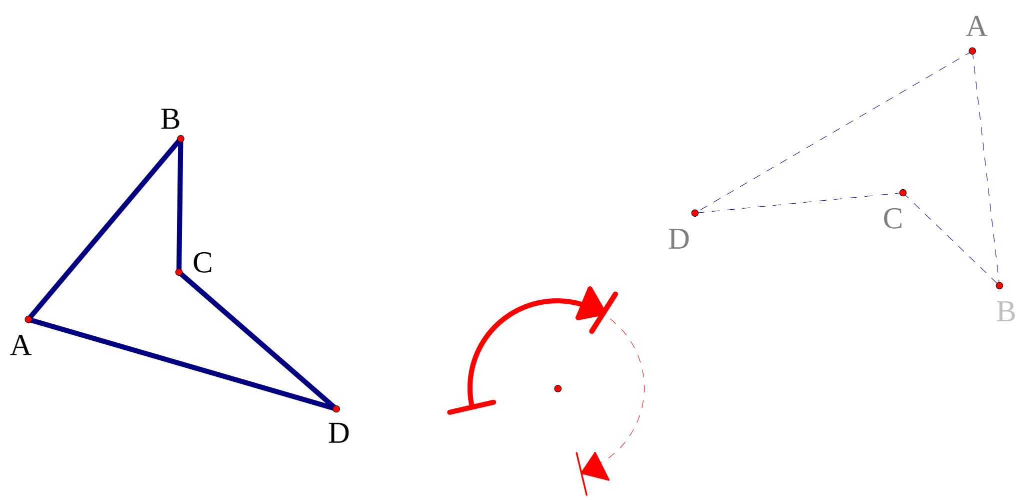
We will be using patty paper (or thin paper) to perform the rotation. First, place a piece of patty paper on top of the original diagram. Trace the pre-image as well as center of rotation and the angle of rotation.

We now have two options as to how to perform the rotation, as described below.

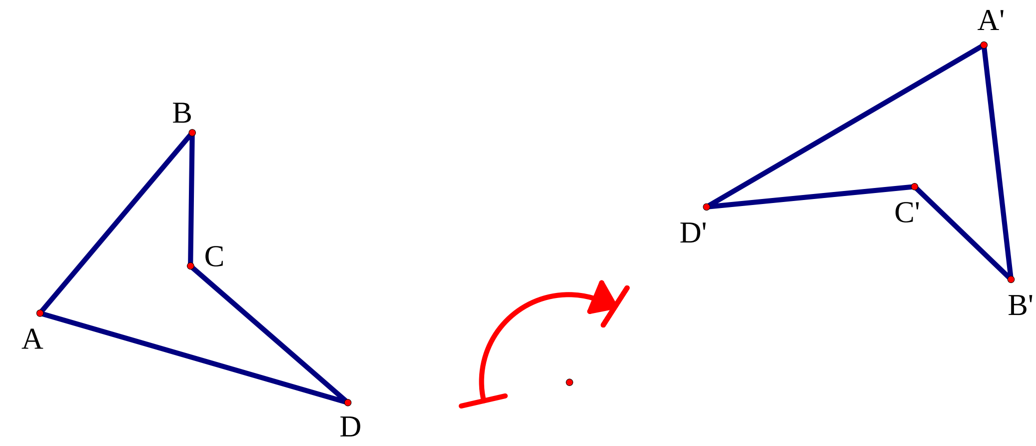
Option 1:
Keeping the patty paper on top of the initial diagram and keeping the center of rotation aligned in the top and bottom pieces of paper, ***rotate the bottom piece of paper (not patty paper) in the direction of the angle of rotation*** and for the entire measure of the angle of rotation.

Option 2:
Keeping the patty paper on top of the initial diagram and keeping the center of rotation aligned in the top and bottom pieces of paper, ***rotate the top piece of paper (the patty paper) in the OPPOSITE direction of the angle of rotation*** and for the entire measure of the angle of rotation.

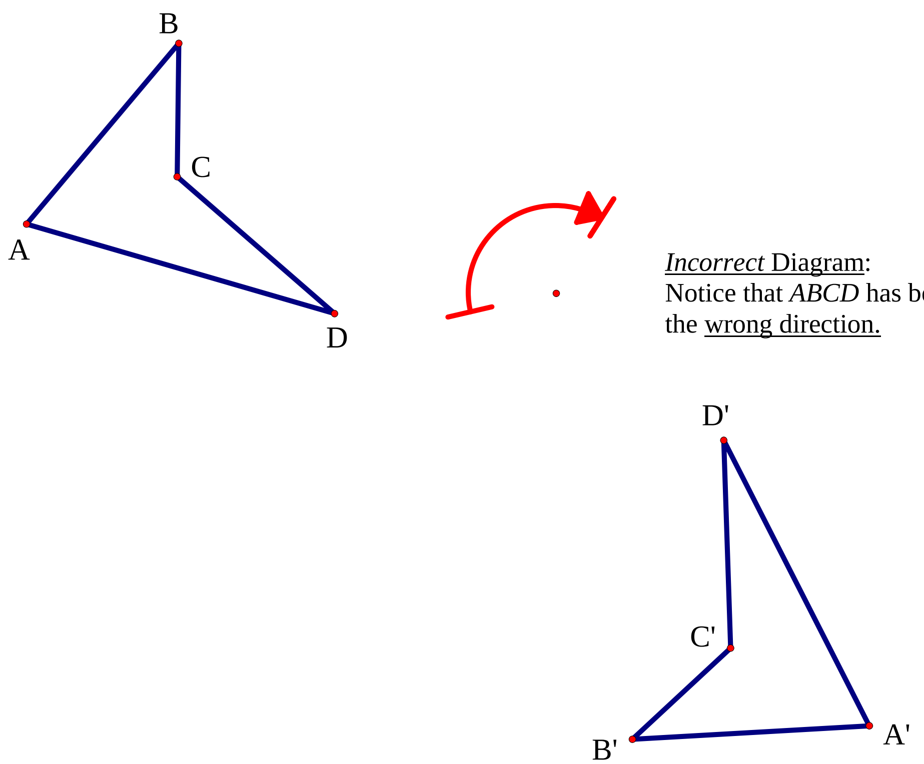
A diagram is shown below which represents what you will see using either of the two methods described above. The dotted figures represent the lower piece of paper and the solid figures represent the patty paper.



Now trace the quadrilateral that you see from the bottom paper on to the patty paper. *We would not recommend re-tracing the image of the angle of rotation, as it may cause confusion.* As you re-trace quadrilateral *ABCD*, label the vertices of the image as A', B', C' and D'. Then remove the patty paper from the bottom sheet of paper. A diagram of the patty paper, which represents the final solution, is shown below.



Notice that the basic methodology is similar to that of the translations. (If moving the bottom paper, move in the direction of the arrow; if moving the top paper, move in the opposite direction of the arrow.)
If you move the paper in the wrong direction, you will obtain the diagram which is found on the next page. (After performing a rotation, ask yourself if the angle of rotation is supposed to be greater or less than 180 degrees; this may help you to gauge whether or not your image is placed correctly.)

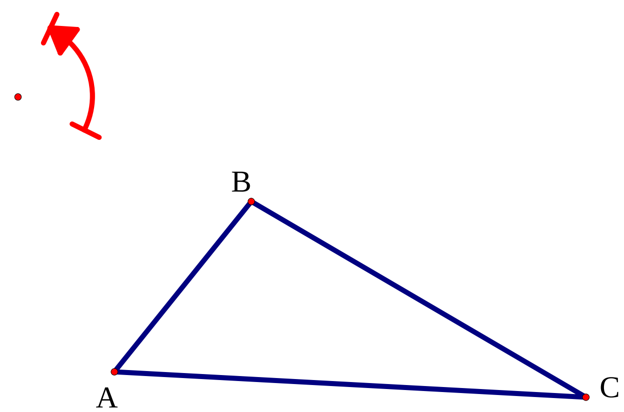


Incorrect Diagram:
Notice that *ABCD* has been rotated in the wrong direction.

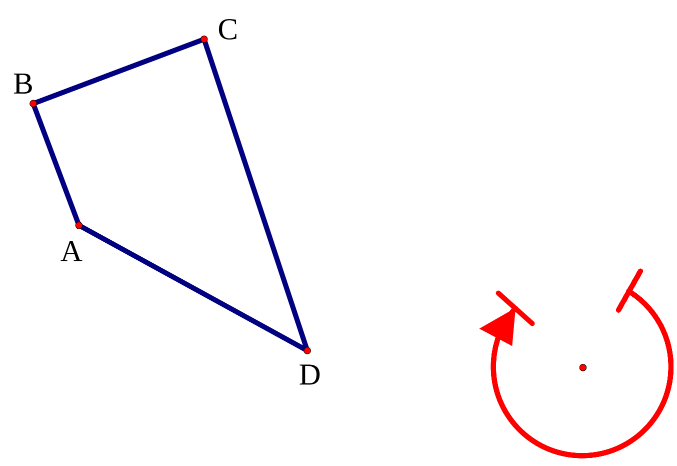
Examples

Rotate the following figures using the given center and angle of rotation.

1.

