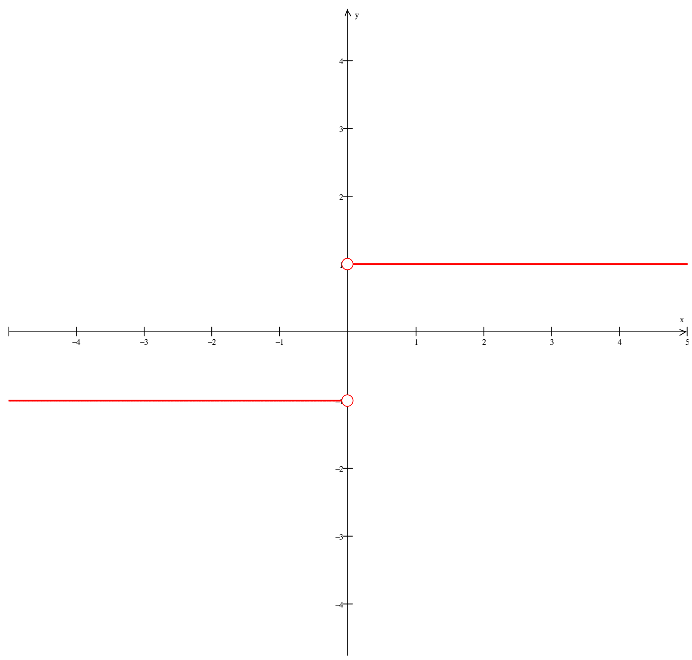
Math 1314
One-Sided Limits and Continuity

One-Sided Limits

Sometimes we are only interested in the behavior of a function when we look from one side and not from the other.

Example 1: Consider the function $f(x) = \frac{|x|}{x}$. Find $\lim_{x \to 0} f(x)$.



Now suppose we are *only* interested in looking at the values of x that are bigger than 0. In this case, we are looking at a **one-sided limit**.

We write $\lim_{x \to 0^+} f(x)$. This is called a right-hand limit, because we are looking at values on the right side of the target number.

In this case,

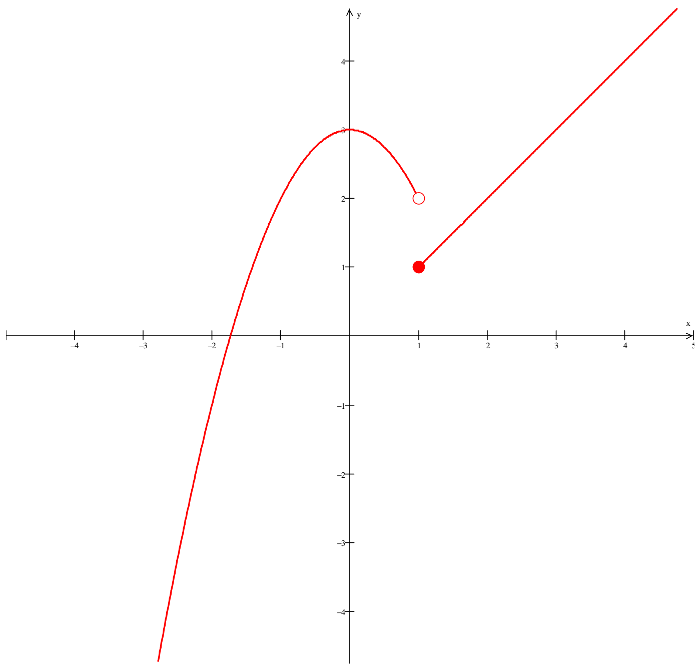
If we are interested in looking only at the values of x that are smaller than 0, then we would be finding the left-hand limit. The values of x that are smaller than 0 are to the left of 0 on the number line, hence the name. We write $\lim_{x \to 0^-} f(x)$.

In this case,

Our definition of a limit from the last lesson is consistent with this information. We say that $\lim_{x\to a} f(x) = L$, if and only if the function approaches the same value, *L*, from both the left side and the right side of the target number. This idea is formalized in this theorem:

Theorem: Let *f* be a function that is defined for all values of *x* close to the target number *a*, except perhaps at *a* itself. Then $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$.

Example 2: Consider this graph:



Find $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1} f(x)$, if it exists.

We can also find one-sided limits from piecewise defined functions.

Example 3: Suppose $f(x) = \begin{cases} x+4, & x > 2 \\ x^2 - 1, & x \le 2 \end{cases}$. Find $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2} f(x)$ (if it exists).

Example 4: Suppose $f(x) = \begin{cases} x^2 + 3, & x < -1 \\ x^3 + 5, & x \geq -1 \end{cases}$.

Find $\lim_{x \to -1^-} f(x)$, $\lim_{x \to -1^+} f(x)$ and $\lim_{x \to -1} f(x)$ (if it exists).

Continuity

A function is continuous on an interval if it has no gaps, holes, breaks or jumps. If you can trace the graph of a function over a given interval without ever lifting your pencil, then the function is continuous over that interval.

We will be interested in finding where a function is continuous and where it is discontinuous. We'll look at continuity over the entire domain of the function, over a given interval and at a specific point.

Continuity at a Point:

A function f is said to be continuous at the point $x = a$ if the following three conditions are met:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

Example 7: Determine if $f(x) = \begin{cases} 5x - 1, & x \geq 1 \\ x^2 + 3, & x < 1 \end{cases}$ is continuous at $x = 1$.

