Math 1314
Lesson 12
Curve Sketching

One of our objectives in this part of the course is to be able to graph functions. In this lesson, we'll add to some tools we already have to be able to sketch an accurate graph of each function.

From prerequisite material, we can find the domain, *y*-intercept and end behavior of the graph of a function, and from the last two sections, we can learn much about a function by analyzing the first and second derivatives. We also know how to find the zeros of some functions. We'll expand that group of function before we continue to curve sketching.

The Rational Zeros of a Polynomial Function

The rational zeros of a function are the zeros of the function that can be written as a fraction, such as 2 or $\frac{-1}{5}$. Sometimes we can find the rational zeros of a function by factoring.

Example 1: Find the rational zeros: $f(x) = x^3 - 4x^2 - 5x$

Example 2: Find the rational zeros: $f(x) = x^4 - 5x^2 - 36$.

Note that zeros that are square roots are NOT rational roots. Imaginary solutions to the equation $f(x)=0$ (such as $\pm 3i$) are NOT rational roots.

Sometimes we won't be able to factor the function. Then we'll need another method. We'll use a theorem called the Rational Zeros Theorem.

First, we'll find all of the possible rational zeros of a given function using the Rational Zeros Theorem. Then we can use a calculator or synthetic division to determine which – if any – of the possible rational zeros are actually zeros of the function.

Here's the theorem:

Rational Zeros Theorem:

Suppose $f(x)=a_n x^n + a_{n-1}x^{x-1} + \cdots + a_0$, where $a_n \neq 0$ and $a_0 \neq 0$, and all of the coefficients of the polynomial are integers.

If $x=\dfrac{p}{q}$ is a rational zero of the function, where p and q have no common factors, then p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

So the possible rational zeros of $f(x)=3x^3-7x^2-2x+6$ are:

Example 3: Find all rational zeros of $f(x)=x^3+6x^2+11x+6$ or state that there are none.

Example 4: Find all rational zeros of $f(x) = x^3 - 3x^2 - 10x + 24$ or state that there are none.

Example 5: Find all rational zeros of $f(x) = x^4 - 2x^3 - 8x^2 + 18x - 9$ or state that there are none.

Example 6: Find all rational zeros of $f(x) = x^3 - 5x^2 - 3$ or state that there are none.

Example 7: Find all rational zeros of $f(x) = 4x^3 + 16x^2 - x - 4$ or state that there are none.

Curve Sketching

Now we'll turn our attention to graphing functions. You will need to be able to use the following guide to sketch the graphs of functions.

A Guide to Curve Sketching

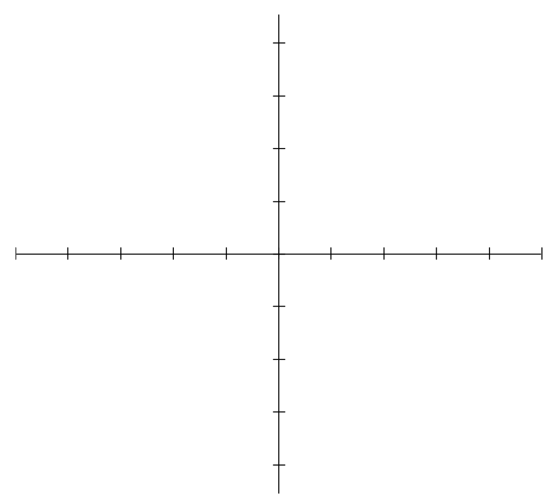
1. Determine the domain of f.
2. Find the rational x intercept(s) and y intercept of the function. If there are no rational x intercepts, say so.
3. Determine the end behavior of the function.
4. For an exponential function, determine any horizontal asymptotes.
5. Determine where the function is increasing and where it is decreasing.
6. Find the x and y coordinates of any relative extrema.
7. Determine where the function is concave upward and where it is concave downward.
8. Find the x and y coordinates of any points of inflection.
9. If necessary, plot a few additional points to determine the shape of the graph.
10. Sketch the function.