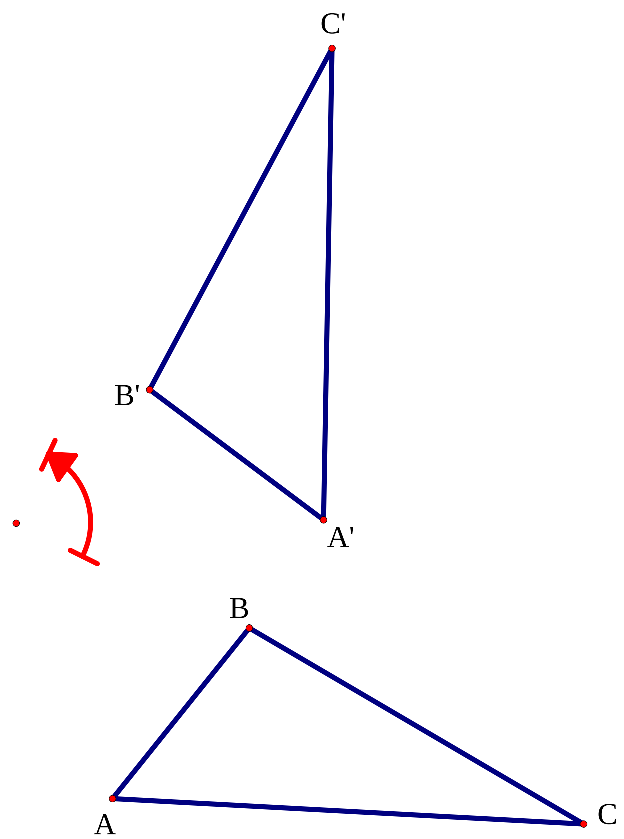


2.

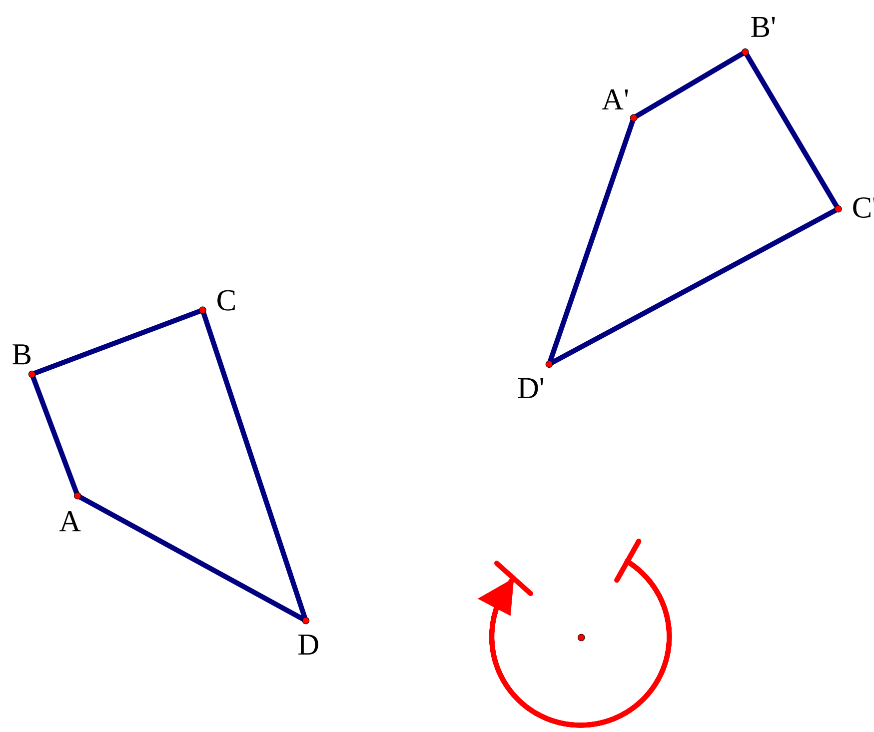


Solutions:

1.

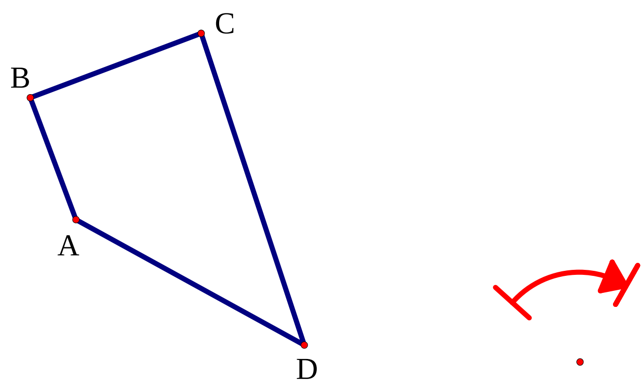


2. Examples such as this one are frequently more difficult, because the angle of rotation is greater than 180 degrees. Two solutions are shown below, a direct solution, and another solution which might be more easily obtained.

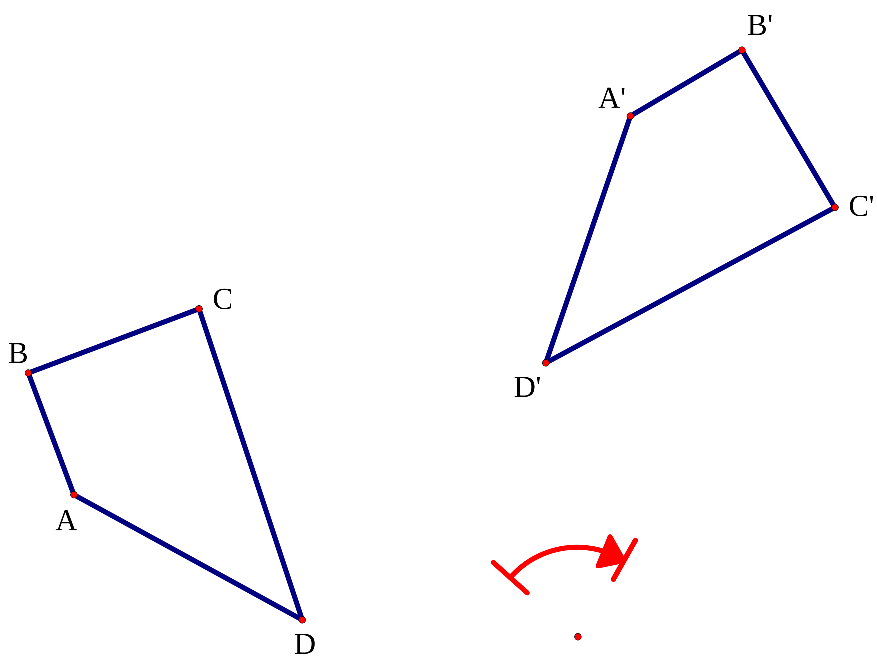


If it is difficult to perform a rotation that is greater than 180 degrees, you may want to consider doing the following. The angle of rotation in the following diagram is a different angle of rotation, but will place the image in the same position as the diagram above.

<u>Equivalent angle of rotation for Example #2:</u>



Below is an equivalent solution for Example 2. Notice that the image is the same as the one shown on the previous page; only the angle of rotation is different.



Activity
When a figure is reflected over a line, and then the intermediate image is reflected over another line which intersects the first, a rotation is obtained. For further exploration of this topic, refer to the activity entitled "Rotations with Patty Paper."

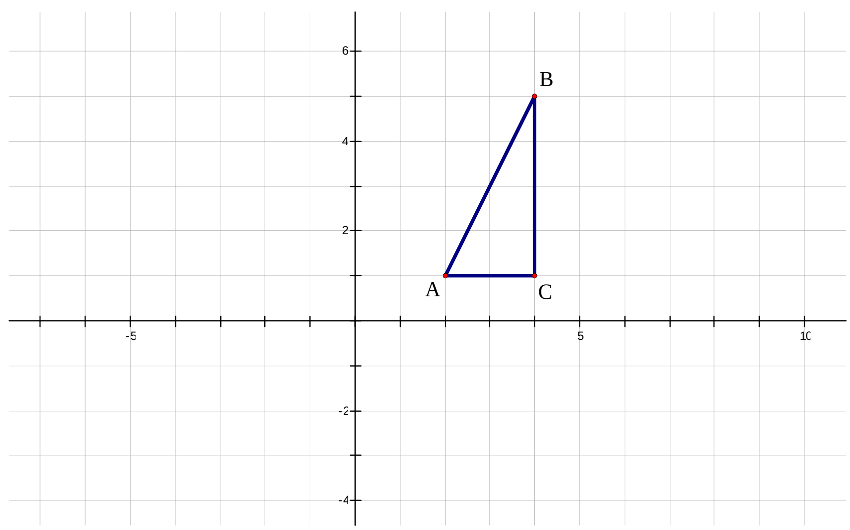
<u>Rotations Within the Coordinate Plane</u>

Rotations are a bit more difficult within the coordinate plane. Because of this fact, we will limit our angle of rotation to multiples of 90 degrees. If the original vertices of the figure have coordinates that are integers and the rotation is a multiple of 90 degrees, then the coordinates of the image will also have coordinates that are integers.

When given the angle of rotation, <u>a positive angle indicates a counterclockwise rotation, and a negative angle indicates a clockwise rotation</u>. (Note: This is the same standard that is used when measuring angles in Trigonometry.)

