PERIMETER AND AREA

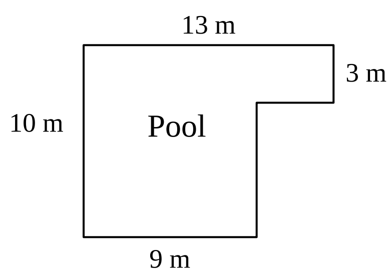
In this unit, we will develop and apply the formulas for the perimeter and area of various two-dimensional figures.

Perimeter

Perimeter
The perimeter of a polygon, denoted by *P*, is the distance around the polygon, i.e. the sum of the lengths of its sides.

Examples
Answer the following. Remember to include units.

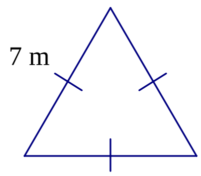
1. Find the perimeter of an equilateral triangle with side length 7 m.
2. Find the perimeter of a square with side length 5 in.
3. Find the perimeter of a regular hexagon with side length 10 m.
4. Find the perimeter of a rectangle with length 8 cm and width 5 cm.
5. If the perimeter of a regular pentagon is 75 in, find the length of one side.
6. If a tennis court has a perimeter of 210 ft and a length of 78 ft, find its width.
7. Find the perimeter of the following swimming pool. (Assume that all intersecting sides in the diagram are perpendicular to each other.)



The solutions to the above examples can be found below.

Solutions to Examples:

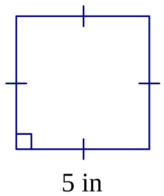
1. Find the perimeter of an equilateral triangle with side length 7 m.



$$P = 7 + 7 + 7 = 3(7) = 21$$

The perimeter of the equilateral triangle is 21 m.

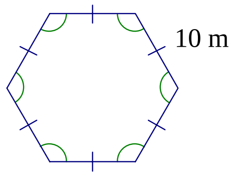
2. Find the perimeter of a square with side length 5 in.



$$P = 5 + 5 + 5 + 5 = 4(5) = 20$$

The perimeter of the square is 20 in.

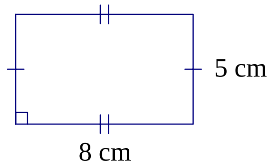
3. Find the perimeter of a regular hexagon with side length 10 m.



$$P = 10 + 10 + 10 + 10 + 10 + 10 = 6(10) = 60$$

The perimeter of the regular hexagon is 60 m.

4. Find the perimeter of a rectangle with length 8 cm and width 5 cm.



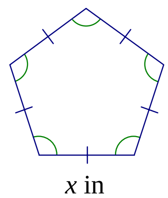
$$P = 8 + 8 + 5 + 5 = 2(8) + 2(5) = 36$$

The perimeter of the rectangle is 36 cm.

Note: The perimeter of a rectangle with length l and width w can be represented by the formula: $P = 2l + 2w$.

5. If the perimeter of a regular pentagon is 75 in, find the length of one side.

Let x = the length of a side of the regular pentagon



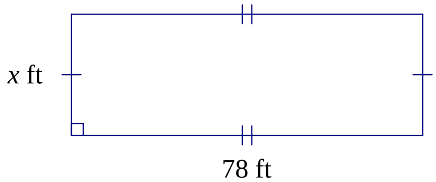
Since the perimeter of the pentagon is 75 inches, we can say that
$x + x + x + x + x = 75$, or equivalently
$5x = 75$.

Next, we solve for x by dividing both sides by 5:
$x = 15$

The length of one side of the regular pentagon is 15 inches.

6 If a tennis court has a perimeter of 210 ft and a length of 78 ft, find its width.

Let x = the width of the tennis court



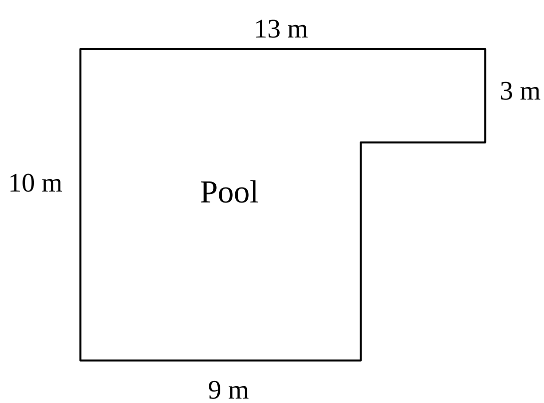
We know that $P = 210$.

Using the formula $2l + 2w = P$,
$2(78) + 2x = 210$.

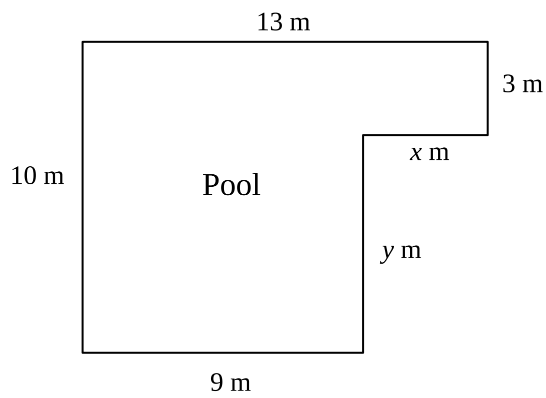
Solving for x:
$156 + 2x = 210$
$2x = 54$
$x = 27$

Therefore, the width of the tennis court is 27 ft.

7. Find the perimeter of the following swimming pool. (Assume that all intersecting sides in the diagram are perpendicular to each other.)



Let x and y represent the unlabeled sides of the pool, as shown in the diagram below.



The horizontal distance across the pool is 13 m (as seen along the top edge of the pool). Therefore, $9 + x = 13$, so $x = 4$.

The vertical distance across the pool is 10 m (as seen along the left edge of the pool). Therefore, $3 + y = 10$, so $y = 7$.

Now, adding the side lengths to find the perimeter of the figure:

$$P = 9 + 10 + 13 + 3 + 4 + 7 = 46$$

The perimeter of the pool is 46 m.

Circumference

The circumference of a circle is the distance around the circle. If a circle is compared to a many-sided polygon, the circumference of a circle can be likened to the perimeter of a polygon.

Let us first begin with an exploration regarding circumference.

Consider the table below, which contains a listing of various circular objects (lid, soda can, drinking glass, jar, etc.) In each of these three-dimensional objects, a plane containing a circle has been isolated, and the diameter and circumference of that circle has been listed. Measurements have been rounded to the nearest tenth of a centimeter.

OBJECT	DIAMETER	CIRCUMFERENCE	$\frac{\text{CIRCUMFERENCE}}{\text{DIAMETER}}$
Lid (outer rim)	6.1 cm	19.3 cm	
Soda Can (top rim)	5.5 cm	17.2 cm	
Drinking Glass (top rim)	8.2 cm	25.7 cm	
Jar (top rim)	4.7 cm	14.9 cm	
Mixing Bowl (top rim)	21.6 cm	67.8 cm	
Bucket (top rim)	31.0 cm	97.5 cm	

Exercises

1. Use a calculator to compute the ratio $\frac{\text{Circumference}}{\text{Diameter}}$ for each object in the last column of the table above. Round this ratio to the nearest thousandth.

2. Find three other circular objects. Add them to the table above, measure the circumference and diameter for each, and then compute the ratio $\frac{\text{Circumference}}{\text{Diameter}}$ for each object. (Note: To measure the circumference of an object, you may want to wrap a piece of string around the object, and then measure the string.)

3. Analyze the last column of numbers in the table. What do you notice?

A completed chart can be found below.

OBJECT	DIAMETER	CIRCUMFERENCE	$\frac{\text{CIRCUMFERENCE}}{\text{DIAMETER}}$
Lid (outer rim)	6.1 cm	19.3 cm	3.164
Soda Can (top rim)	5.5 cm	17.2 cm	3.127
Drinking Glass (top rim)	8.2 cm	25.7 cm	3.134
Jar (top rim)	4.7 cm	14.9 cm	3.170
Mixing Bowl (top rim)	21.6 cm	67.8 cm	3.139
Bucket (top rim)	31.0 cm	97.5 cm	3.145

Notice that the numbers in the last column are very close to each other in their numerical value. Ancient mathematicians noticed that the ratio $\frac{\text{Circumference}}{\text{Diameter}}$ always seemed to give the same number, and it became a challenge to try to determine the value of this ratio. Since we rounded to the nearest tenth (and measurement is not completely accurate due to human error and/or measuring devices), the values in our table are not exactly the same. If we were to be able to measure perfectly, we would find that the ratio $\frac{\text{Circumference}}{\text{Diameter}}$ is always equal to the number π (pronounced "pi").