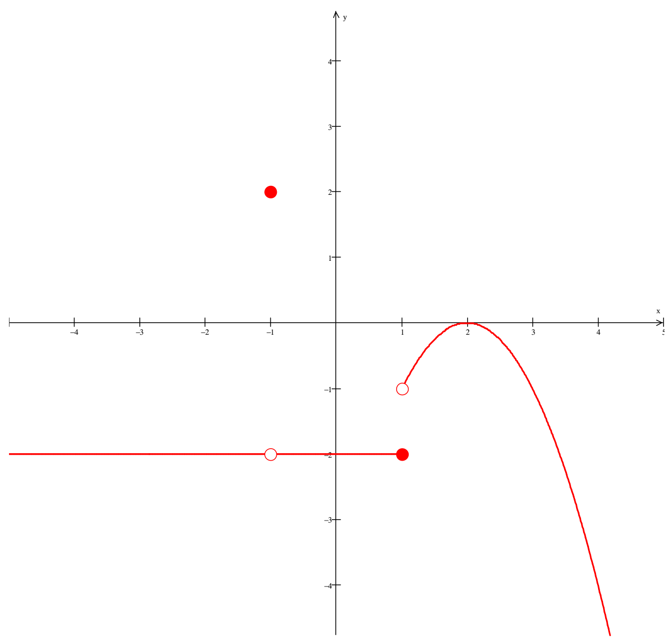
If a function is not continuous at a point, then we say it is discontinuous at that point.

Continuity over an Interval

A function is continuous over the interval [a, b] if it is continuous at every point in the interval.

We find points of discontinuity by examining the function that we are given.

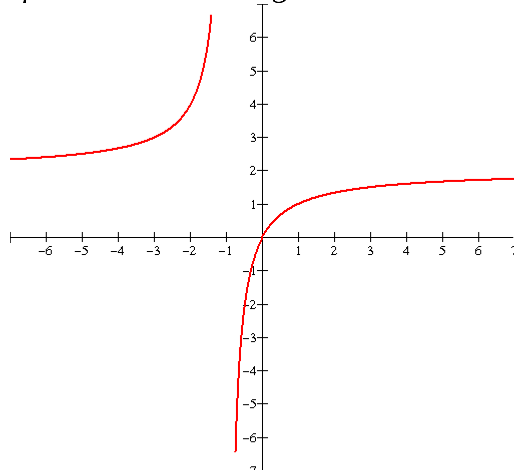
Example 8: Find any points of discontinuity. State why the function is discontinuous.



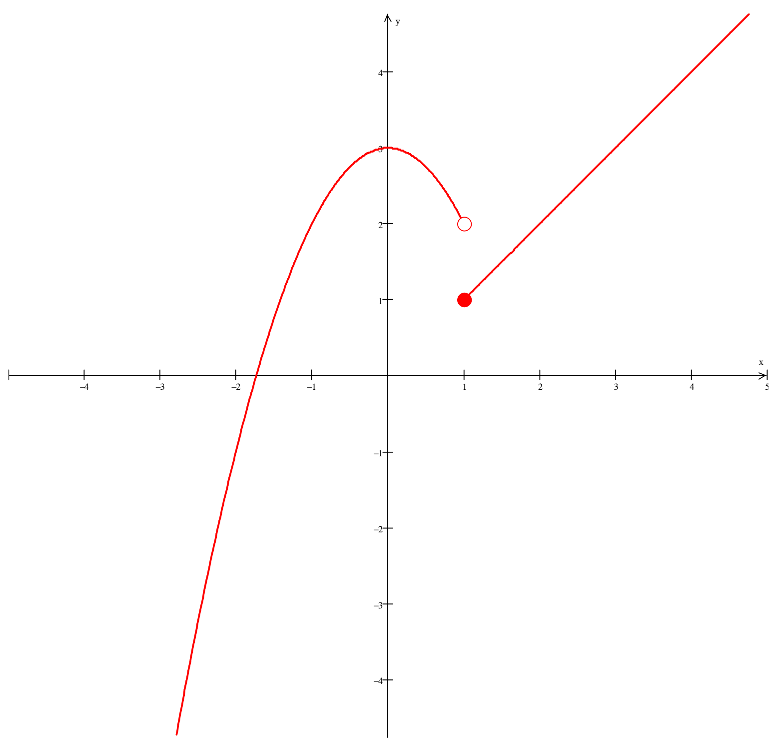
Examples

Once we know where a function is discontinuous, we know it is continuous everywhere else.

Example 11: State where *f* is continuous using interval notation:



Example 12: State where *f* is continuous using interval notation:



Example 13: State where f is continuous using interval notation: $f(x)=\dfrac{3x-7}{x^2-25}$.

Sometimes we consider continuity over the entire domain of the function. For many functions, this is the entire set of real numbers. A function is continuous on its domain if there are no points of discontinuity in the domain of the function.

Example 14: State where $f(x)=5x^4+6x^2-3x+1$ is continuous.

From this lesson you should be able to

- Say what we mean by a one-sided limits
- Find a one-sided limit from the graph of a function
- Find a one-sided limit from a piecewise-defined function
- Find a one-sided limit from a function
- Determine if a function is continuous at a point
- Find points of discontinuity over an interval or over the domain of the function given either a graph of the function or the function itself
- State intervals where a function is continuous given either a graph of the function or the function itself