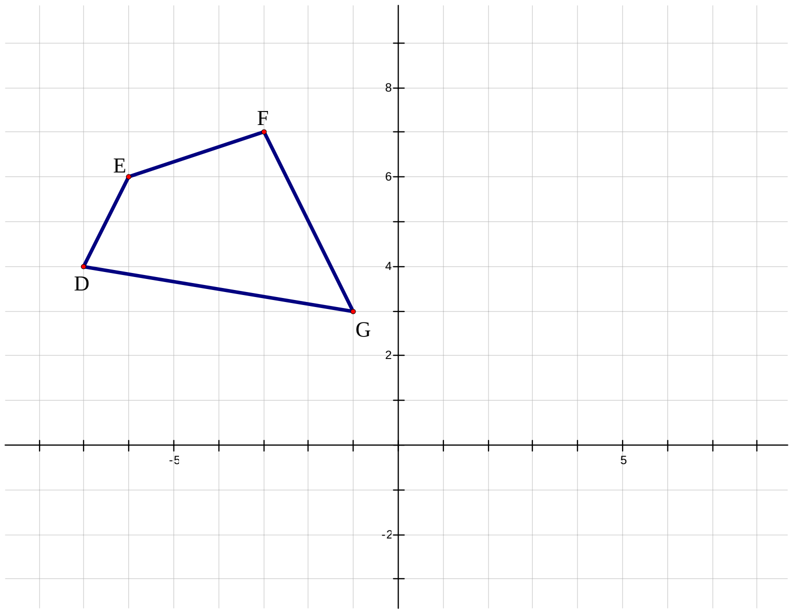
Examples

1. Rotate ΔABC with the given center and angle of rotation. Draw each image on a new coordinate plane along with its pre-image and rotation arrow.



a) Center of rotation: $(0, 0)$ Angle of rotation: 90°
b) Center of rotation: $(0, 0)$ Angle of rotation: 180°
c) Center of rotation: $(-3, 2)$ Angle of rotation: -90°
d) Center of rotation: $(2, -1)$ Angle of rotation: -270°

2. Rotate quadrilateral *DEFG* with the given center and angle of rotation. Draw each image on a new coordinate plane along with its pre-image and rotation arrow.



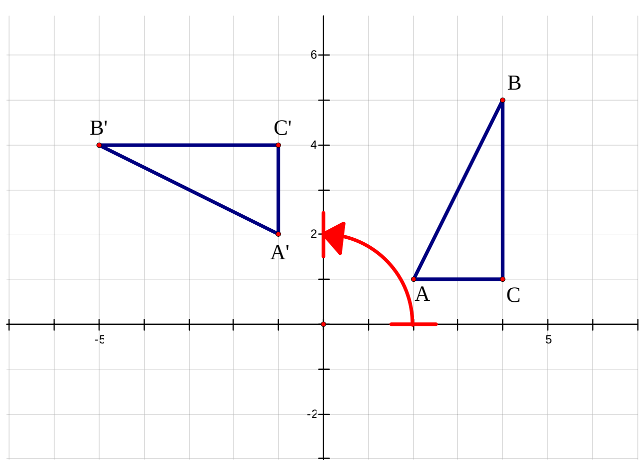
a) Center of rotation: (0, 0) Angle of rotation: 90°

b) Center of rotation: (–4, – 2) Angle of rotation: –90°

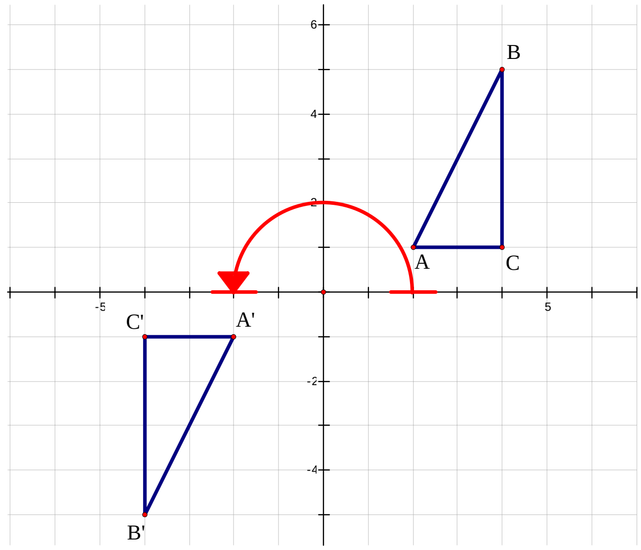
<u>Solutions:</u>

The first step for each rotation is to draw the center and angle of rotation. Use a compass to draw the arc for the angle of rotation, and use the gridlines to draw the angle of rotation, since all angles in these examples are multiples of 90 degrees. (If other angles were given, you would need to use a protractor.) To draw the image, you can then use the same method of tracing as we used with the non-coordinate plane rotations. (You should use thin graph paper instead of patty paper, however, so that your final solutions are within the coordinate plane.)

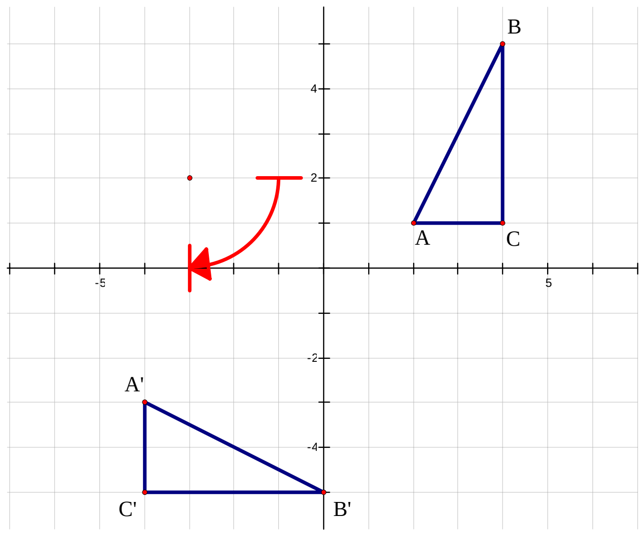
1. a)



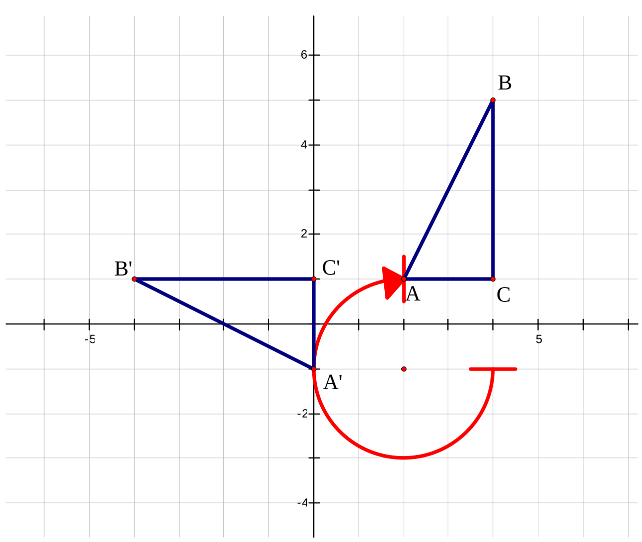
b)



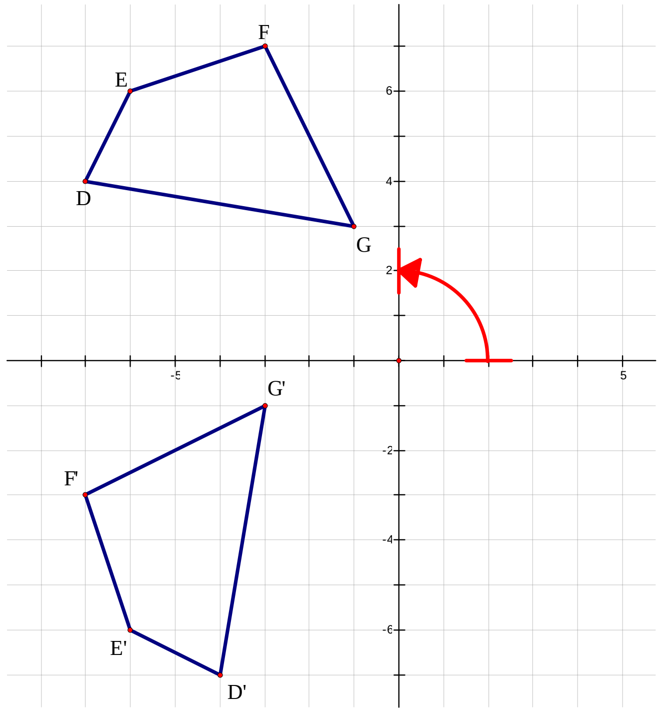
c)



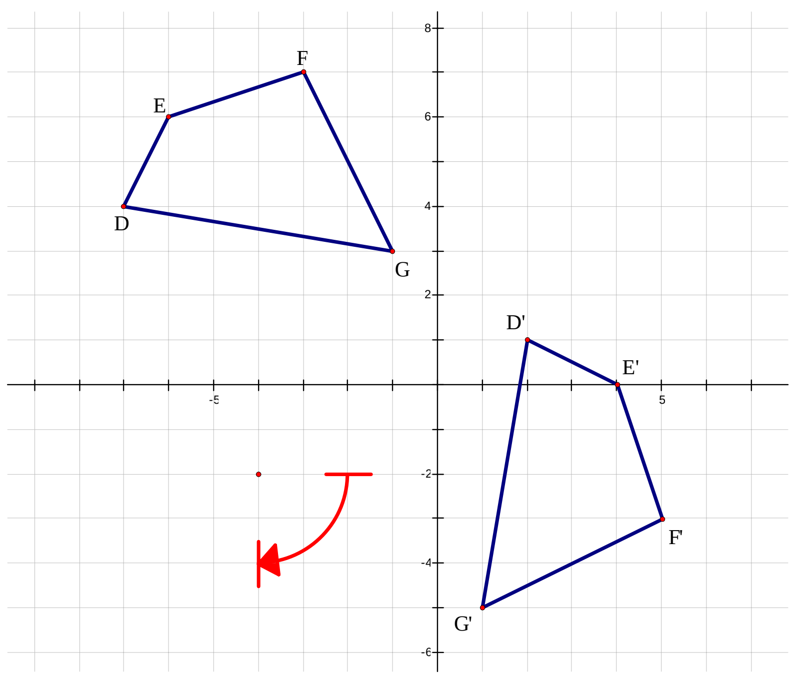
d)