π is an irrational number, but its value is shown below to 50 decimal places:

$$\pi \approx 3.14159265358979323846264338327950288419716939937510...$$

Note: For more information on irrational numbers, see the "Irrational Numbers" tutorial in the Additional Materials section.

Since the ratio $\frac{\text{Circumference}}{\text{Diameter}}$ is equal to π, we can write the following equation:

$$\frac{C}{d} = \pi$$ (Where C represents circumference and d represents diameter.)

Multiplying both sides of the equation by d, we obtain the following equation:

$$C = \pi d$$

Since the diameter d of a circle is twice the radius r, i.e. $d = 2r$, we can also write:

$$C = \pi d = \pi(2r) = 2\pi r$$

<u>Circumference</u>

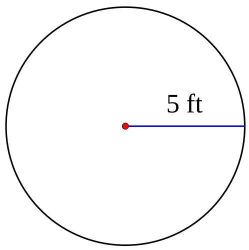
The <u>circumference</u> of a circle, denoted by C, is the distance around the circle. The circumference is given by either of the following formulas:

$C = \pi d$, or equivalently, $C = 2\pi r$,

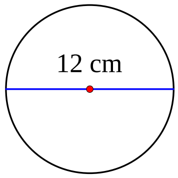
where d and r represent the diameter and radius of the circle, respectively.

Examples
Find the circumference of the following circles. First, write the circumference of each circle in terms of π. Then use the approximation $\pi \approx 3.14$ and compute the circumference of each circle to the nearest hundredth. (Be sure to include units in each answer.)

1.



2.



Solutions:

1. The radius of the circle is 5 ft $(r = 5)$ so we use the equation $C = 2\pi r$.

$C = 2\pi r = 2\pi(5) = 10\pi$ Exact answer: $C = 10\pi$ ft

Using 3.14 for π:
$C = 10\pi \approx 10(3.14)$ Approximate answer: $C \approx 31.4$ ft

2. The diameter of the circle is 12 cm $(d = 12)$ so we use the equation $C = \pi d$.

$C = \pi d = \pi(12) = 12\pi$ Exact answer: $C = 12\pi$ cm

Using 3.14 for π:
$C = 12\pi \approx 12(3.14)$ Approximate answer: $C \approx 37.68$ cm

Area of a Circle

The formula for the area of a circle can be found below.

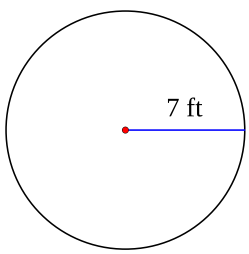
<u>Area of a Circle</u>

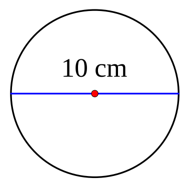
The <u>area of a circle</u> with radius r is given by the formula

$$A = \pi r^2$$

Examples

Find the area of the following circles. First, write the area of each circle in terms of π. Then use the approximation $\pi \approx 3.14$ and compute the area of each circle to the nearest hundredth. (Be sure to include units in each answer.)

1. 

2. 

<u>Solutions:</u>

1. $A = \pi r^2 = \pi \cdot 7^2 = 49\pi \ \text{ft}^2$. Exact answer: $A = 49\pi \ \text{ft}^2$

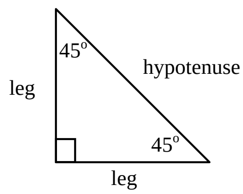
   Using 3.14 for π:
   $A = 49\pi \approx 49(3.14)$ Approximate answer: $A \approx 153.86 \text{ ft}^2$

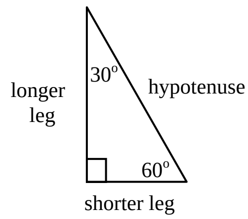
2. The diameter of the circle is 10 cm $(d = 10)$ so $r = 5$ cm.
   $A = \pi r^2 = \pi \cdot 5^2 = 25\pi \ \text{cm}^2$. Exact answer: $A = 25\pi \ \text{cm}^2$

   Using 3.14 for π:
   $A = 25\pi \approx 25(3.14)$ Approximate answer: $A \approx 78.5 \text{ cm}^2$

Special Right Triangles
(Needed for upcoming sections involving area)

There are two special right triangles which are useful to us as we study the area of polygons. These triangles are named by the measures of their angles, and are known as 45^o-45^o-90^o triangles and 30^o-60^o-90^o triangles. A diagram of each triangle is shown below:





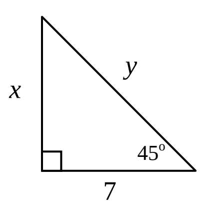
Tutorial:
For a more detailed exploration of this section along with additional examples and exercises, see the tutorial entitled "Special Right Triangles" in the Additional Material section.

The theorems relating to special right triangles can be found below, along with examples of each.

<u>Theorem:</u> In a 45^o-45^o-90^o triangle, the legs are congruent, and the length of the hypotenuse is $\sqrt{2}$ times the length of either leg.

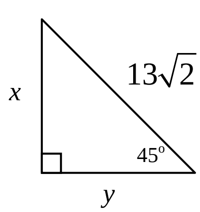
Examples
Find x and y by using the theorem above. Write answers in simplest radical form.

1.



<u>Solution:</u>
The legs of the triangle are congruent, so $x = 7.$ The hypotenuse is $\sqrt{2}$ times the length of either leg, so $y = 7\sqrt{2}.$

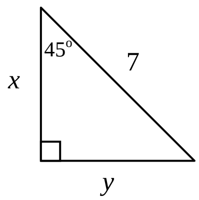
2.



<u>Solution:</u>
The hypotenuse is $\sqrt{2}$ times the length of either leg, so the length of the hypotenuse is $x\sqrt{2}.$ We are given that

the length of the hypotenuse is $13\sqrt{2}$, so $x\sqrt{2} = 13\sqrt{2}$, and we obtain $x = 13$. Since the legs of the triangle are congruent, $x = y$ and $y = 13$.

3.



Solution:

The hypotenuse is $\sqrt{2}$ times the length of either leg, so the length of the hypotenuse is $x\sqrt{2}$. We are given that the length of the hypotenuse is 7, so $x\sqrt{2} = 7$, and we obtain $x = \frac{7}{\sqrt{2}}$. Rationalizing the denominator, $x = \frac{7}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{7\sqrt{2}}{2}$. Since the legs of the triangle are congruent, $x = y$ and $y = \frac{7\sqrt{2}}{2}$.

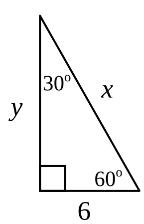
Note: For more information on rationalizing denominators, see the "Rationalizing Denominators" tutorial in the Additional Materials section.

Theorem:	In a 30°-60°-90° triangle, the length of the hypotenuse is twice the length of the shorter leg, and the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg.

Examples

Find *x* and *y* by using the theorem above. Write answers in simplest radical form.

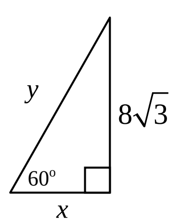
1.



Solution:

The length of the shorter leg is 6. Since the length of the hypotenuse is twice the length of the shorter leg, $x = 2 \cdot 6 = 12$. The length of the longer leg is $\sqrt{3}$ times the length of the shorter leg, so $y = 6\sqrt{3}$.

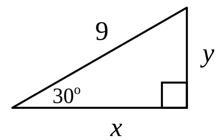
2.



Solution:

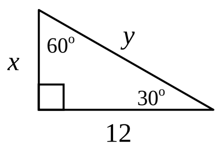
The length of the shorter leg is *x*. Since the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg, the length of the longer leg is $x\sqrt{3}$. We are given that the length of the longer leg is $8\sqrt{3}$, so $x\sqrt{3} = 8\sqrt{3}$, and therefore $x = 8$. The length of the hypotenuse is twice the length of the shorter leg, so $y = 2x = 2 \cdot 8 = 16$.

3.



Solution:
The length of the shorter leg is *y*. Since the length of the hypotenuse is twice the length of the shorter leg, $9 = 2y$, so $y = \frac{9}{2} = 4.5$. Since the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg, $x = y\sqrt{3} = \frac{9}{2}\sqrt{3} = 4.5\sqrt{3}$.