Recall the generalizations about end behavior of a polynomial function from College Algebra:

PEHH   NELL

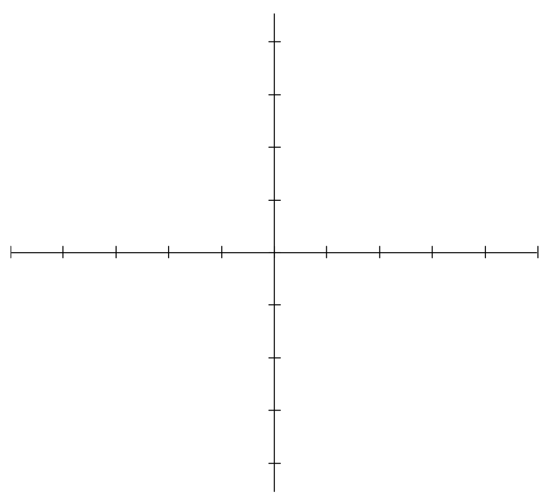
POLH   NOHL

Example 8: Use the guide to curve sketching to sketch $f(x) = x^4 - 4x^3$.

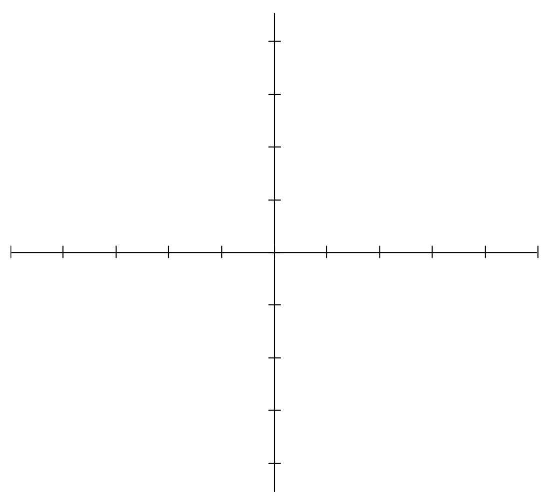


Sometimes a function has some zeros that are not rational. We may occasionally give you the approximate zeros of the function and ask you to complete the rest of the guide to curve sketching.

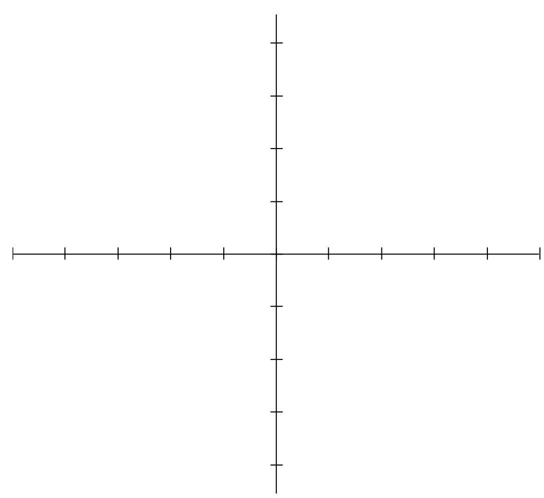
Example 9: Use the guide to curve sketching to sketch. $f(x) = x^3 + 6x^2 - 15x + 3$. Note, the approximate zeros of the function are 0.22, 1.72, and -7.94.



Example 10: Use the guide to curve sketching to sketch $f(x) = x^3 - 8x^2 + 19x - 12$.



