The Locker Problem

Purpose:
Participants will determine attributes of numbers and their factors.

Overview:

Participants will determine which locker numbers have exactly two factors, an odd number of factors, an even number of factors, and a large number of factors. They will determine which lockers will be left open and which will be left closed. They will investigate which locker numbers are visited the most frequently and why. They will determine which lockers are visited exactly twice and why.

TExES Mathematics 4-8 Competencies. The beginning teacher:

1.002.B	Analyzes and describes relationships between number properties, operations, and algorithms for the four basic operations involving integers, rational numbers, and real numbers.
I.003.A	Demonstrates an understanding of ideas from number theory (e.g., prime factorization, greatest common divisor) as they apply to whole numbers, integers, and rational numbers, and uses these ideas in problem situations.
I.003.E	Applies properties of the real numbers to solve a variety of theoretical and applied problems.
1.004.C	Makes, tests, validates, and uses conjectures about patterns and relationships in data presented in tables, sequences, or graphs.

TEKS Mathematics Objectives. The student is expected to:

4.4.A	Model factors and products using arrays and area models.
4.4.B	Represent multiplication and division situations in picture, word, and number form.
4.4.C	Recall and apply multiplication facts through 12 x 12.
4.4.D	Use multiplication to solve problems involving two-digit numbers.
5.3.B	Use multiplication to solve problems involving whole numbers (no more than three digits times two digits without technology).
5.3.C	Use division to solve problems involving whole numbers.
5.3.D	Identify prime factors of a whole number and common factors of a set of whole numbers.
5.5.B	Use lists, tables, charts, and diagrams to find patterns and make generalizations.
5.5.C	Identify prime and composite numbers using concrete models and patterns in factor pairs.
6.1.E	Identify factors and multiples including common factors and common multiples.
6.2.C	Use multiplication and division of whole numbers to solve problems.
7.2.E	Simplify numerical expressions involving order of operations and exponents.
7.2.F	Select and use appropriate operations to solve problems and justify the selections.
8.2.A	Select and use appropriate operations to solve problems and justify the selections.

Terms.
Factor, divisor, multiple, exponent, perfect square number, prime factors

Materials.

- Transparencies
- Activity Sheets
- Graph paper

Transparencies.

- *Transparency: The Locker Problem*
- *Transparency: Solution*

Activity Sheet(s).

- *Activity Sheet: The Locker Problem*

Procedure:

Steps	Questions/Math Notes
1. Place the transparency *The Locker Problem* on the overhead projector and have the participants read the problem.	*State the rules of the game in your own words?*
2. With the help of the participants, act out the first few steps of *The Locker Problem*. Have the participants form a circle and explain that each person represents a locker. Number the participants (they will each represent a locker). The locker is closed when a person's back faces the inside of the circle; the locker is open when the person's front faces the inside of the circle. Ask Student #1 to role-play the part of the 1st student in the problem. That person is to walk around the circle and open every locker. (It is effective to have Student #1 hand each person in the circle a 3x5 card with the number 1 written on it.) Ask Student #2 to role-play the part of the 2nd student in the problem. That person is to walk around the circle and change the position of every 2nd locker (beginning with locker #2). (It is effective to have Student #2 hand each even-numbered person in the circle a 3x5 card with the number 2 written on it.	*What number locker do you represent?* *What is the initial position of the lockers?* (All lockers are closed so all persons should face the outside of the circle.) *What does the 1st student in this problem do?* (The first student opens every locker.) Have Student #1 (starting at locker #1) tap each person on the shoulder to open the locker. As participants are tapped, they should rotate so that they are facing the inside of the circle. *What does the 2nd student do? Are his/her actions different from the 1st student's?* (The 2nd student closes every other locker beginning with locker #2.) Ask Student #2 to begin with locker #2 and tap every other person on the shoulder. Those who are tapped should *change the position* of their locker. Half the circle should be facing inward and half the circle (those who were just