4.



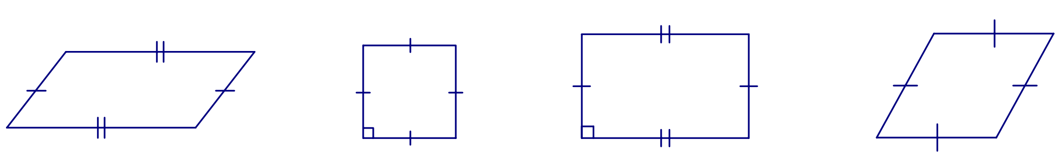
Solution:
The length of the shorter leg is *x*. Since the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg, the length of the longer leg is $x\sqrt{3}$. We are given that the length of the longer leg is 12, so $x\sqrt{3} = 12$. Solving for *x* and rationalizing the denominator, we obtain $x = \frac{12}{\sqrt{3}} = \frac{12}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$. The length of the hypotenuse is twice the length of the shorter leg, so $y = 2x = 2 \cdot 4\sqrt{3} = 8\sqrt{3}$.

Area of a Parallelogram

Before we explore how to find the area of a parallelogram, let us first review the definition of a parallelogram:

Parallelogram
A parallelogram is a quadrilateral with two pairs of parallel sides.

The following quadrilaterals are examples of parallelograms:

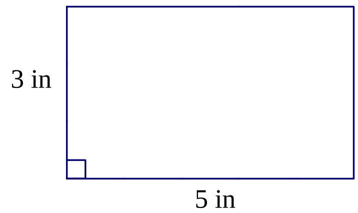


The second, third, and fourth diagrams above represent a square, rectangle, and rhombus, respectively. All of these quadrilaterals can still be classified as parallelograms, since they each have two pairs of parallel sides.

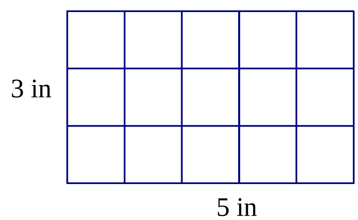
The *area* of a figure is represented by the number of square units which can be contained inside of it.

Area of a Rectangle

Let us first examine the following rectangle and determine its area:



If we divide the rectangle into one-inch squares, we can quickly see that the area of the rectangle is 15 square inches, as shown below.



We can see from this example that in order to find the area of a rectangle, we can just multiply the base times the height. In the previous example, $A = 5(3) = 15\ \text{in}^2$.

> Area of a Rectangle
>
> The area of a rectangle with base b and height h is given by the formula
>
> $$A = bh$$

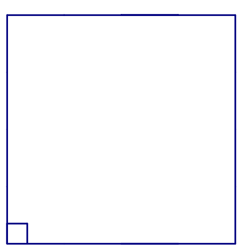
Area of a Square

A square is simply a special case of a rectangle. The formula for the area of a rectangle can be used for squares, but because it is a special case, there is a special formula for the area of the square as well.

Suppose that a square has side *s*. This represents both the base and the height of the square. Since the formula for the area of a rectangle is $A = bh$, we can say that $A = s \cdot s = s^2$.

> Area of a Square
>
> The area of a square of side *s* is given by the formula
>
> $$A = s^2.$$

Consider a square with side length 4 cm. The area of the square is16 cm^2, as illustrated by the following diagrams and equations.



4 cm

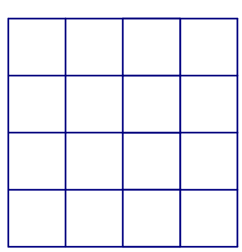
4 cm

$A = s^2 = 4^2 = 16$

We also could use the formula $A = bh$:
$A = 4(4) = 16$

The area of the square is 16 cm^2.

This is further illustrated below by dividing the diagram into one-inch squares.

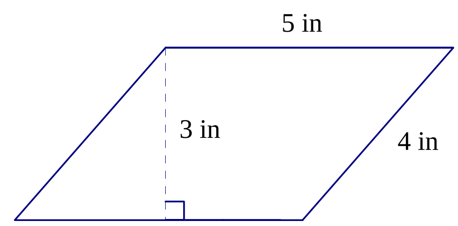


4 cm

4 cm

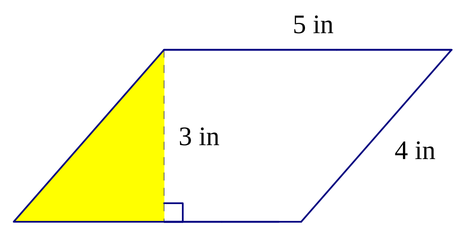
<u>Area of a Parallelogram</u>

Suppose that we want to find the area of the following parallelogram.

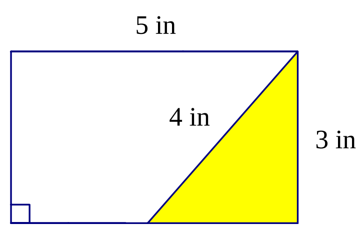


We want to divide the parallelogram into square units, but the sides of the parallelogram are not perpendicular to each other.

Look at the triangle which is shaded in the figure below.



Imagine slicing the triangle off of the parallelogram above, and translating it to the other side of the parallelogram, as shown below. This forms a rectangle which has the same area as the original parallelogram.



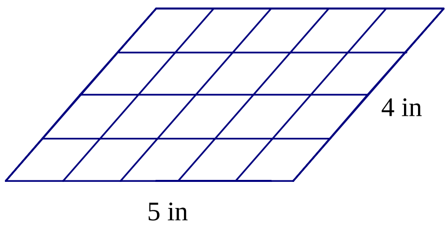
To find the area of the rectangle, we need only to multiply the base and the height.

$$A = 5(3) = 15 \text{ in}^2$$

Since the area of the original parallelogram is the same as the area of the rectangle, the area of the parallelogram is also 15 in^2.

Notice that the length of the 4 cm side of the parallelogram was extraneous in terms of finding its area; all that was needed was the base and the height. A common error in finding the area of the parallelogram above is to multiply 5 times 4; the reason this is incorrect is that the answer (20) represents “slanted” units, not square units (where each unit is a rhombus). A diagram is shown below to illustrate this concept.

INCORRECT method of computing area:



An answer of 20 square units for the area (multiplying 5 times 4) is **<u>incorrect</u>**. Although 20 units of area can be seen in the diagram at the left, they are not square units and are not a standard means of measuring area.

The proper formula for the area of a parallelogram can be found below.

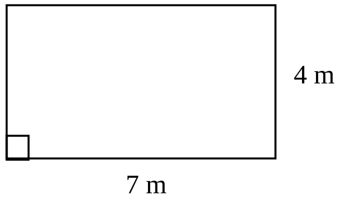
<u>Area of a Parallelogram</u>

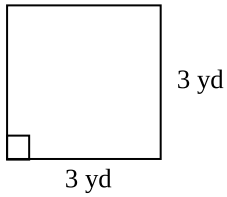
The <u>area of a parallelogram</u> with base b and height h is given by the formula

$$A = bh$$

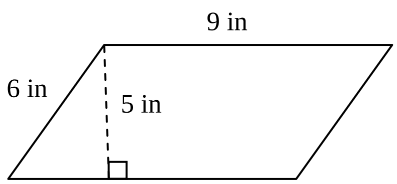
Examples

Find the area of each of the following parallelograms. Be sure to include units.
(Note: Some figures may not be drawn to scale.)

1. 

2. 

3.



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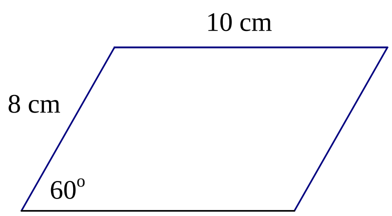

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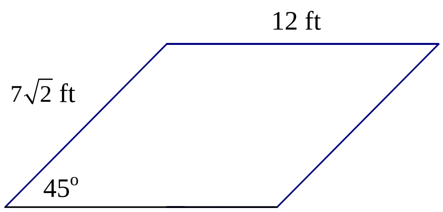
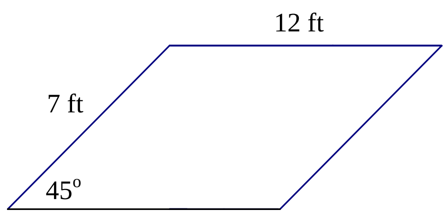


6.



