Math 1314
Lesson 11
Applications of the Second Derivative

## Concavity

As mentioned earlier, the second derivative is the rate of change of the first derivative. The second derivative can tell us if the rate of change of the function is increasing or decreasing. In business, for example, the first derivative might tell us that our profits are increasing, but the second derivative will tell us if the pace of the increase is increasing or decreasing.

Example 1: Draw some lines tangent to each of these graphs. What do you notice about the slopes of the tangent lines?



From these graphs, you can see that the shape of the curve change differs depending on whether the slopes of tangent lines are increasing or decreasing. This is the idea of concavity.

Definition: Let the function $f$ be differentiable on an interval $(a, b)$. Then $f$ is concave upward on $(a, b)$ if $f^{\prime}$ is increasing on $(a, b)$ and $f$ is concave downward on $(a, b)$ if $f^{\prime}$ is decreasing on $(a, b)$.

## Determining Where a Function is Concave Upward and Where it is Concave Downward By Analyzing the Sign of the Second Derivative Algebraically

We can determine concavity intervals by doing the following:

1. Find $f^{\prime \prime}(x)$.
2. Solve $f^{\prime \prime}(x)=0$.
3. Draw a number line and subdivide it at each number you found in step 2.
4. Choose a test value in each interval.
5. Determine the sign of $f^{\prime \prime}(x)$ (either + or - ) in each interval.
6. Apply the following Theorem to determine the intervals where the function is concave upward and the intervals where it is concave downward.

## Theorem:

If $f^{\prime}(x)>0$ for each value of $x$ in $(a, b)$, then $f$ is concave upward on $(a, b)$.
If $f^{\prime \prime}(x)<0$ for each value of $x$ in $(a, b)$, then $f$ is concave downward on $(a, b)$.

Example 2: Determine where the function $f(x)=x^{3}-6 x^{2}+18 x-15$ is concave upward and where it is concave downward.

Example 3: Determine where the function $f(x)=3 x e^{4 x}$ is concave upward and where it is concave downward.

Example 4: Determine where the function $f(x)=x^{2}+8 \ln (5 x)$ is concave upward and where it is concave downward.

## Inflection Points

The point where a function changes from being concave upward to concave downward (or from being concave downward to concave upward) is called a point of inflection or an inflection point. We'll show the significance of this point by an example.

Example 5: Draw some lines tangent to the graph and compare their slopes.


Definition: A point on the graph for a differentiable function $f$ at which the concavity changes is called an inflection point.

## Finding Inflection Points

To find the inflection point(s) of a function,

1. Compute $f^{\prime \prime}(x)$.
2. Find all points in the domain of $f$ for which $f^{\prime \prime}(x)=0$.
3. Determine the sign of $f^{\prime \prime}(x)$ to the left and to the right of each point $x=c$ found in step 2 . If there is a change in the sign of $f^{\prime \prime}(x)$ as we move across the point $x=c$ from left to right, then the point $(c, f(c))$ is an inflection point of $f$.

## Finding Inflection Points by Analyzing the Sign of the Second Derivative

Example 6: Determine any points of inflection if $f(x)=x^{4}-2 x^{3}+6$.

Example 7: Determine any points of inflection if $f(x)=4 x e^{2 x}$.

Example 8: Determine any points of inflection if $f(x)=x^{2}+2 \ln (6 x)$.

## The Second Derivative Test

Recall that we use the first derivative test to determine if a critical number is a relative extremum. There is also a second derivative test to find relative extrema. It is sometimes convenient to use; however, it can be inconclusive.

The Second Derivative Test:

1. Find $f^{\prime}(x)$ and $f^{\prime \prime}(x)$.
2. Find all critical numbers.
3. Compute $f^{\prime \prime}(c)$ for each critical number $c$.
(a) If $f^{\prime \prime}(c)>0$, then $f$ has a relative minimum at $c$.
(b) If $f^{\prime \prime}(c)<0$, then $f$ has a relative maximum at $c$.
(c) If $f^{\prime}(c)=0$, then the test fails. It is inconclusive. Try the First Derivative Test.

Example 9: Find the $x$ coordinate all relative extrema using the Second Derivative Test if $f(x)=\frac{1}{3} x^{3}-2 x^{2}-5 x-10$. .

Example 10: Find the $x$ coordinate and the $y$ coordinate of all relative extrema using the Second Derivative Test if $f(x)=x^{2}+\frac{2}{x}$.

From this lesson, you should be able to
Explain what we mean by concavity and inflection point.
Find interval(s) where a function is concave upward (concave downward) algebraically for a polynomial, exponential or logarithmic function.

Find inflection point(s) algebraically for a polynomial, exponential or logarithmic function.

Use the second derivative test to determine relative extrema of a function.