2. a)

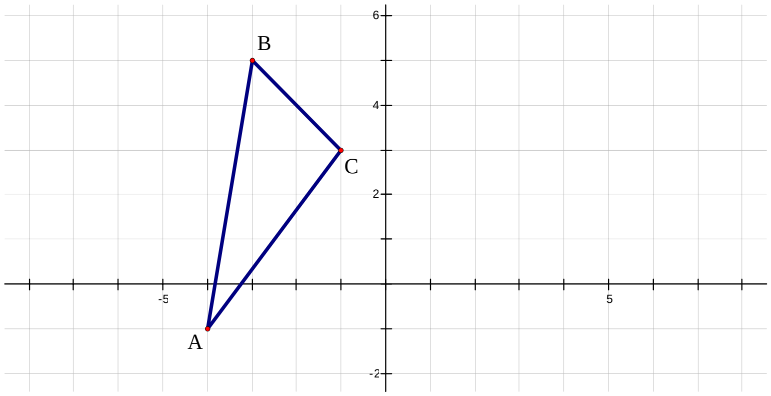


b)



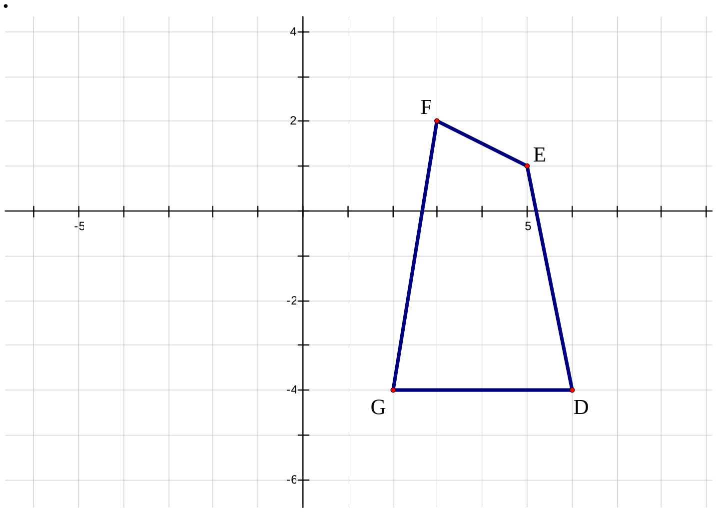
Exercises

1. Rotate ΔABC with the given center and angle of rotation. Draw each image on a new coordinate plane along with its pre-image and rotation arrow.



a) Center of rotation: $(0, 0)$ Angle of rotation: 90°
b) Center of rotation: $(0, 0)$ Angle of rotation: 270°
c) Center of rotation: $(3, 2)$ Angle of rotation: -180°
d) Center of rotation: $(-2, 0)$ Angle of rotation: -90°

2. Rotate quadrilateral $DEFG$ with the given center and angle of rotation. Draw each image on a new coordinate plane along with its pre-image and rotation arrow.



a) Center of rotation: $(0, 0)$ Angle of rotation: 180°
b) Center of rotation: $(-1, 2)$ Angle of rotation: 90°

3. Draw two intersecting lines and label them as ℓ and m. (These do not have to be drawn within a coordinate plane.) Then draw ΔTRA. Reflect ΔTRA over line ℓ and label the image as $\Delta T'R'A'$. Then reflect $\Delta T'R'A'$ over line m and label the image as $\Delta T''R''A''$. Compare ΔTRA with $\Delta T''R''A''$. What do you notice?

Dilations

In the English language, the word "dilate" means to widen or enlarge. In transformational geometry, however, the word takes on a broader meaning (although it sometimes does enlarge a figure.) Earlier in this module, we studied the concept of similarity. A dilation is known as a *similarity transformation* because it creates an image that is similar to the pre-image. It is possible that the image will in fact be larger than the pre-image, but the image can also be smaller or even congruent to the pre-image.

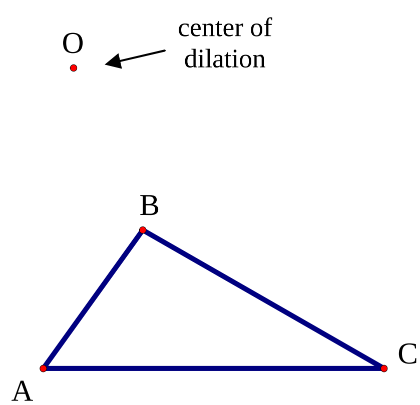
To dilate a pre-image, we need the following information:

1. A center of dilation
2. A scale factor

Just as the center of rotation was a crucial part of determining the location of the image for rotations, the center of dilation is an equally critical element in a dilation.

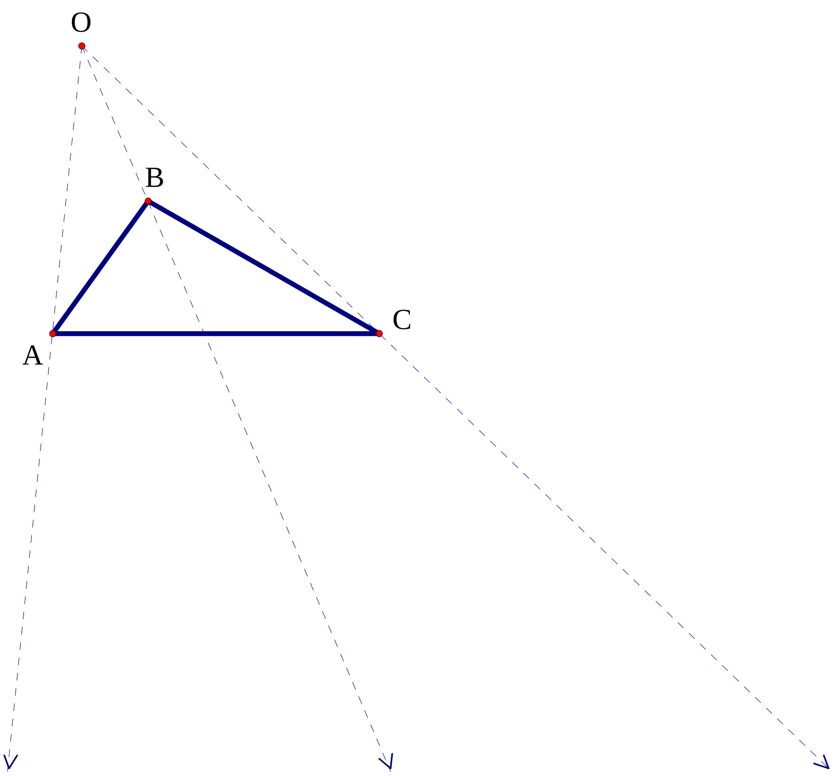
In discussing previous transformations, we have listed properties of the transformation at the beginning of the section. In this section, we will list the properties later in the section, after the concept of dilations has been more thoroughly explored.

Consider ΔABC below with the given center of dilation, *O*. Suppose that we want to dilate ΔABC by a scale factor of 2.

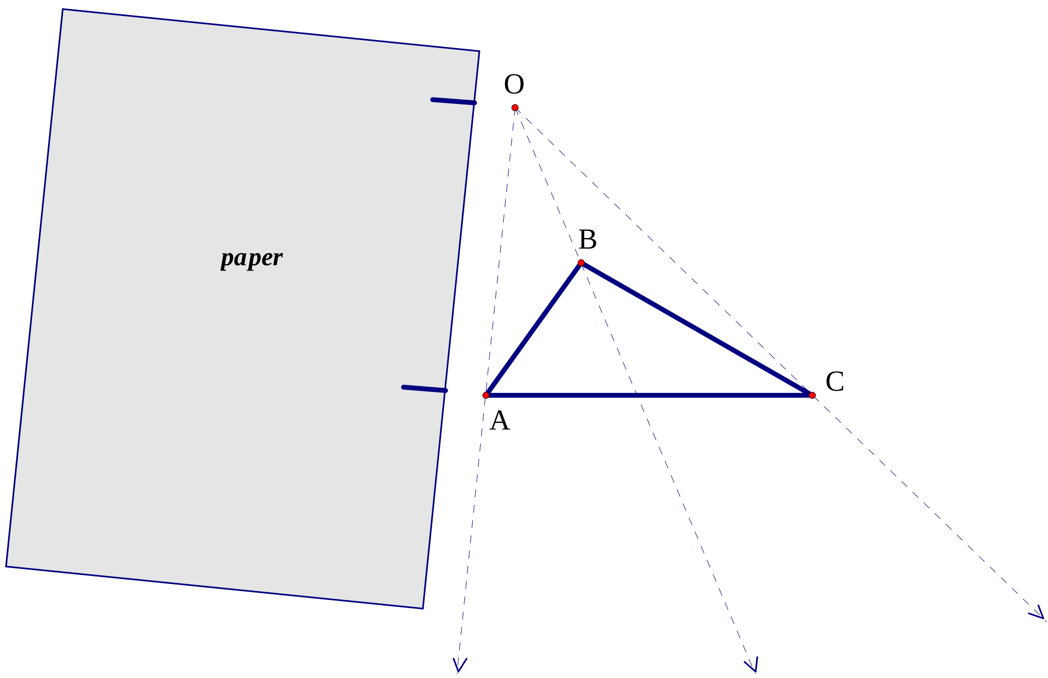


First, trace the pre-image onto patty paper, along with the center of dilation.

To perform this dilation, we will draw rays originating at the center of dilation which pass through each of the vertices of the original figure, as shown below. (Note: When the scale factor is negative, we need to apply a slightly different technique, which will be discussed later in this section.)