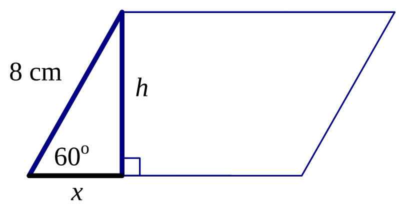
<u>Solutions:</u>

1. Since the base of the parallelogram is 7 m and the height is 4 m,
   $A = bh = 7(4) = 28\,\text{m}^2$.

2. Since the base of the parallelogram is 3 yd and the height is 3 yd,
   $A = bh = 3(3) = 3^2 = 9\ \text{yd}^2$.

3. Since the base of the parallelogram is 9 in and the height is 5 in,
   $A = bh = 9(5) = 45\ \text{in}^2$.

4. We first need to find the height of the parallelogram. When an altitude is drawn, it forms a 30^o-60^o-90^o triangle, as shown below.

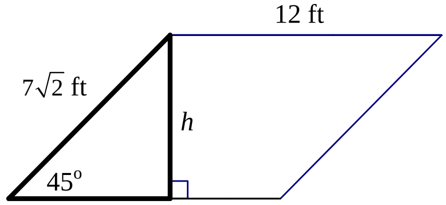


The length of the shorter leg is x. Since the length of the hypotenuse is twice the length of the shorter leg, $8 = 2x$, so $x = \frac{8}{2} = 4$. Since the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg, $h = x\sqrt{3} = 4\sqrt{3}$.

Since the base is 10 cm and the height is $4\sqrt{3}$ cm,

$$A = bh = 10\left(4\sqrt{3}\right) = 40\sqrt{3}\ \text{cm}^2.$$

5. We first need to find the height of the parallelogram. When an altitude is drawn, it forms a 45^o-45^o-90^o triangle, as shown below.

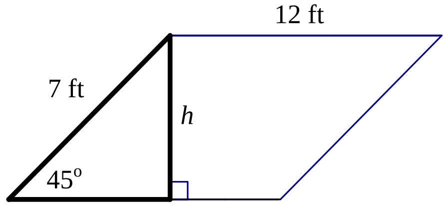


The hypotenuse is $\sqrt{2}$ times the length of either leg, so the length of the hypotenuse is $h\sqrt{2}$. We are given that the length of the hypotenuse is $7\sqrt{2}$, so $h\sqrt{2} = 7\sqrt{2}$, and we obtain $h = 7$ ft.

Since the base of the parallelogram is 12 ft and the height is 7 ft,

$$A = bh = 12(7) = 84\ \text{ft}^2.$$

6 We first need to find the height of the parallelogram. When an altitude is drawn, it forms a 45^o-45^o-90^o triangle, as shown below.



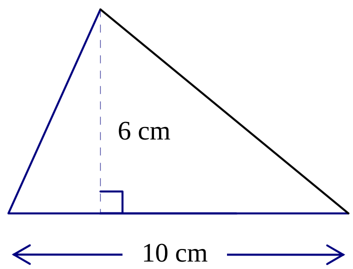
The hypotenuse is $\sqrt{2}$ times the length of either leg, so the length of the hypotenuse is $h\sqrt{2}$. We are given that the length of the hypotenuse is 7, so $h\sqrt{2} = 7$, and we obtain $h = \frac{7}{\sqrt{2}}$. Rationalizing the denominator, $h = \frac{7}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{7\sqrt{2}}{2}$.

Since the base of the parallelogram is 12 ft and the height is $\frac{7\sqrt{2}}{2}$ ft,

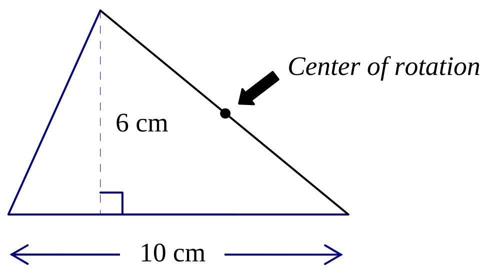
$$A = bh = 12\left(\frac{7\sqrt{2}}{2}\right) = \frac{84\sqrt{2}}{2} = 42\sqrt{2}\ \text{ft}^2.$$

Area of a Triangle

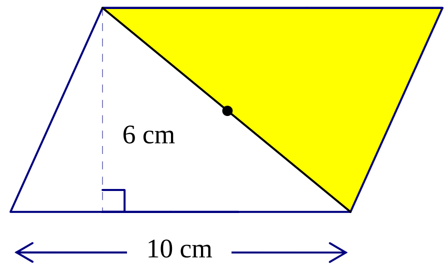
Suppose that we want to find the area of the following triangle.



First, we will mark the midpoint of one of the sides (other than the base) as the center of rotation, as shown below.



We now rotate the triangle 180° around the center of rotation. The original triangle and its image are shown below. (The image is shown as a shaded region.)



Notice that the two triangles together form a parallelogram. The base of the parallelogram is 10 cm and its height is 6 cm. To find the area of the parallelogram, we multiply the base and the height, so the parallelogram has an area of 60 cm^2.

The original triangle comprises half of the parallelogram, so the area of the original triangle is $\frac{1}{2}(60) = 30\ \text{cm}^2$.

The above process could be repeated for any triangle. We conclude that the area of a triangle is equal to one-half of the area of a parallelogram. Since the formula for the area of a parallelogram is $A = bh$, the formula for the area of a triangle is $A = \frac{1}{2}bh$.

<u>Area of a Triangle</u>

The <u>area of a triangle</u> with base b and height h is given by the formula:

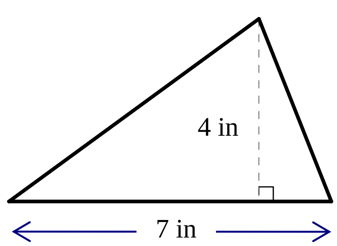
$$A = \frac{1}{2}bh.$$

Examples

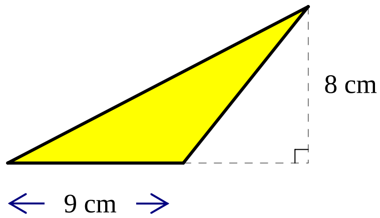
Find the area of each of the following triangles. Be sure to include units.

(Note: Some figures may not be drawn to scale.)

1.

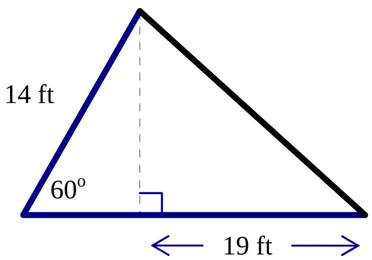


2.



Find the area of the shaded triangle.

3.

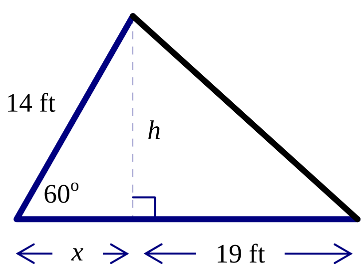


Solutions:

1. Since the base of the triangle is 7 in and the height is 4 in,
   $A = \frac{1}{2}bh = \frac{1}{2}(7)(4) = 14\,\text{in}^2.$

2. Since the base of the triangle is 9 cm and the height is 8 cm,
   $A = \frac{1}{2}bh = \frac{1}{2}(9)(8) = 36\,\text{cm}^2.$

3. We first need to find the base and the height of the triangle. Let us label the diagram with x and h as shown below:



Notice that the altitude of the original creates a 30°-60°-90° triangle with hypotenuse 14 ft and short leg x. Since the length of the hypotenuse is twice the length of the shorter leg, $14 = 2x$, so $x = \frac{14}{2} = 7$. Since the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg, $h = x\sqrt{3} = 7\sqrt{3}$.

The base of the original triangle is 26 ft (since $7+19=26$). The height of the triangle is $7\sqrt{3}$ ft. Therefore,

$$A=\tfrac{1}{2}bh=\tfrac{1}{2}(26)(7\sqrt{3})=91\sqrt{3} \text{ ft}^2.$$

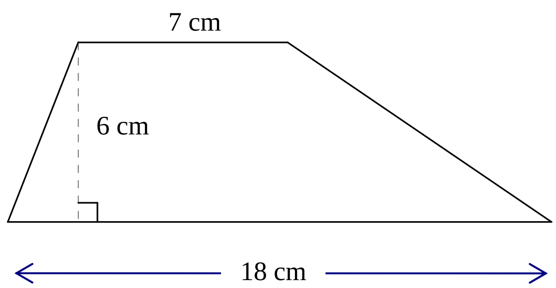
Area of a Trapezoid

Let us first review the definition of a trapezoid.