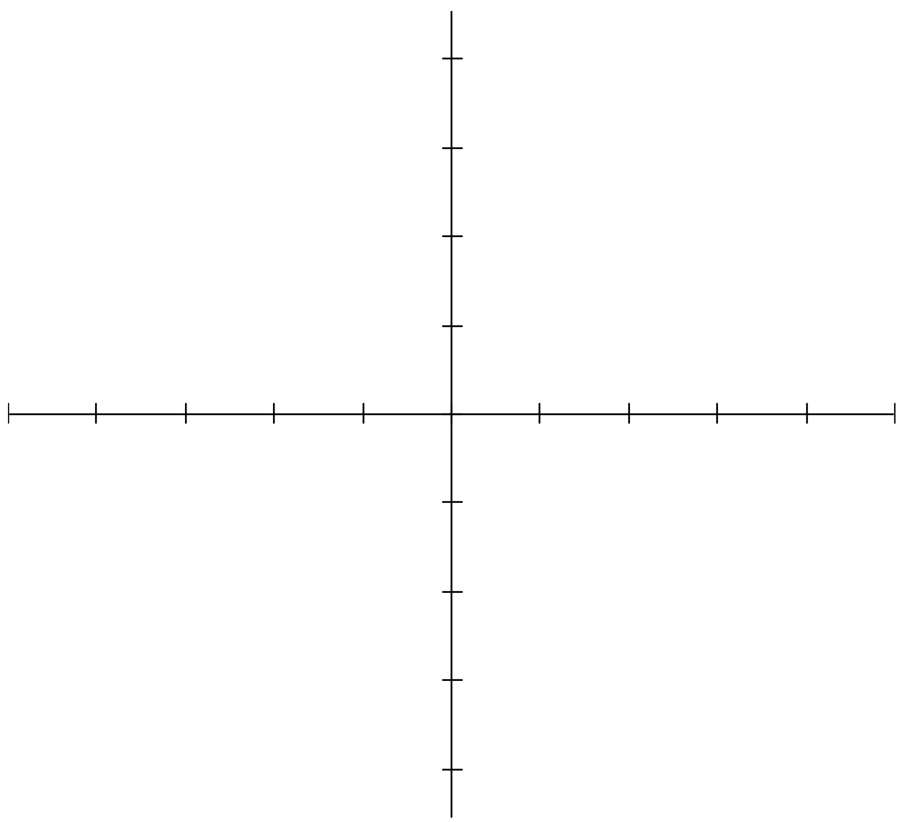
Example 11: Use the guide to curve sketching to sketch $f(x) = 2xe^{x}$.



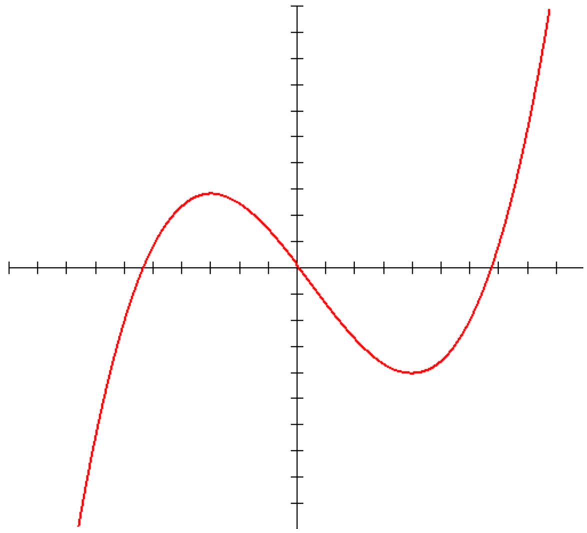
For the next problem, you are given all of the information listed in the guide to curve sketching. You just need to use it to graph the function.

Example 12: Sketch the function if you are given the following information:

x intercept	(0, 0), (4, 0)
y intercept	(0, 0)
end behavior	↑ ↑
relative minimum	(3, -27)
increasing intervals	$(3, \infty)$
decreasing intervals	$(-\infty, 0), (0, 3)$
points of inflections	(0, 0), (2, -16)
concave upward intervals	$(-\infty, 0)$ and $(2, \infty)$
concave downward intervals	(0, 2)



Example 13: Here is the graph of a polynomial function. Which of the statements below is/are true?



1. The function has three zeros
2. The graph of the function is increasing on one interval and decreasing on two intervals.
3. The graph of the function has one relative maximum and one relative minimum.
4. The graph of the function has two inflection points.
5. The function could be a quartic function (4^{th} degree) with a positive leading coefficient.

From this section, you should be able to

- Find any rational zeros of a 3rd or 4th degree polynomial
- Use the guide to curve sketching to sketch the graph of a polynomial or exponential
- Sketch a graph of a function given all of the information from the guide to curve sketching
- Answer questions about the graph of a function given the graph of the function