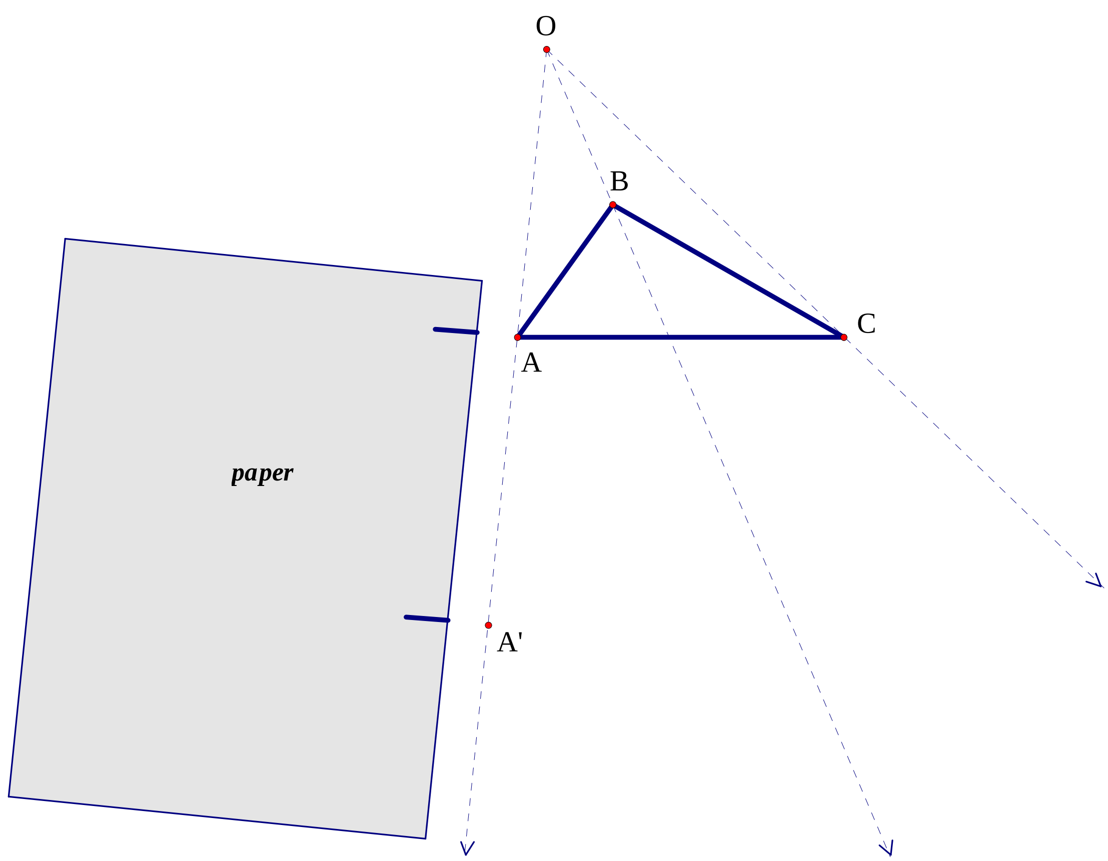


Since the scale factor is 2, we want each of the vertices of the image to be twice as far away from the center of dilation as the original vertices of the pre-image. To accomplish this goal, we can simply use the edge of a piece of paper as our tool. Take a piece of paper and align it with $\overrightarrow{OA}$, and put "tick marks" on the edge of the paper at points O and A. This method can be illustrated by the diagram on the next page. (The piece of paper is shaded in gray and in reality would be placed even closer to $\overrightarrow{OA}$.)



Using this measurement as your unit, slide the edge of the paper downward so that the top tick mark is aligned with point A, and then place a point on $\overrightarrow{OA}$ at the position of the second tick mark. Label this new point as A', as shown below.

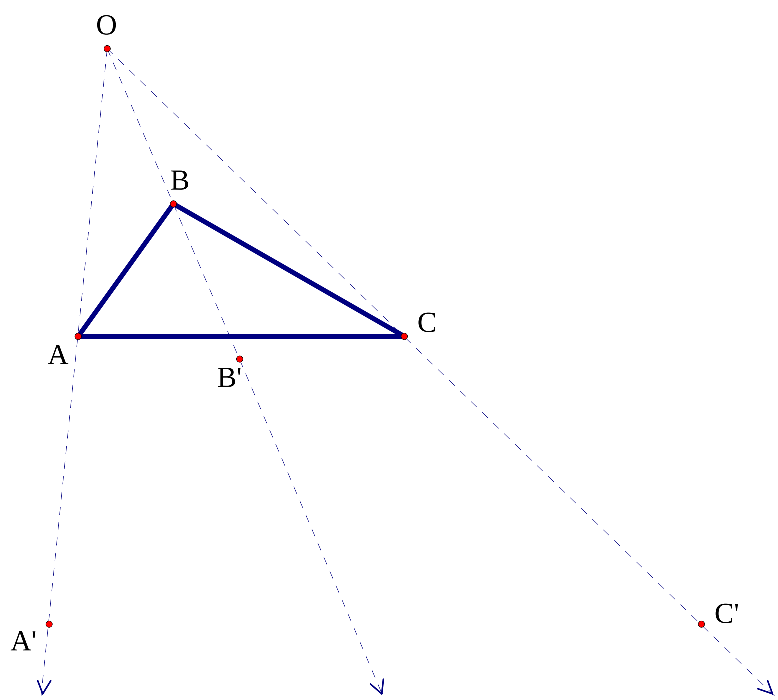
\\



Repeat the same process to identify the location of B': Measure the length of $\overline{OB}$ using the edge of your paper and tick marks. (The tick marks will be spaced differently than $\overline{OA}$; you may want to use a different edge of the paper.) Using this as your unit, slide the paper downward one more unit to determine the location of B'.

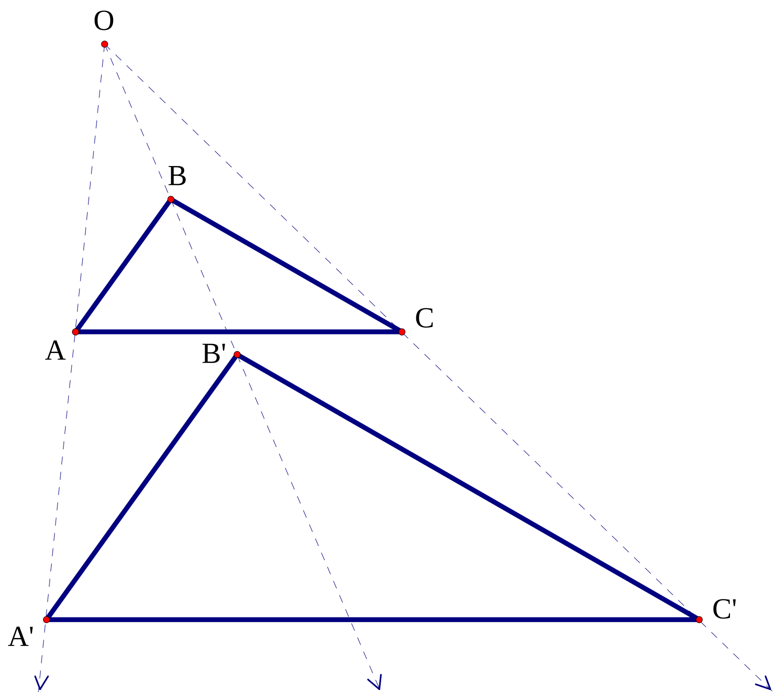
Finally, repeat the process to find the location of C': Measure the length of $\overline{OC}$ using the edge of your paper and tick marks. Using this as your unit, slide the paper downward one more unit to determine the location of C'.

A diagram showing the location of A', B', and C' is shown below.



Now that we have points A', B', and C', the final step is simply to connect the points with segments to complete the image, $\Delta A'B'C'$. A diagram of the dilation is shown on the next page.

Solution: ΔABC has been dilated about point O, with a scale factor of 2.



Let us examine the above diagram more closely to examine the properties of dilations.

Exploration

1. Measure the three angles of ΔABC with a protractor, and then measure the three angles of $\Delta A'B'C'$. (Round your measurements to the nearest degree.) What do you notice?

2. Measure the three sides of ΔABC with a ruler, and then measure the three sides of $\Delta A'B'C'$. (Round your measurements to the nearest centimeter.) What do you notice?

3. Compare $\overline{AB}$ with $\overline{A'B'}$, $\overline{BC}$ with $\overline{B'C'}$, and $\overline{AC}$ with $\overline{A'C'}$. Can you make any additional observations about the corresponding sides that were not made in number 2 above?

Solutions to this exploration can be found on the following page.

Solutions:

1. $\angle A \approx 54^\circ$ $\angle B \approx 96^\circ$ $\angle C \approx 30^\circ$
   $\angle A' \approx 54^\circ$ $\angle B' \approx 96^\circ$ $\angle C' \approx 30^\circ$

   The corresponding angles of ΔABC and $\Delta A'B'C'$ are congruent. This means that the two triangles are similar to each other. (They are similar by Angle-Angle Similarity, also known as AA Similarity. For more information about similarity, refer to the unit on similar polygons.)

2. $AB \approx 2.7$ cm $BC \approx 4.4$ cm $AC \approx 5.3$ cm
   $A'B' \approx 5.3$ cm $B'C' \approx 8.7$ cm $A'C' \approx 10.7$ cm

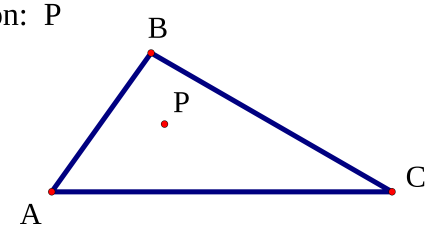
   The sides of $\Delta A'B'C'$ appear to be (and in fact are) twice the length of each of their corresponding sides of ΔABC. (Realize that the measurements above are only given to the nearest tenth of a centimeter.) The ratio of the side lengths (of the image to the pre-image) is the same as the scale factor of the dilation.