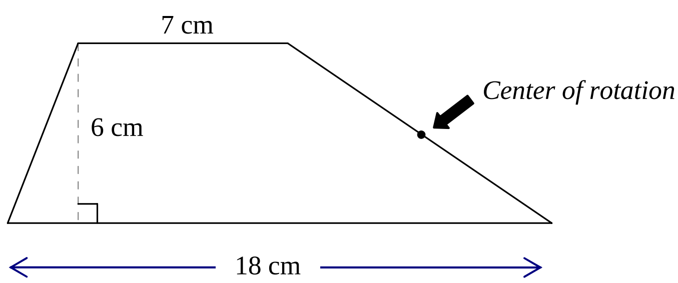
<u>Trapezoid</u>

A <u>trapezoid</u> is a quadrilateral with exactly one pair of parallel sides. The parallel sides are known as the <u>bases</u> and the other two sides are known as <u>legs</u>.

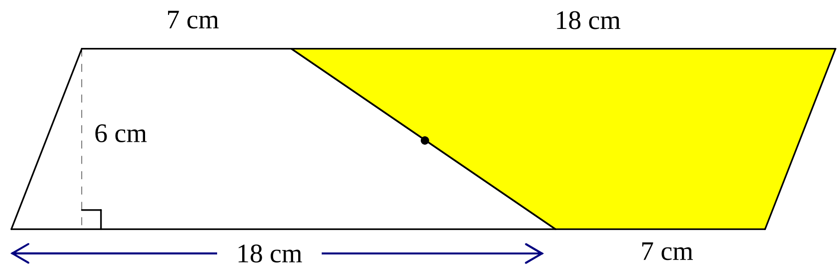
Suppose that we want to find the area of the following trapezoid.



First, we will mark the midpoint of one of the legs as the center of rotation, as shown below.



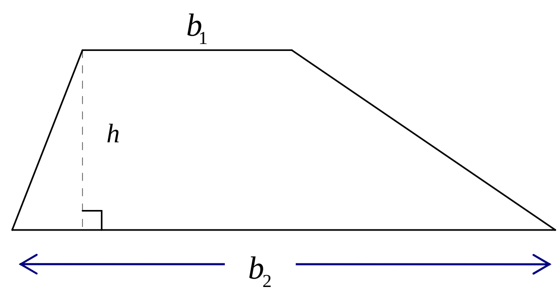
We now rotate the trapezoid 180^{o} around the center of rotation. The original trapezoid and its image are shown below. (The image is shown as a shaded region.)



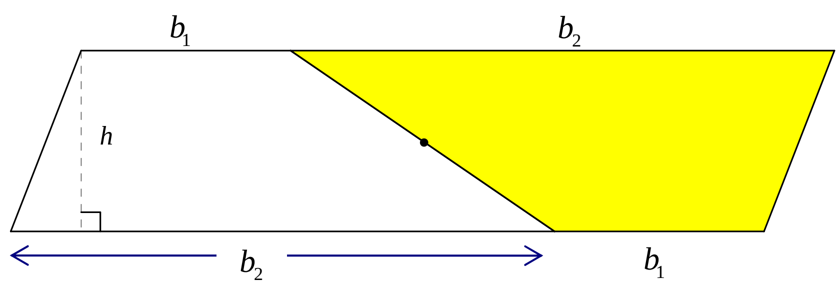
Notice that the two trapezoids together form a parallelogram. The base of the parallelogram is 25 cm (since $18+7=25$) and its height is 6 cm. To find the area of the parallelogram, we multiply the base and the height. $25 \cdot 6 = 150$, so the parallelogram has an area of 150 cm^2.

The original trapezoid comprises half of the parallelogram, so the area of the original trapezoid is $\frac{1}{2}(150) = 75\,\text{cm}^2$.

The above process could be repeated for any trapezoid, as shown below in general form. In the diagram below, the bases of the trapezoid are denoted as b_1 and b_2, and the height is denoted by h.



We again rotate the trapezoid 180$^\text{o}$ around the midpoint of one of the legs; the two trapezoids together form a parallelogram.



The area of the trapezoid is one-half the area of the parallelogram formed by the two trapezoids. The parallelogram has a base represented by $(b_1 + b_2)$ and has height *h*. Since the formula for the area of a parallelogram is $A = bh$, the area of the parallelogram is $(b_1 + b_2)h$. The area of the trapezoid is one-half the area of the parallelogram; therefore the area of the trapezoid can be represented by the formula $A = \frac{1}{2}(b_1 + b_2)h$.

Area of a Trapezoid

The area of a trapezoid with bases b_1 and b_2 and height h is given by the formula:

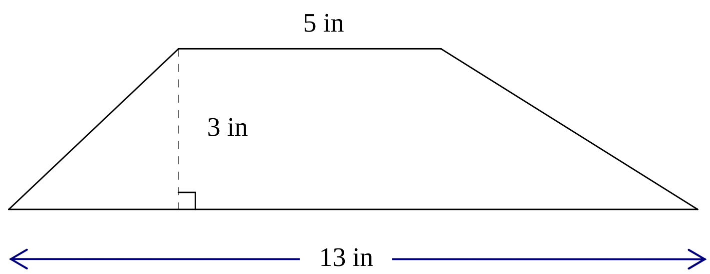
$$A = \tfrac{1}{2}(b_1 + b_2)h = \left(\frac{b_1 + b_2}{2}\right)h.$$

One way to remember this formula is

$$A = (\text{The average of the bases}) \cdot (\text{The height}).$$

Example

Find the area of the following trapezoid. Be sure to include units.



Solution:

The bases of the trapezoid are 5 in and 13 in, and the height is 3 in.

$$A = \tfrac{1}{2}(b_1 + b_2)h = \tfrac{1}{2}(5+13)(3) = \tfrac{1}{2}(18)(3) = 27 \text{ in}^2.$$

Suppose that we want to use the equivalent formula

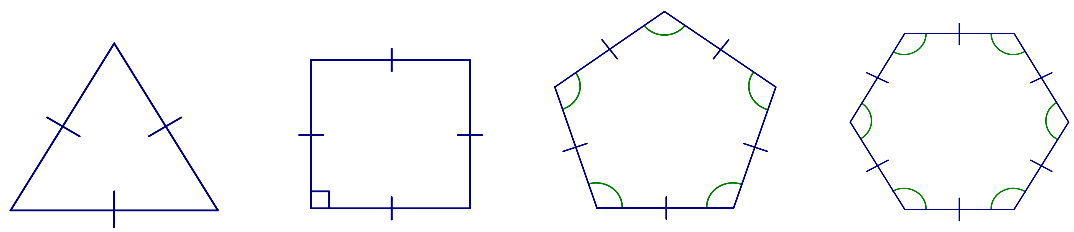
$$A = (\text{The average of the bases}) \cdot (\text{The height}) = \left(\frac{b_1 + b_2}{2}\right)h$$

We can first find the average of the bases, $\left(\frac{b_1 + b_2}{2}\right) = \left(\frac{5+13}{2}\right) = 9$.

$$A = \left(\text{The average of the bases}\right) \cdot (\text{The height}) = 9(3) = 27 \text{ in}^2.$$

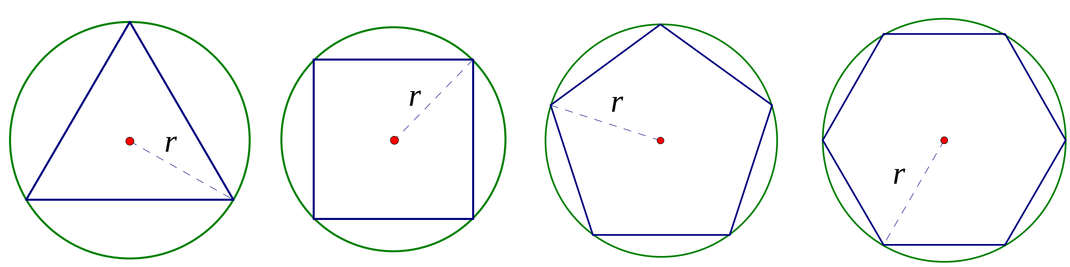
Area of a Regular Polygon

If a polygon is both equilateral (all sides are congruent) and equiangular (all angles are congruent), it is called a regular polygon. Some examples of regular polygons can be found below.

