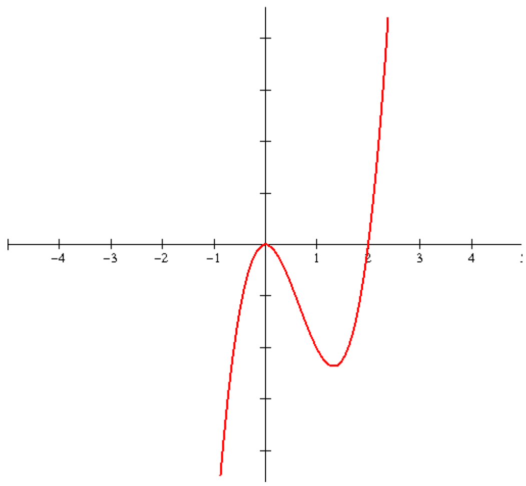
Math 1314
Applications of the First Derivative

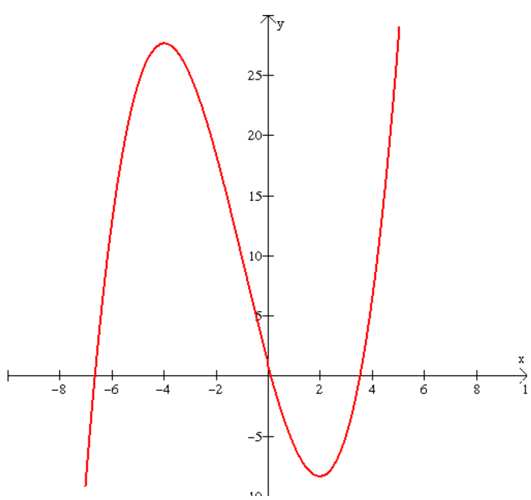
Determining the Intervals on Which a Function is Increasing or Decreasing From the Graph of *f*

Definition: A function is **increasing** on an interval (*a*, *b*) if, for any two numbers x_1 and x_2 in (*a*, *b*), $f(x_1) < f(x_2)$, whenever $x_1 < x_2$. A function is **decreasing** on an interval (*a*, *b*) if, for any two numbers x_1 and x_2 in (*a*, *b*), $f(x_1) > f(x_2)$, whenever $x_1 < x_2$.

Example 1: You are given the graph of *f*(*x*). State the interval(s) on which *f* is increasing and the interval(s) on which *f* is decreasing.



Example 2: You are given the graph of *f*(*x*). State the interval(s) on which *f* is increasing and the interval(s) on which *f* is decreasing.

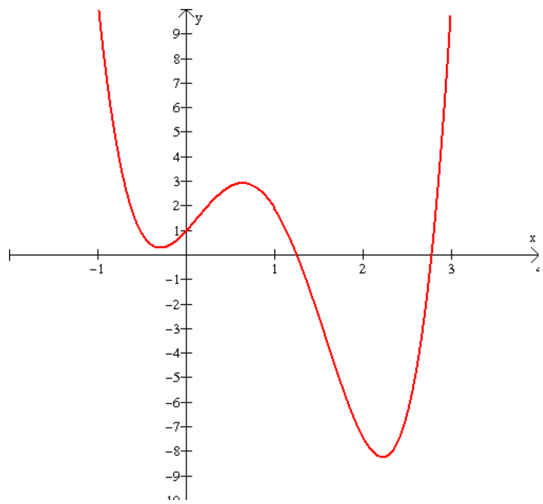


The rate of change of a function at a point is given by the derivative of the function at that point. So we can use the derivative to determine where a function is increasing and where a function is decreasing.

At a point where the derivative is positive, a function is increasing.

At a point where the derivative is negative, a function is decreasing.