3. The corresponding sides of ΔABC and $\Delta A'B'C'$ are parallel:
   $\overline{AB} \parallel \overline{A'B'}$ $\overline{BC} \parallel \overline{B'C'}$ $\overline{AC} \parallel \overline{A'C'}$

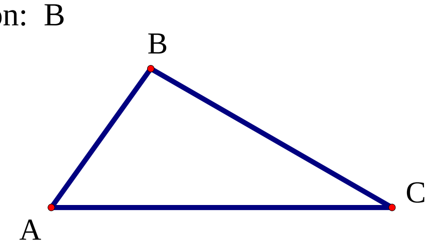
Suppose that we wanted to again dilate ΔABC, but with a different center of dilation. Several examples are shown below.

Examples
Dilate ΔABC by a scale factor of 2, using the given center of dilation.

1. Center of dilation: P

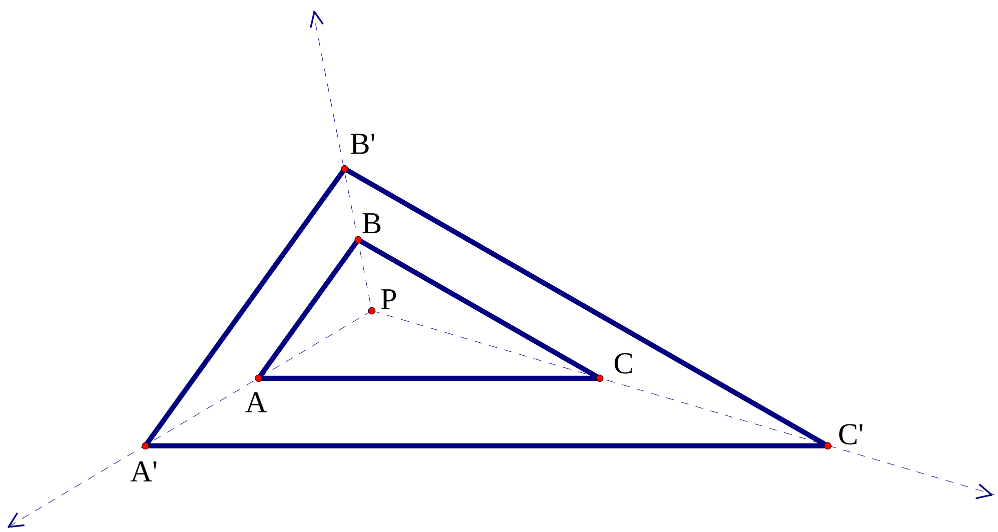


2. Center of dilation: B

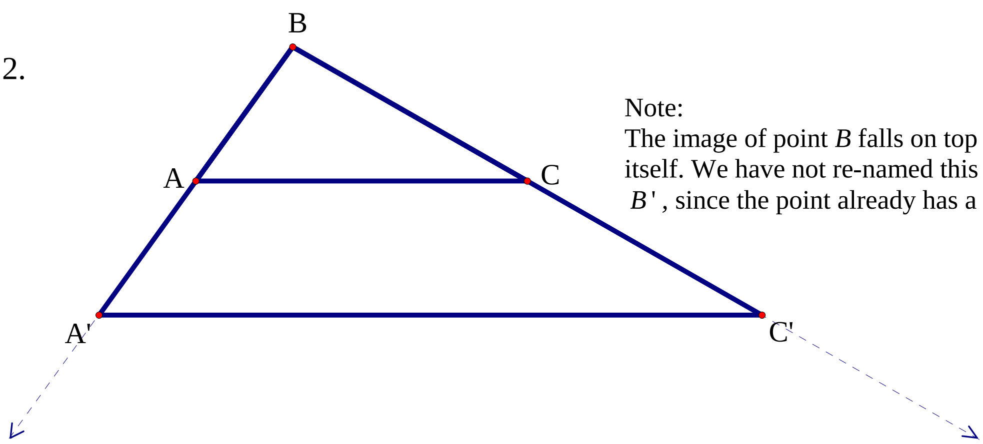


<u>Solutions:</u> (The dashed dilation rays are not necessary for the solution, but are drawn to help the reader to understand how the dilation was constructed.)

1.

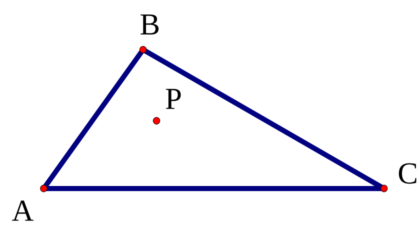


2.

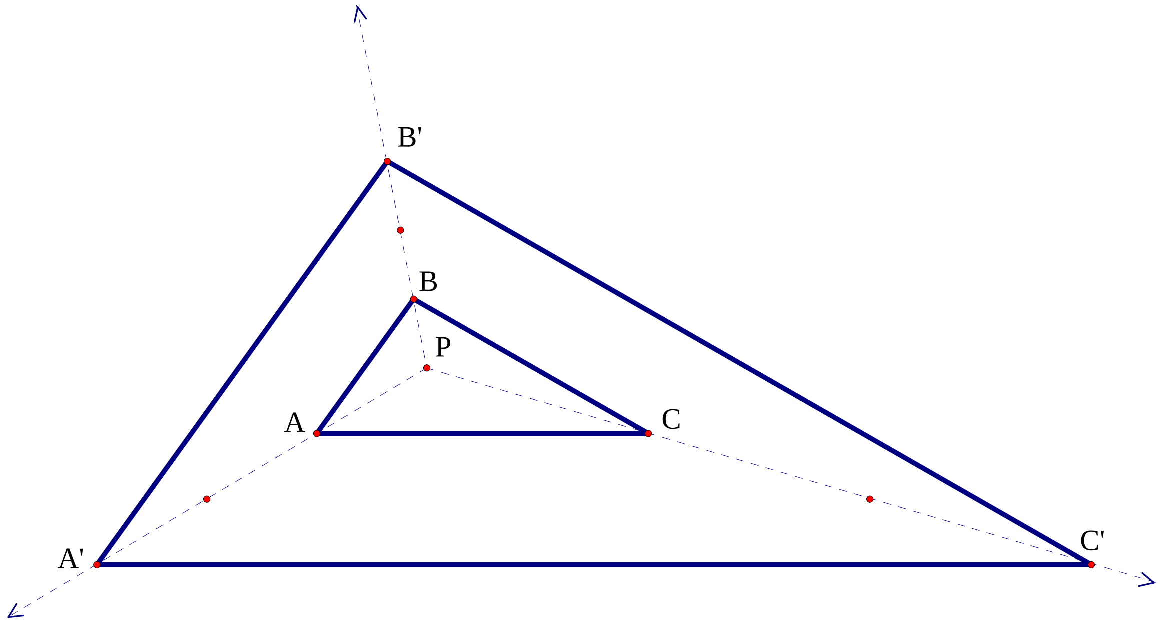


Note:
The image of point *B* falls on top of itself. We have not re-named this point as B', since the point already has a name.

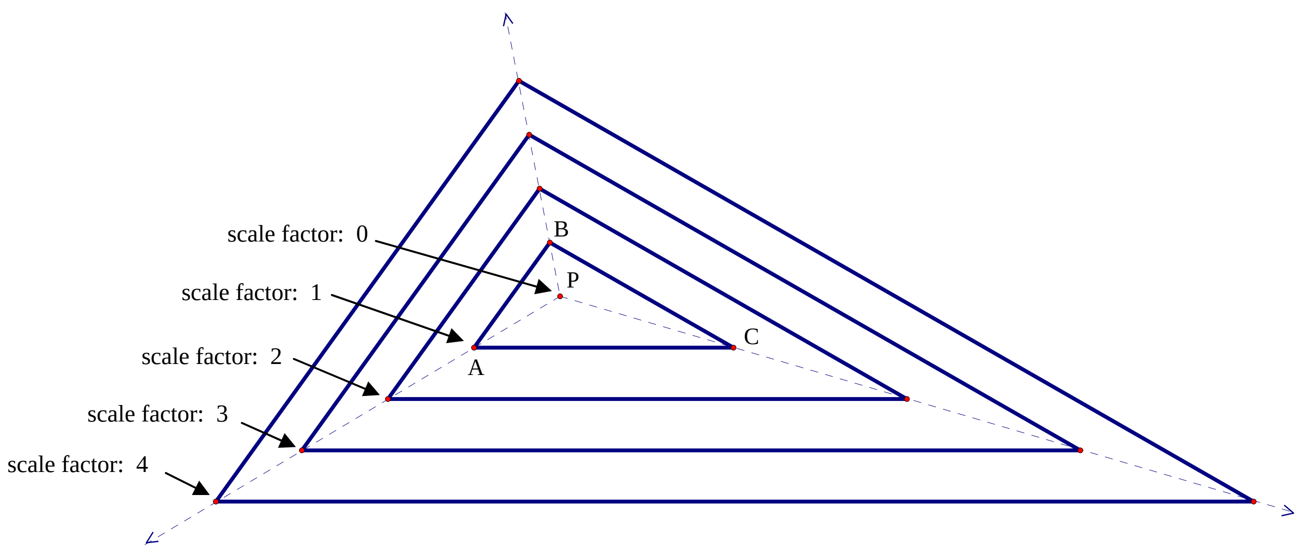
Instead of changing centers, let us now look at dilations with different scale factors (and the same center of dilation). Consider again ΔABC with center *P*.