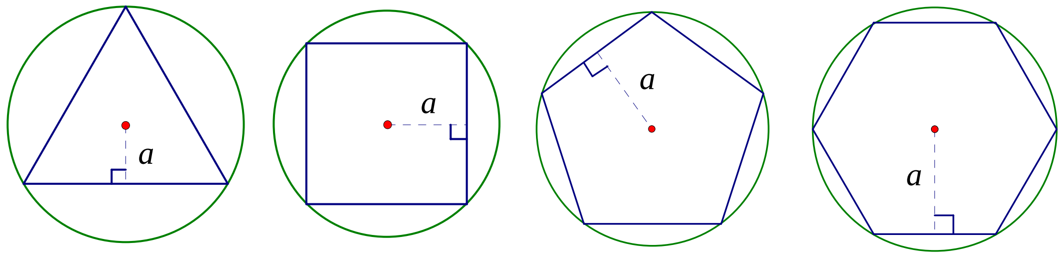


Equilateral Triangle | Square | Regular Pentagon | Regular Hexagon

We will now define some other terms which relate to regular polygons. A circle can be circumscribed around any regular polygon, and the radius of the regular polygon is the same as the radius of its circumscribed circle. The radius of the regular polygon can also be defined as the distance from the center of the regular polygon to one of its vertices. (Note: The center of a regular polygon is the same as the center of its circumscribed circle.) Examples are shown below, with a radius labeled r in each diagram. (Assume that all polygons in this section are regular polygons; congruence marks have been omitted to keep the diagrams from being too cluttered.)



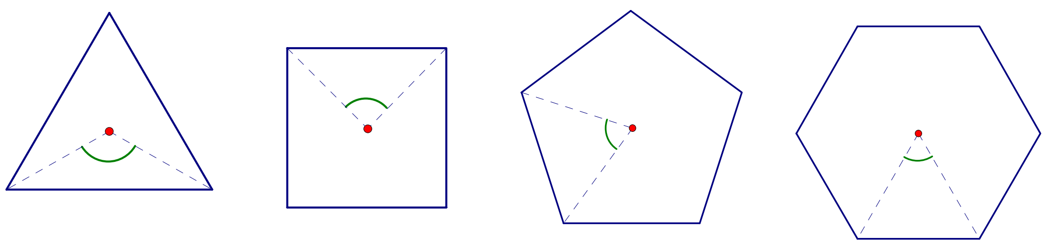
A circle can be inscribed within any regular polygon, and the apothem of the regular polygon is the same as the radius of its inscribed circle. The apothem of the regular polygon can also be defined as the distance from the center of the regular polygon to the midpoint of one of its sides. Examples are shown below, with an apothem labeled a in each diagram.



A <u>central angle of a regular polygon</u> has the following properties:

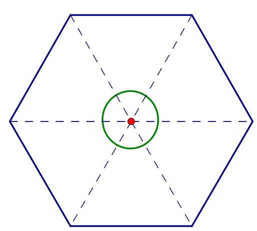
1. Its vertex is at the center of the regular polygon.
2. Its sides are two consecutive radii of the regular polygon.

Examples are shown below, with a central angle shown in each diagram.



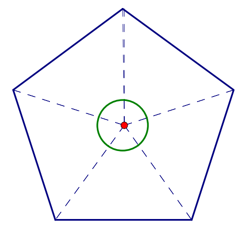
In order to find the area of a regular polygon, we must first learn to find the measure of one of its central angles.

Suppose that we want to find the measure of a central angle of a regular hexagon. In the diagram below, all of the central angles of a regular hexagon are drawn.



A regular hexagon has six central angles which are all congruent. Since the degree measure of a complete rotation is 360^{o}, the measure of one central angle is $\frac{360^\circ}{6} = 60^\circ$.

We repeat the same process for a regular pentagon below.



A regular pentagon has five central angles which are all congruent. Since the degree measure of a complete rotation is 360^o, the measure of one central angle is $\frac{360^\circ}{5} = 72^\circ$.

<u>The Measure of a Central Angle of a Regular Polygon</u>

If a regular polygon has *n* sides, then the measure of one of its central angles is

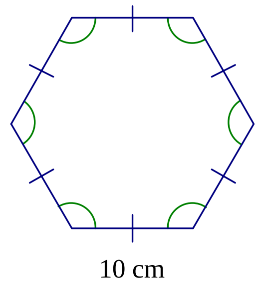
$$\frac{360^\circ}{n}.$$

Exercises
Answer the following:

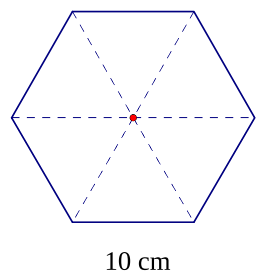
1. Sketch each of the following regular polygons. Then draw a central angle for each and find its measure.
   a) Equilateral Triangle
   b) Square
   c) Regular Pentagon
   d) Regular Hexagon
   e) Regular Octagon

2. Find the measure of a central angle for the following regular polygons.
   a) Regular Decagon
   b) Regular Dodecagon (a dodecagon has 12 sides)
   c) Regular 36-gon
   d) Regular 120-gon

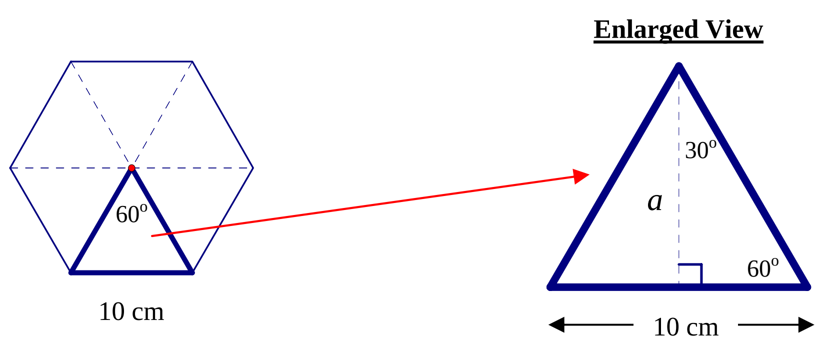
Suppose that we want to find the area of the following regular hexagon.



First, we draw all radii of the regular hexagon to divide it into six congruent triangles. (We will use this approach to discover a general formula for the area of any regular polygon.)



Let us examine the properties of one of the triangles above. Since a regular hexagon has 6 sides, the measure of one of its central angles is $\frac{360^\circ}{6} = 60^\circ$. If an apothem is drawn, we can see that it forms a 30^o-60^o-90^o triangle, as shown in the enlarged view of the triangle below.



We want to find the length of the apothem. The shorter leg of the 30^o-60^o-90^o triangle has length 5 cm, since the apothem intersects the side at its midpoint. The length of the longer leg, a, is $\sqrt{3}$ times the length of the shorter leg, so $a = 5\sqrt{3}$ cm.

We now rearrange the six individual triangles to form a parallelogram. (The figure is again enlarged to show detail.)

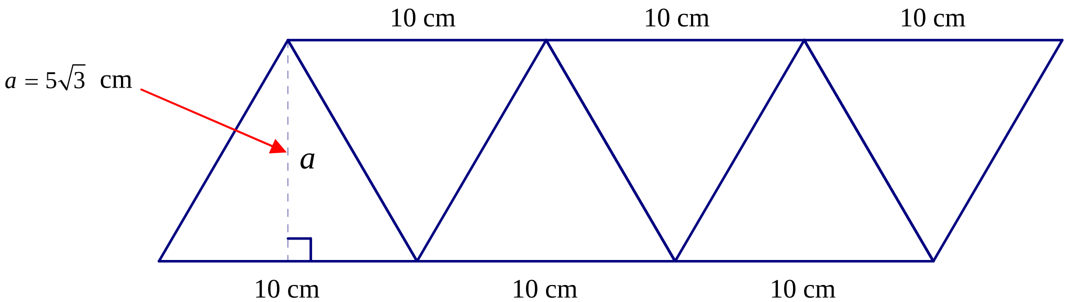


Figure enlarged to show detail.

