Math 1314
Antiderivatives

So far in this course, we have been interested in finding derivatives and in the applications of derivatives. In this chapter, we will look at the "reverse" process. Here we will be given the "answer" and we'll have to find the problem. In other words, if we are given a function and told that it is the derivative, we'll want to find the original function.

Antiderivatives

Definition: A function F is an antiderivative of f on interval I if $F'(x) = f(x)$ for all x in I.

The process of finding an antiderivative is called **antidifferentiation** or **finding an indefinite integral**.

Example 1: Determine if F is an antiderivative of f if $F(x) = x^3 - x^2 + 4x + 1$ and $f(x) = 3x^2 - 2x + 4$.

Example 2: Suppose $F(x) = x^3 + 2$, $G(x) = x^3 - 5$, $H(x) = x^3 + 10$ and $K(x) = x^3 - 27$. If $f(x) = 3x^2$, show that each of F, G, H and K is an antiderivative, and draw a conclusion.

Notation: We will use the integral sign $\int$ to indicate integration (antidifferentiation). Problems will be written in the form $\int f(x)\,dx = F(x) + C$. This indicates that the indefinite integral of $f(x)$ with respect to the variable x is $F(x) + C$ where $F(x)$ is an antiderivative of f.

Basic Rules

Rule 1: The Indefinite Integral of a Constant

$\int k\,dx = kx + C$

Example 3: $\int (-9)\,dx$

Rule 2: The Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\; n \neq -1$$

Example 4: $\int x^5\,dx$

Example 5: $\int \sqrt[3]{x}\,dx$

Example 6: $\int \frac{1}{x^{\frac{7}{3}}} dx$

Rule 3: The Indefinite Integral of a Constant Multiple of a Function

$\int cf(x)dx = c\int f(x)dx$

Example 7: $\int 5x^4 dx$

Example 8: $\int 4x^{\frac{3}{2}} dx$

Example 9: $\int \frac{-8}{x^5} dx$

Rule 4: The Sum (Difference) Rule

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

Example 10: $\int (4x^2 - 7x + 3)dx$

Rule 5: The Indefinite Integral of the Exponential Function

$$\int e^x dx = e^x + C$$

Example 11: $\int (2e^x - 3x^5)dx$

Rule 6: The Indefinite Integral of the Function $f(x) = \dfrac{1}{x}$

$$\int \frac{1}{x} dx = \ln|x| + C, \quad x \neq 0$$

Example 12: $\int \left(6x + \dfrac{2}{x} - \dfrac{4}{x^2} \right) dx$

Applying the Rules

Example 13: $\int \frac{9x+7x^2-2x^3}{x}\,dx$

Example 14: $\int \left(\sqrt{x} + \frac{6}{\sqrt[4]{x}} \right) dx$

Example 15: $\int x^2 \left(\frac{3}{x} - \frac{8}{x^2} + \frac{2}{x^3} \right) dx$

Example 16: $\int (\sqrt{x^7} - 2e^x - \frac{8}{x})dx$

Differential Equations

A differential equation is an equation that involves the derivative (or differential) of some function. So, if we write $f'(x) = 3x + 5$, we have a differential equation. We will be interested in solving these.

A **solution** of a differential equation is any function that satisfies the differential equation. So, for the example above, $f(x) = \frac{3}{2}x^2 + 5x + 3$ is a solution of the differential equation, since $f'(x) = 3x + 5$.

The **general solution** of a differential equation is one which gives all of the solutions, so the general solution for the example above will be $f(x) = \frac{3}{2}x^2 + 5x + C$.

If we are given a point that lies on the function, we can find a **particular solution**; that is, we can find C. If we know that $f(-2) = 1$, we can substitute this information into our general solution and solve for C:

$f(-2) = 1$ is called an **initial condition**.

Initial Value Problems

An **initial value problem** is a differential equation together with one or more initial conditions. If we are given this information, we can find the function *f* by first finding the general solution and then finding the value of *C* that satisfies the initial condition.

Example 17: Solve the initial value problem:

$$\left.\begin{array}{l} f'(x) = 3x + 1 \\ f(3) = 2 \end{array}\right\}$$

Example 18: Solve the initial value problem:

$$\left.\begin{array}{l} f'(x) = 6x^2 - 9x + 1 \\ f(3) = 0 \end{array}\right\}$$

Example 19: Solve the initial value problem:

$$\left.\begin{array}{l} f'(x) = 3e^x - 4x \\ f(0) = -3 \end{array}\right\}$$

From this section, you should be able to

- Explain what we mean by an antiderivative (indefinite integral), a differential equation and an initial value problem
- Determine if one function is an antiderivative of another function
- Use the basic rules to find antiderivatives
- Simplify (if necessary) before applying the basic rules
- Solve initial value problems