Example 3: Let's look at the slopes of some lines that are tangent to this graph at various points:



Determining the Intervals on Which a Function is Increasing or Decreasing
By Finding the Derivative and Analyzing its Sign

We can also determine where a function is increasing and where it is decreasing algebraically. Here's how:

1. Find the derivative of the function.
2. Determine all values of x for which $f'(x) = 0$ or is undefined.
3. Use the values found in step 2 to divide the number line into open intervals.
4. Choose a test value, c, in each open interval.
5. Substitute each test value, c, into the derivative to determine the sign of $f'(c)$.
6. Apply the following theorem:

Theorem:

(a) If $f'(x) > 0$ for each value of x in an interval (a, b), then f is increasing on (a, b).

(b) If $f'(x) < 0$ for each value of x in an interval (a, b), then f is decreasing on (a, b).

(c) If $f'(x) = 0$ for each value of x in an interval (a, b), then f is constant on (a, b).

Example 4: Determine the interval(s) where *f* is increasing and the interval(s) where *f* is decreasing if $f(x) = x^4 - 8x^2 + 4$.

Example 5: Determine the interval(s) where f is increasing and the interval(s) where f is decreasing if $f(x) = xe^{x}$.

Example 6: Determine the interval(s) where f is increasing and the interval(s) where f is decreasing if $f(x) = \ln(x-3)$.

Finding Relative Extrema

The first derivative can also help up find the *x* coordinate(s) of any high points or low points on the graph. This will allow us to find the *x* coordinates of the "humps" in the graphs. Once we find the *x* coordinate of any high or low points, we can substitute that number into the original function to find the *y* coordinate of the high or low point.

These high points and/or low points are called **relative extrema** of a function. An extremum is called a **relative (local) maximum** if it is higher than the points located nearby. An extremum is called a **relative (local) minimum** if it is lower than the points located nearby.

Definition: A function *f* has a **relative maximum** at $x = c$ if there exists an open interval (a, b) containing *c* such that $f(x) \leq f(c)$ for all *x* in (a, b). A function *f* has a **relative minimum** at $x = c$ if there exists an open interval (a, b) containing *c* such that $f(x) \geq f(c)$ for all *x* in (a, b).

To find the relative extrema, we must first find the **critical numbers** of the function.

Definition: A **critical number** of a function *f* is any number *x* in the domain of *f* such that $f'(x) = 0$ or $f'(x)$ does not exist.

Once we find the critical numbers, we can use a line test to determine whether the critical numbert gives us a maximum, a minimum or neither.

The First Derivative Test

To find the relative extrema of a function,

1. Determine the critical numbers of *f*.
2. Determine the sign of *f'*(*x*) to the left and to the right of each critical number.

(a) if *f'*(*x*) changes sign from positive to negative as we move across a critical number $x = c$ from left to right, then $f(c)$ is a relative maximum.

(b) if *f'*(*x*) changes sign from negative to positive as we move across a critical number $x = c$ from left to right, then $f(c)$ is a relative minimum.

(c) if *f'*(*x*) does not change sign as we move across a critical number $x = c$ from left to right, then $f(c)$ is not a relative extremum.

Finding Relative Extrema by Finding the Derivative of *f* and Analyzing its Sign

We can find relative extrema algebraically:

Example 7: Find the relative extrema if $f(x) = x^3 - 3x^2 - 24x + 32$.

Example 8: Find the relative extrema if $f(x) = xe^x$.

Example 9: Find the relative extrema if $f(x) = \ln(x-3)$.

Example 10: After birth, an infant normally will lose weight for a few days and then start gaining. A model for the average W (in pounds) of infants over the first two week following birth is $W(t) = .033t^2 - .3974t + 7.3032,\ 0 \le t \le 14,$ where t is measured in days. Find the interval(s) on which weight is expected to increase and the interval(s) on which weight is expected to decrease based on this model.

From this section, you should be able to

Explain what we mean by an increasing (decreasing) function and relative extremum

State intervals where a function is increasing (decreasing) from a graph of the function

State intervals where a polynomial, exponential or logarithmic function is increasing (decreasing) algebraically

Find relative extrema of a polynomial, exponential or logarithmic function algebraically

Solve word problems involving interval(s) where a function is increasing (decreasing)