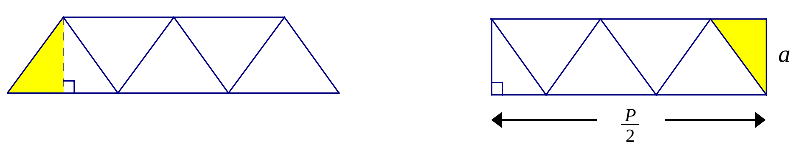
Notice that the parallelogram has a base of 30 cm and a height of $5\sqrt{3}$ cm. To find the area of a parallelogram, we use the formula $A = bh = 30\left(5\sqrt{3}\right) = 150\sqrt{3}$ cm^2. Therefore, the regular hexagon with side length 10 cm has area $150\sqrt{3}$ cm^2.

We now want to use these results to find a general formula for the area of a regular polygon. Notice that the base of the parallelogram is half of the perimeter of the regular hexagon. (The perimeter of the regular hexagon is 60 cm, and the base of the parallelogram is 30 cm.) If we denote the base of the parallelogram as b and the perimeter of the regular polygon as P, then $b = \frac{P}{2}$. We can see from the diagram above that the height, h, of the parallelogram is the same as the apothem, a, of the regular hexagon.

Using the formula $A = bh$ and substituting in $b = \frac{P}{2}$ and $h = a$, we obtain the equation:

$$A = bh = \frac{P}{2} \cdot a = \frac{1}{2} aP$$

Side Note: If the original regular polygon had an odd number of sides (let's say a regular pentagon, for example), we would have obtained a trapezoid instead of a parallelogram when we pieced the triangles together. This diagram could still be formed into a parallelogram (more specifically, a rectangle), by 'slicing' off one of the ends and repositioning it as shown below.



A different approach:

Another method of finding the area of the regular hexagon is to find the area of one of the triangles and then multiply by 6 to find the total area (since there are six triangles formed by all of the central angles). In this case,

$$\text{The area of one triangle} = \frac{bh}{2} = \frac{10\left(5\sqrt{3}\right)}{2} = 25\sqrt{3}\ \text{cm}^2.$$

Therefore, the area A of the entire hexagon is

$$A = 6\left(25\sqrt{3}\right) = 150\sqrt{3}\ \text{cm}^2.$$

In general terms, if x represents a side of the regular polygon, and we form n triangles by drawing all radii of the regular polygon, then

$$\text{The area of one triangle} = \frac{bh}{2} = \frac{xa}{2}.$$

So for an n-sided regular polygon, the area A of the entire polygon (formed by n triangles) is

$$A = n\left(\frac{xa}{2}\right) = \frac{nxa}{2}$$

Since the perimeter P of the regular polygon can be represented by $P = nx$, we substitute this into the above equation and obtain

$$A = \frac{nxa}{2} = \frac{Pa}{2} = \tfrac{1}{2}aP$$

Notice that in both methods used above, we obtained the same general formula for the area of a regular polygon. This formula can be used as a 'shortcut' instead of first dividing the regular polygon into triangles in order to find its area.

The Area of a Regular Polygon

If the apothem and perimeter of a regular polygon are denoted as a and P, respectively, then the area of the regular polygon is given by the formula:

$$A = \tfrac{1}{2}aP$$

Examples
Answer the following.

1. Use the formula for the area of a regular polygon to find the area of a regular hexagon with side length 16 in.

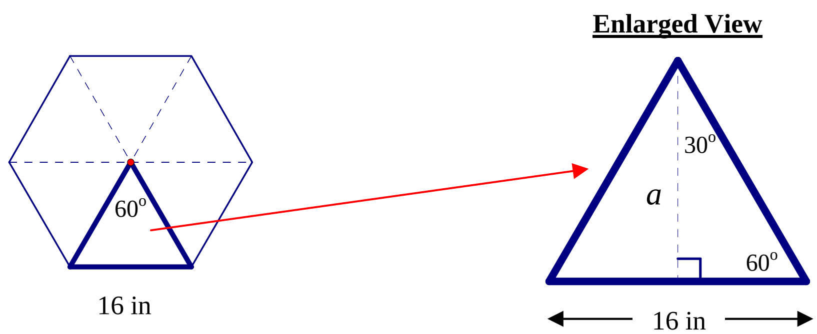
2. Find the area of an equilateral triangle with side length 10 in by using the following two methods:
   a) Use the formula for the area of a regular polygon.
   b) Find the height of the triangle and use the general formula for the area of any triangle.

3. Find the area of a square with side length 7 in by using the following two methods:
   a) Use the formula for the area of a regular polygon.
   b) Use the formula for the area of a square.

<u>Solutions:</u>

1. The perimeter, P, of the regular hexagon is $P = 6(16) = 96$ in.

   The measure of one of its central angles is $\frac{360^\circ}{6} = 60^\circ$. If an apothem is drawn, we can see that it forms a 30^o-60^o-90^o triangle, as shown in the enlarged view of the triangle below.

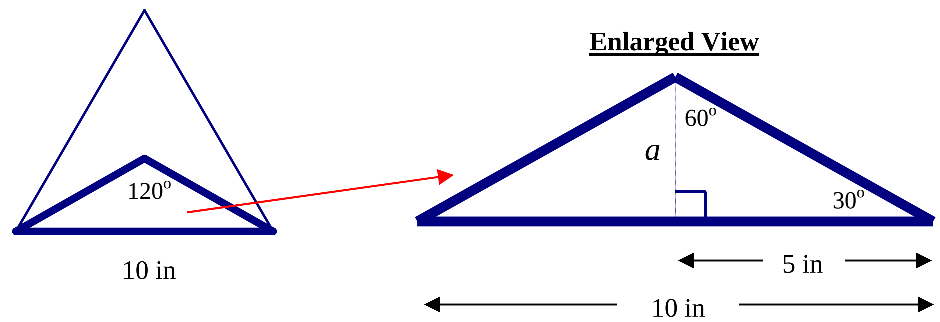


   The shorter leg of the 30^o-60^o-90^o triangle has length 8 cm, since the apothem intersects the side at its midpoint. The length of the longer leg, a, is $\sqrt{3}$ times the length of the shorter leg, so $a = 8\sqrt{3}$ cm.

   We now have the information that we need to find the area of the regular hexagon:

$$A = \tfrac{1}{2}aP = \tfrac{1}{2}\left(8\sqrt{3}\right)\left(96\right) = 384\sqrt{3}\ \text{in}^2.$$

2. a) The perimeter, P, of the equilateral triangle is $P = 3(10) = 30$ in. The measure of one of its central angles is $\frac{360^\circ}{3} = 120^\circ$. If an apothem is drawn, we can see that it forms a 30^o-60^o-90^o triangle, as shown in the enlarged view of the triangle below.



Focusing now on the 30^o-60^o-90^o triangle, the length of the shorter leg is a. Since the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg, the length of the longer leg is $a\sqrt{3}$. We know that the length of the longer leg is 5 in (since the apothem intersects the side at its midpoint), so $a\sqrt{3} = 5$. Solving for a and rationalizing the denominator, we obtain

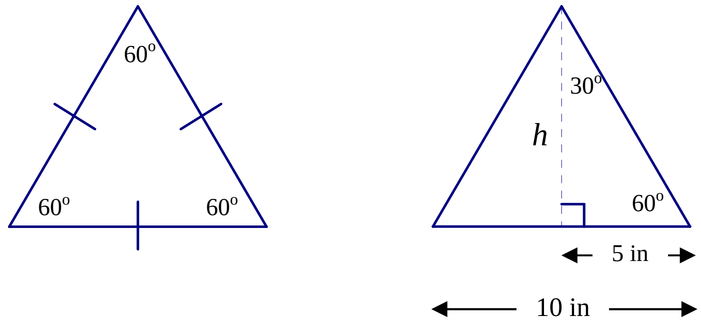
$$a = \frac{5}{\sqrt{3}} = \frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{3}.$$

We now have the information that we need to find the area of the equilateral triangle:

$$A = \frac{1}{2}aP = \frac{1}{2}\left(\frac{5\sqrt{3}}{3}\right)(30) = \frac{150\sqrt{3}}{6} = 25\sqrt{3} \text{ in}^2.$$

b) We will now find the height of the equilateral triangle and then use the general formula $A = \frac{1}{2}bh$ for the area of a triangle.

An equilateral triangle has three angles of equal measure. Since the sum of the measures of a triangle is 180^o, each angle of an equilateral triangle measures 60^o. We then draw an altitude of the triangle (i.e. its height), which forms a 30^o-60^o-90^o triangle, as shown below.