Suppose that we instead wanted to perform a dilation with center *P*, but with a scale factor of 3. To find the position of A', for example, we would follow the same steps, as in Example 1 above, but would slide our "unit" (the length of $\overline{PA}$) down one additional time. The complete diagram is shown below, with intermediate points along the dashed rays to show the intermediate steps of the dilation.



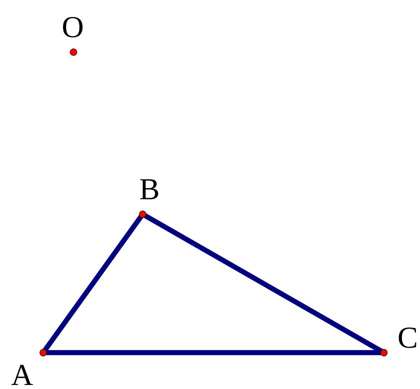
A diagram is shown below which shows dilations of ΔABC with scale factors of 1, 2, 3, and 4. The vertices of the images have not been labeled, but an arrow points to the image of point *A* for each scale factor which was used.



In the above diagram, notice that a scale factor of 1 puts the image directly on top of the pre-image. (In this case, the image and pre-image are congruent). A scale factor of 0, on the other hand, "shrinks" the triangle to a single point.

Let us now consider dilations with more complicated scale factors.

Examples
Dilate ΔABC about center O, using each of the scale factors listed below. (Draw a separate diagram for each dilation.)



1. Scale factor: $\frac{1}{2}$

2. Scale factor: 1.5

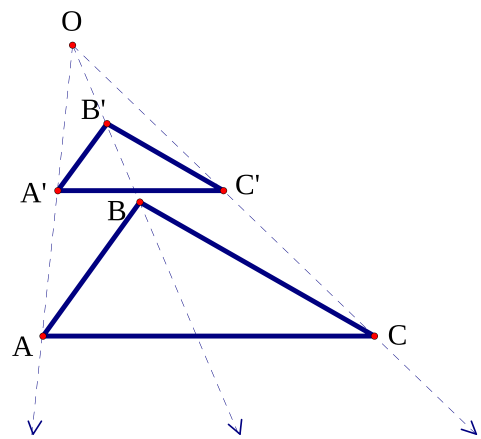
3. Scale factor: -1

4. Scale factor: -2

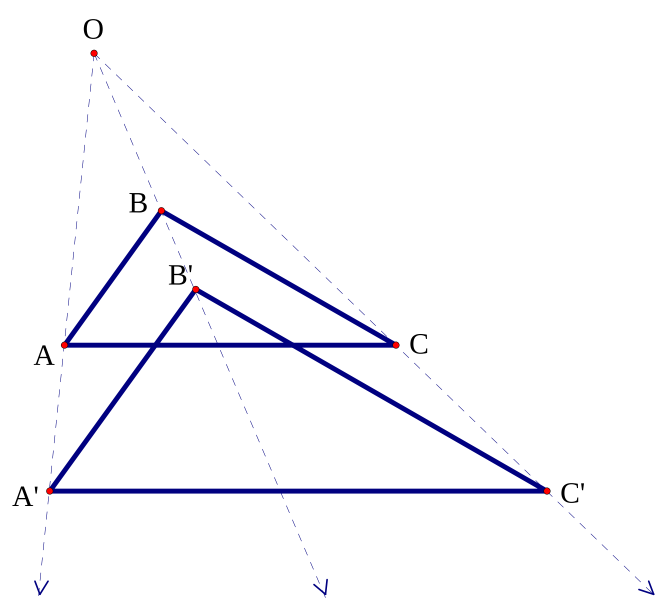
Solutions to the above examples can be found on the following page.

Solutions:

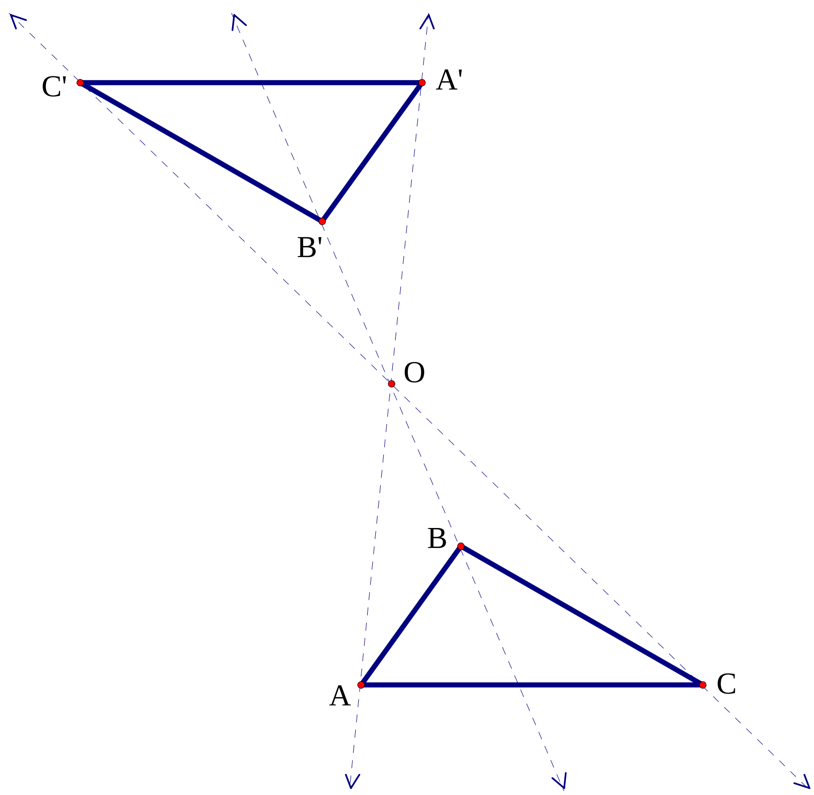
1. To dilate with a scale factor of $\frac{1}{2}$, we again begin by drawing rays from the center of dilation through the vertices of the pre-image. To find the location of A', we measure the distance from *O* to *A* and find the midpoint. (This can be done without a ruler; you can again place tick marks on the edge of a paper to represent the distance between *O* and *A*, and then fold those tick marks on top of each other. The crease of the fold represents the midpoint between the two points, and gives the location of A'. The solution can be found below.



2. To dilate with a scale factor of 1.5, we will use the distances measured in Example 1 above (for a scale factor of $\frac{1}{2}$, or 0.5). We know that the original triangle, ΔABC, represents a scale factor of 1, so we need to dilate 0.5 "units" beyond each of the original vertices (where a "unit" represents the distance between *O* and each vertex of the pre-image). The solution can be found below.

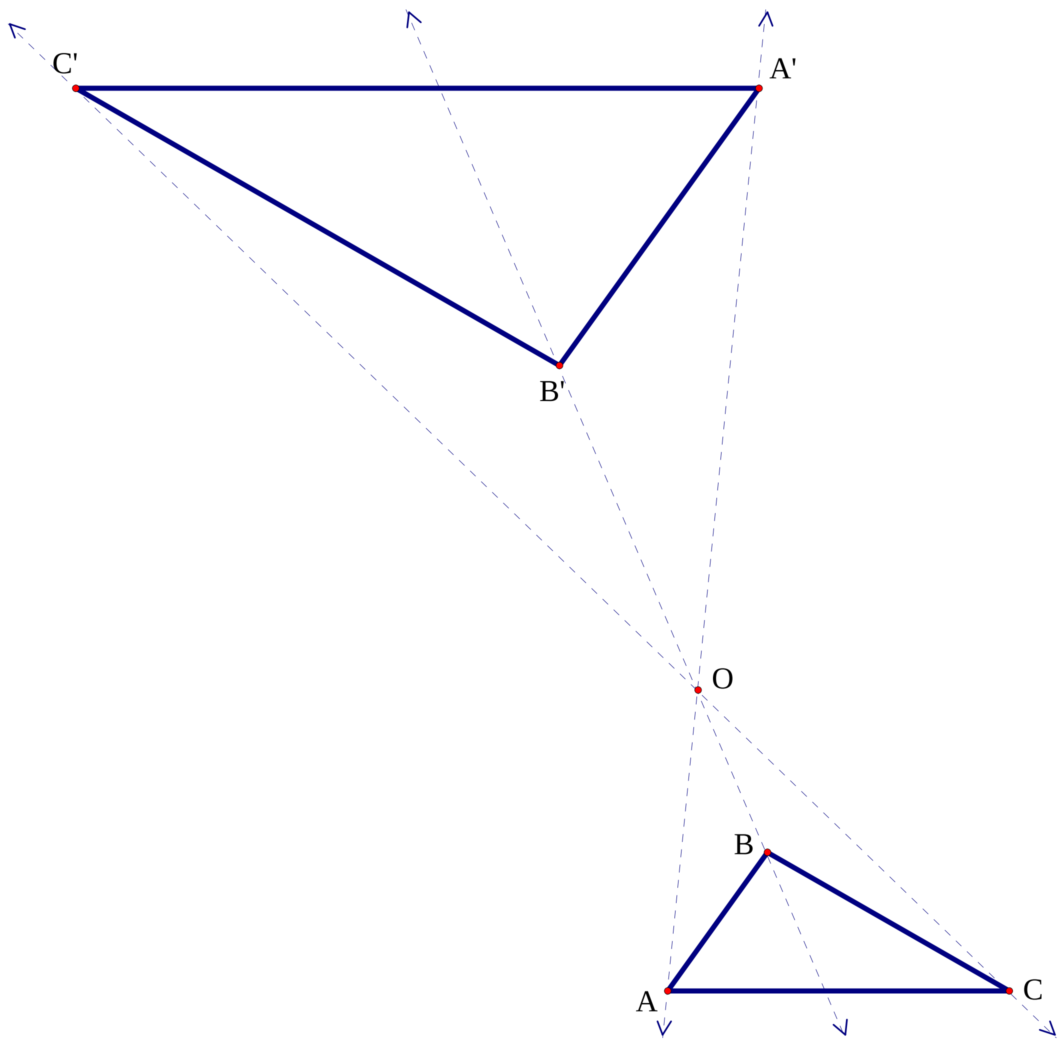


3. To dilate with a scale factor of -1, we begin by drawing *lines* (instead of rays) that pass through the center of dilation as well as the vertices of the pre-image. To find, for example, the location of A', we measure the distance from *O* to *A*, and then measure that same distance *backwards* on $\overleftrightarrow{OA}$ (i.e. from point *O* away from the pre-image). The solution can be found below.



