Math 1314
Exponential Functions as Mathematical Models
In this lesson, we will look at a few applications involving exponential functions. We'll first consider some word problems having to do with money. Next, we'll consider exponential growth and decay problems.

## Interest Problems

From previous course work, you may have encountered the compound interest formula:

$$
\begin{aligned}
& A=P\left(1+\frac{r}{m}\right)^{m t} \\
& P=\text { principal mount invested } \\
& A=\text { accumulated amount } \\
& r=\text { interest rate } \\
& m=\text { number of times interest is compounded per } \\
& \quad \text { year } \\
& t=\text { time in years }
\end{aligned}
$$

Now suppose we let the number of compounding periods increase, that is, we'll take the limit of this function as $m$ goes to infinity:

$$
\lim _{m \rightarrow \infty} P\left(1+\frac{r}{m}\right)^{m t}
$$

This is a fairly complicated limit to evaluate, so we will omit the details.

$$
\lim _{m \rightarrow \infty} P\left(1+\frac{r}{m}\right)^{m t}=P e^{r t}
$$

You may also have seen this formula before. This is the interest formula to use when interest is compounded continuously.

We'll be interested in two kinds of problems, those that ask for an accumulated amount and those that ask for present value. We'll use two formulas:

Accumulated amount:

$$
\begin{aligned}
& A=P e^{r t} \\
& P=A e^{-r t}
\end{aligned}
$$

Present value:
All values are as defined above.
Example 1: Find the accumulated amount when 53000 is invested for 8 years in an account that pays $2.5 \%$ annual interest compounded continuously.

Example 2: Suppose your current annual salary is $\$ 50,000$ per year. Suppose inflation remains at a flat rate of $3 \%$ per year for the next 12 years. What will your salary need to be in order to have the same purchasing power you have now? Assume that inflation compounds continuously.

## Exponential Functions

Recall the graph of an exponential function, such as $f(x)=3^{x}$.


This is an exponential growth function. The function increases rapidly: each time you increase the $x$ value by 1 unit, you multiply the preceeding $y$ value by 3 .

This kind of growth will occur for any exponential function where $b>1$, including $f(x)=e^{x}$. If $f(x)=\left(\frac{1}{3}\right)^{x}$, we'll have the reflection of this graph about the $y$ axis:


This is an exponential decay function. This kind of decay will occur for any exponential function where $0<b<1$.

We'll look at a function, $Q(t)=Q_{0} e^{k t}$, for exponential growth problems and a different function $Q(t)=Q_{0} e^{-k t}$, for exponential decay problems. In these formulas, $Q_{0}$ is the original amount of the substance or population under study, $Q(t)$ is the amount of the substance or population at time $t$ and $k$ is the growth constant or $-k$ is the decay constant, depending on whether your problem is a growth problem or a decay problem.

We can find the rate of growth or rate of decay by finding the derivative of the growth or decay functions. Thus, the growth rate can be found using $Q(t)=k Q_{0} e^{k t}$ and the decay rate can be found using $Q(t)=-k Q_{0} e^{-k t}$

## Exponential Growth

Example 3: A biologist wants to study the growth of a certain strain of bacteria. She starts with a culture containing 35,000 bacteria. After two hours, the number of bacteria has grown to 53,000 . How many bacterial will be present in the culture 7 hours after she started her study? What will be the rate of growth 7 hours after she started her study? Assume the population grows exponentially.

Example 4: A think tank began a study of population growth in a small country 3 years ago. At the beginning of the study, the population was $4,200,000$. Two years later, it was $5,100,000$. What will the population be 4 years form now? What will the growth rate be in 4 years? Assume the population grows exponentially.

## Exponential Decay

Example 5: At the beginning of a study, there are 150 grams of a substance present. After 23 days, there are 111.2 grams remaining. How much of the substance will be present after 65 days? What will be the rate of decay on day 65 of the study? Assume the substance decays exponentially.

Example 6: A certain drug has a half life of 4 hours. Suppose you take a dose of 1000 milligrams of the drug. How much of it is left in your bloodstream 45 hours later?

From this lesson, you should be able to
Solve problems involving continuously compounded interest, including problems that ask for accumulated amount and present value
Solve problems involving exponential growth
Solve problems involving exponential decay
Find a rate of growth or decay