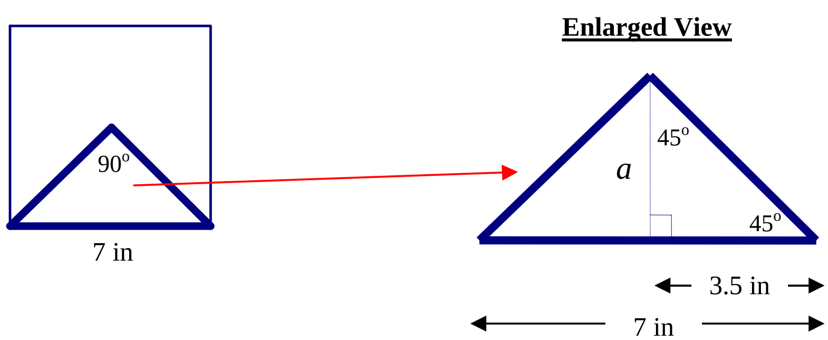


The shorter leg of the 30^o-60^o-90^o triangle has length 5 in, since the altitude intersects the base ($b = 10$ in) at its midpoint. The length of the longer leg, h, is $\sqrt{3}$ times the length of the shorter leg, so $h = 5\sqrt{3}$ in.

We can now find the area of the equilateral triangle by using the formula for the area of a triangle:

$$A = \tfrac{1}{2}bh = \tfrac{1}{2}(10)\left(5\sqrt{3}\right) = 25\sqrt{3}\ \text{in}^2$$

3. a) The perimeter, P, of the square is $P = 4(7) = 28$ in. The measure of one of its central angles is $\frac{360^\circ}{4} = 90^\circ$. If an apothem is drawn, we can see that it forms a 45^o-45^o-90^o triangle, as shown in the enlarged view of the triangle below.



Focusing now on the 45^o-45^o-90^o triangle, the legs of the triangle are congruent, so $a = 3.5$ in.

We now have the information that we need to find the area of the square:

$$A = \tfrac{1}{2}aP = \tfrac{1}{2}(3.5)(28) = 49\ \text{in}^2.$$

b) Since the side, s, of the square is 7 in, we can use the formula $A = s^2$ to find the area of the square.

$$A = s^2 = 7^2 = 49 \text{ in}^2$$

In Example 3 above, it was significantly shorter to use the direct formula for the area of a square (part b) than to use the formula for a regular polygon (part a). In Example 2, the two methods were more equivalent. In Example 1, however, (the regular hexagon) there was only method available to find the area – since we have not learned a formula for the area of a regular hexagon apart from the general formula for any regular polygon.

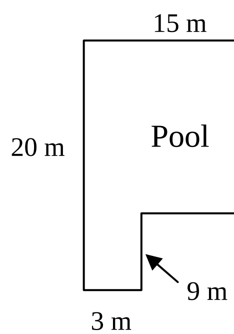
Notice that the only detailed examples discussed in this section were the equilateral triangle, the square, and the regular hexagon. The reason for this is because those are the only regular polygons for which we can use special right triangles (30^o-60^o-90^o and 45^o-45^o-90^o triangles) to find the length of the apothem. If we wanted to find the area, for example, or a regular pentagon (or any other regular polygon not mentioned above), we could still use the formula $A = \frac{1}{2}aP$ to compute its area, but we would need to use Trigonometry to find the length of the apothem. (Refer to the upcoming unit on applications of Right Triangle Trigonometry for more details on how to do this.)

Exercises

Answer the following. Be sure to include units in each answer. (Figures may not be drawn to scale.)

1. Find the perimeter of a square with side length 100 cm.
2. Find the side length of a square with perimeter 100 cm.
3. Find the area of a square with side length 100 cm.
4. Find the side length of a square with area 100 cm.
5. Find the perimeter of an equilateral triangle with side length 6 in.
6. Find the side length of an equilateral triangle with perimeter 6 in.
7. Find the area of an equilateral triangle with side length 6 in.
8. Find the perimeter of a rectangle with length 7 m and width 5 m.
9. Find the area of a rectangle with length 7 m and width 5 m.
10. Find the width of a rectangle with length 8 in and perimeter 40 in.
11. Find the width of a rectangle with length 8 in and area 40 in^2.
12. Find the perimeter of a regular hexagon with side length 6 cm.
13. Find the area of a regular hexagon with side length 6 cm.

14. Find the side length of a regular hexagon with perimeter 6 cm.
15. If a photo frame has perimeter 54 in and width 12 in, find its length.
16. If a photo frame has area 54 in^2 and length 12 in, find its width.
17. If a square has side length $(x+3)$ ft, write an algebraic expression for the area of the square in terms of x.
18. If a rectangle has width $(x+3)$ ft and length $(x+7)$ ft, write an algebraic expression for the area of the rectangle in terms of x.
19. If a rectangle has width $(x+3)$ ft, length $(x+7)$ ft, and area 165 ft^2,
    a) Find the value of x.
    b) Find the width.
    c) Find the length.
20. If a rectangle has width $(x+3)$ ft, length $(x+7)$ ft, and perimeter 104 ft,
    a) Find the value of x.
    b) Find the width.
    c) Find the length.
21. If a rectangle has area $(2x^2+5x+3)\text{ ft}^2$ and width $(x+1)\text{ ft}$, find the length in terms of x.
22. Find the perimeter of the following swimming pool. (Assume that all intersecting sides in the diagram are perpendicular to each other.)



23. Find the area of the swimming pool in the diagram above.
24. Find the circumference of a circle with radius 16m.
25. Find the circumference of a circle with diameter 16m.
26. Find the area of a circle with radius 16m.
27. Find the area of a circle with diameter 16m.

28. Find the radius of a circle with area $16\pi \text{ m}^2$.

29. Find the radius of a circle with circumference 16π m.

30. Find the area of a circle with circumference 36π cm.

31. Find the circumference of a circle with area $36\pi \text{ cm}^2$.

32. Find the area of a parallelogram with base 5 cm and height 6 cm.

33. Find the area of a triangle with base 5 cm and height 6 cm.

34. Find the area of a trapezoid with bases 5 cm and 12 cm, and height 6 cm.

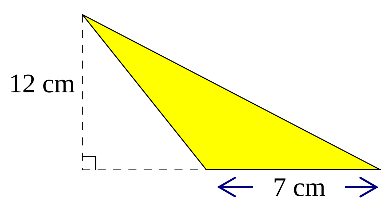
35. If a parallelogram has area 28 in^2 and base 8 in, find the height.

36. If a triangle has area 28 in^2 and base 8 in, find the height.

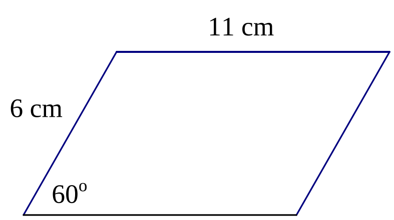
37. If a trapezoid has area 28 in^2 and bases 8 in and 12 in, find the height.

38. Find the area of the following figures:

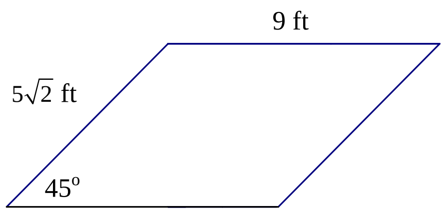
a)



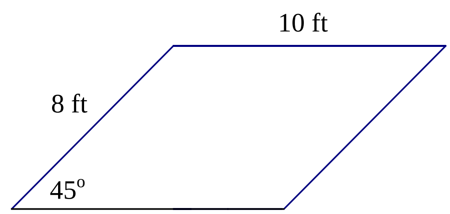
b)



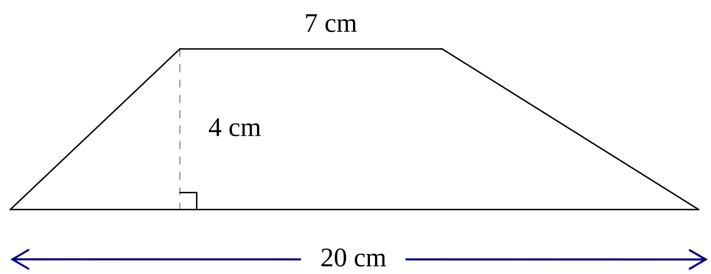
c)



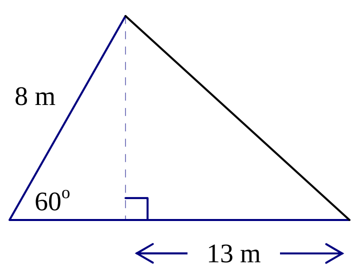
d)



e)



f)



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