	Ask Student #3 to role-play the part of the 3rd student in the problem. That person is to walk around the circle and change the position of every 3rd locker (beginning with locker #3). (It is effective to have Student #3 hand each person whose position is changed a 3x5 card with the number 3 written on it.	tapped) should be facing outward. *How do the actions of the 3rd student differ from the 2nd student's?* (The 3rd student *changes the state* of every 3rd locker, beginning with locker #3.) Have Student #3 begin with locker #3 and tap every 3rd person on the shoulder. Those who are tapped should *change the state* of their locker. Continue the pattern through at least Student #8.
3.	Have participants work with a partner on the Activity Sheet: *The Locker Problem* using graph paper to build a table that shows the actions of the first 30-40 students. Monitor their work and ask scaffolding questions as needed to clarify and extend their thinking about the actions of the students in the problem and the results of their actions.	*What information do you need to collect in order to answer the questions?* *How are you going to record the actions of each student? Will this approach provide you with the information you need to answer the questions?*
4.	Debrief the activity by having several groups share how they solved different pieces of the problem.	*How did you determine how many lockers will be open?* *How did the method you used for recording the data assist you in answering this question?* *How did you determine which lockers will be open? Why are these lockers open?* *How did you determine which lockers switched positions the most times? What are these lockers? Why did their position change so many times?* *Which lockers switched positions exactly twice? How did you determine the numbers of the lockers? How did you count these lockers?*
5.	Have participants summarize in writing three key ideas that can be gleaned from this activity.	

Possible Solution:

1. The lockers that will be open have numbers that are perfect squares and are less than 1000. These lockers are: $(1)^2$, $(2)^2$, $(3)^2$, $(4)^2$, $(5)^2$, … $(31)^2$. Hence, there are 31 lockers of the 1000 total that will be open.
   Open: 31 lockers   Closed: 969 lockers

2. Open lockers are numbered: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961.
   These lockers are open because they have an odd number of factors. They are open because they have a double factor (e.g., 4 x 4).
   16 = 1 x 16
   16 = 2 x 8
   16 = 4 x 4
   16 has 5 factors. It requires an odd number of factors for a locker to end in an open position. An even number of factors leaves a locker in a closed position.

3. Lockers which change their position the most are the ones whose numbers have the most factors.
   840 has 32 factors. $840 = 2^3 \times 3^1 \times 5^1 \times 7^1$

840 = 1 x 840	840 = 10 x 84
840 = 2 x 420	840 = 12 x 70
840 = 3 x 280	840 = 14 x 60
840 = 4 x 210	840 = 15 x 56
840 = 5 x 168	840 = 20 x 42
840 = 6 x 140	840 = 21 x 40
840 = 7 x 120	840 = 24 x 35
840 = 8 x 105	840 = 28 x 30

   Can you find one that has more factors?

4. Lockers that will change their state exactly twice are numbered with prime numbers. The prime numbers between 1 and 1000 are:
   2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 739, 743, 751, 757, 761, 769, 773, 787, 797, 809, 811, 821, 823, 827, 829, 839, 853, 857, 859, 863, 877, 881, 883, 887, 907, 911, 919, 929, 937, 941, 947, 953, 967, 971, 977, 983, 991, 997

   Hence, there are 158 lockers that will change their position exactly twice.

   Visit http://www.utm.edu/research/primes/lists/small/1000.txt to see a listing of the first 1000 primes.

Supporting Websites:

The Locker Problem

Use Joe O'Reilly's applet to simulate The Locker Problem for the first 100 lockers:
http://hydra.educ.queensu.ca/maths/Locker/en.shtml

Visit the Math Forum for several interesting ideas on how to use this problem with Middle School students. Check out the "Locker Boards" and the clear instructions on how to use Claris Works Spreadsheets for recording the data.
http://mathforum.org/alejandre/frisbie/student.locker.html

See Middle School student responses to The Locker Problem when it was posted as a Math Forum Problem of the Week (POW).
http://mathforum.org/midpow/solutions/19980309.midpow.html

Listings of Prime Numbers

For a listing of the first 1000 prime numbers, visit:
http://www.utm.edu/research/primes/lists/small/1000.txt

Sieve of Eratosthenes

To find the prime numbers less than 400, use the Eratosthenes' Sieve:
http://www.faust.fr.bw.schule.de/mhb/eratosiv.htm
The classroom version is excellent also
http://www.faust.fr.bw.schule.de/mhb/eratclass.htm

Reference:

House, P. A. (1980). Making a Problem of Junior High School Mathematics. *The Arithmetic Teacher, 28*(2), 20-23.