The solution to Example 4 can be found on the following page.

4. To dilate with a scale factor of -2, we begin by drawing *lines* (instead of rays) that pass through the center of dilation as well as the vertices of the pre-image. To find, for example, the location of A', we measure the distance from *O* to *A*, and then measure that same distance *twice backwards* on $\overleftrightarrow{OA}$ (i.e. from point *O* away from the pre-image). The solution can be found below.



Look at what happened to the orientation of the figures in Examples 3 and 4 above. When the scale factor is negative, the dilation reverses the orientation of the figure. On the other hand, when the scale factor is positive (as with all of the other examples within this section), the orientation of the image and pre-image are the same.

A summary of the properties of dilations can be found in the following table.

Properties of Dilations

If a polygon is dilated with scale factor *k*,

1. The image is similar to the pre-image.
2. If the length of a segment in the pre-image is *n*, then the length of a corresponding segment in the image is *kn*.
3. The corresponding sides of the pre-image and the image are parallel.
4. If $0 < |k| < 1$, then the image is smaller than the pre-image.

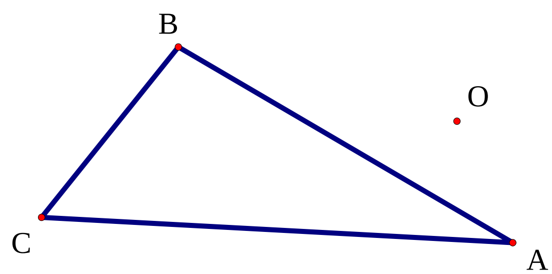
   If $|k| > 1$, then the image is larger than the pre-image.

   If $k = 1$, then the image is congruent to the pre-image.
   If $k = 0$, then the image is reduced to a single point.

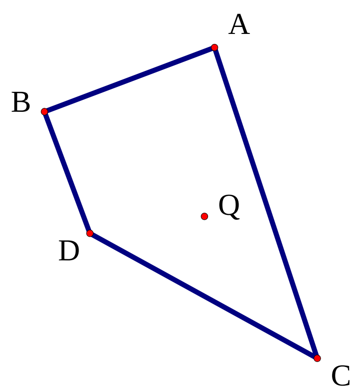
5. If $k > 0$, then the orientation of the image is the same as that of the pre-image.
   If $k < 0$, then the orientation of the image is reversed from that of the pre-image.

Exercises
Answer the following.

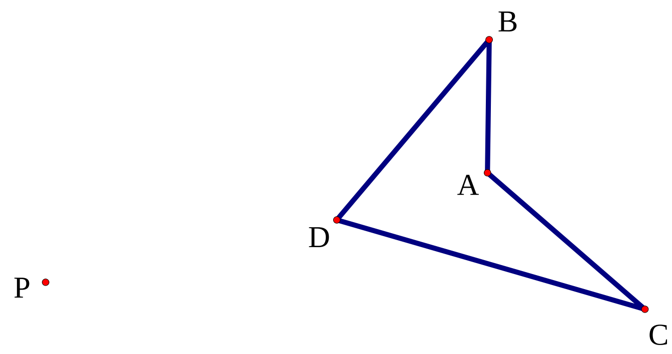
1. Dilate ΔABC about center *O*, using a scale factor of 2.



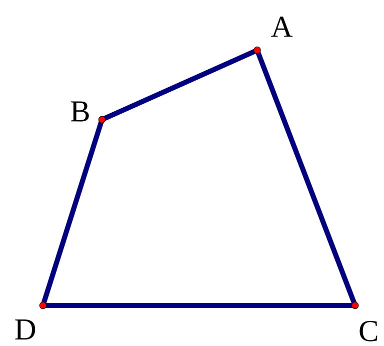
2. Dilate quadrilateral *ABCD* about center *Q*, using a scale factor of 3.



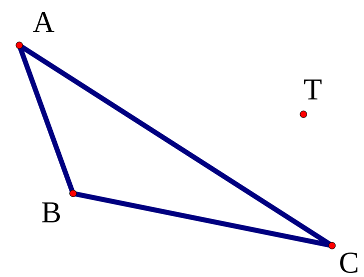
3. Dilate quadrilateral *ABCD* about center *Q*, using a scale factor of 0.5.



4. Dilate quadrilateral *ABCD* about center *B*, using a scale factor of 2.5.



5. Dilate ΔABC about center *T*, using a scale factor of -2.



Dilations in the Coordinate Plane

Our discussion of dilations in the coordinate plane will be quite limited, focusing only on dilations with the point $(0, 0)$ as the center of dilation.

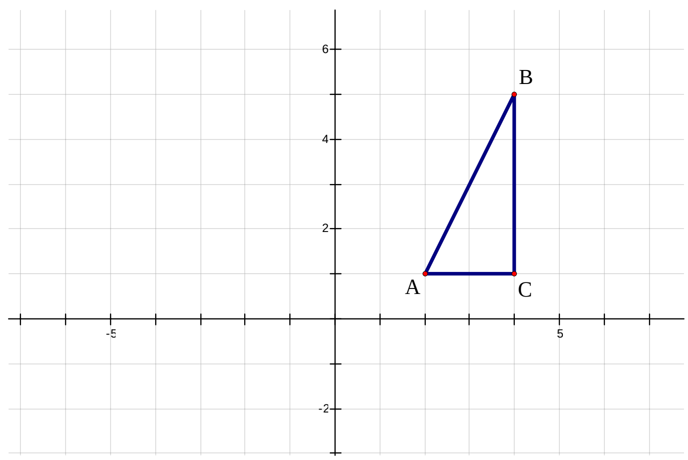
Examples

Consider ΔABC with vertices $A(2,1)$, $B(4,1)$, and $C(4,5)$.

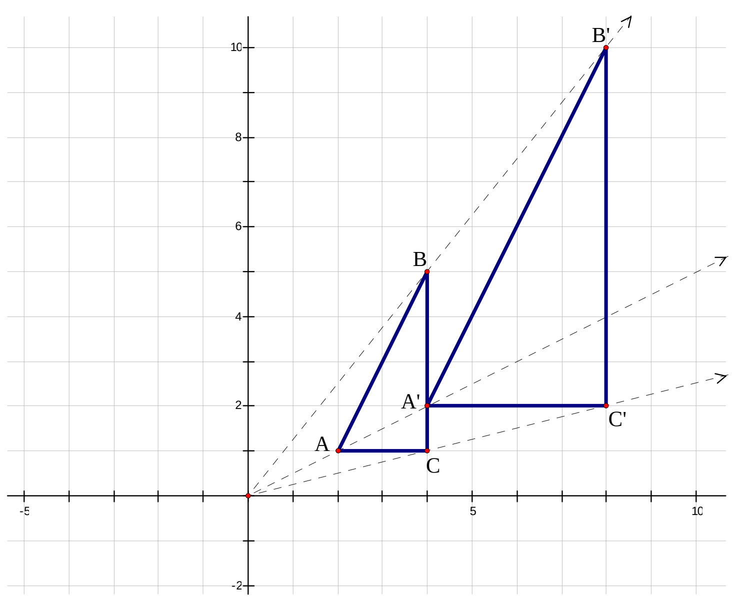
1. Draw ΔABC in the coordinate plane.
2. Dilate ΔABC using center of dilation $(0, 0)$ and a scale factor of 2.
3. Write down the coordinates of $A', B',$ and C', and compare them to the coordinates of $A, B,$ and C. What do you notice?

Solutions:

1.



2.



3. The coordinates of $\Delta A'B'C'$ are $A'(4, 2)$, $B'(8, 2)$, and $C'(8, 10)$.
   The coordinates of ΔABC are $A(2, 1)$, $B(4, 1)$, and $C(4, 5)$.
   Each coordinate of $\Delta A'B'C'$ is twice as large as its corresponding coordinate of ΔABC. (In other words, since the *x*-coordinate of *C* is 4, the *x*-coordinate of C' is $2(4) = 8$. Since the *y*-coordinate of *C* is 5, the *x*-coordinate of C' is $2(5) = 10$.

The previous set of examples gives us a significant shortcut. When a dilation in the coordinate plane is centered at the origin, we need not draw rays and perform measurements as we have done throughout all of our previous examples of dilations outside of the coordinate plane. If the scale factor for a dilation is 5, for example, we need only multiply each coordinate of the pre-image by 5.

Exercises

ΔDEF has vertices $D(-2, -1)$, $E(3, -4)$, and $F(0, 1)$. Answer the following.

1. Draw ΔDEF in the coordinate plane.
2. If ΔDEF is dilated using center of dilation $(0, 0)$ and a scale factor of 3, give the coordinates of D', E', and F'.
3. Draw the image $\Delta D'E'F'$ in the same coordinate plane as ΔDEF from above.