Math 1314
Limits

## What is calculus?

The body of mathematics that we call calculus resulted from the investigation of two basic questions by mathematicians in the $18^{\text {th }}$ century.

1. How can we find the line tangent to a curve at a given point on the curve?
2. How can we find the area of a region bounded by an arbitrary curve?



The investigation of each of these questions relies on the process of finding a limit, so we first turn our attention to finding limits.

## Limits

Finding a limit amounts to answering the following question: What is happening to the $y$ value of a function as the $x$-value approaches a specific target number? If the $y$-value is approaching a specific number, then we can state the limit of the function as $x$ gets close to the target number.

Example 1:


Example 2


It does not matter whether or not the $x$ value every reaches the target number. It might, or it might not!

Example 3:


## When can a limit fail to exist?

We will look at two cases where a limit fails to exist (note: there are more, but some are beyond the scope of this course).

Case 1: The $y$ value approaches one number from numbers smaller than the target number and it approaches a second number from numbers larger than the target number:


Case 2: At the target number for the $x$-value, the graph of the function has an asymptote.


For either of these two cases, we would say that the limit as $x$ approaches the target number "does not exist."

## Definition:

We say that a function $f$ has limit $L$ as $x$ approaches the target number $a$, written

$$
\lim _{x \rightarrow a} f(x)=L
$$

if the value $f(x)$ can be made as close to the number $L$ as we please by taking $x$ sufficiently close to (but not equal to) $a$.

Note that $L$ is a single real number.

## Evaluating Limits

There are several methods for evaluating limits. We will discuss these three:

1. substituting
2. factoring and reducing
3. finding limits at infinity

To use the first two of these methods, we will need to apply several properties of limits.
Properties of limits:
Suppose $\lim _{x \rightarrow a} f(x)=L$ and $\lim _{x \rightarrow a} g(x)=M$. Then,

1. $\lim _{x \rightarrow a}[f(x)]^{r}=\left[\lim _{x \rightarrow a} f(x)\right]^{r}=L^{r}$ for any real number $r$.
2. $\lim _{x \rightarrow a} c f(x)=c \lim _{x \rightarrow a} f(x)=c L$ for any real number $c$.
3. $\lim _{x \rightarrow a}[f(x) \pm g(x)]=\lim _{x \rightarrow a} f(x) \pm \lim _{x \rightarrow a} g(x)=L+M$.
4. $\lim _{x \rightarrow a}[f(x) g(x)]=\left[\lim _{x \rightarrow a} f(x)\right]\left[\lim _{x \rightarrow a} g(x)\right]=L M$.
5. $\lim _{x \rightarrow a} \frac{f(x)}{g(x)}=\frac{\lim _{x \rightarrow a} f(x)}{\lim _{x \rightarrow a} g(x)}=\frac{L}{M}$, provided $M \neq 0$.

We'll use these properties to evaluate limits.

## Substitution

Example 3: Suppose $f(x)=x^{3}-4$. Find $\lim _{x \rightarrow 3} f(x)$.

Example 4: Evaluate: $\lim _{x \rightarrow 0} \frac{4 x+5}{x+3}$

Example 5: Evaluate: $\lim _{x \rightarrow 8} 5 x^{\frac{2}{3}}$

Example 6: Evaluate: $\lim _{x \rightarrow 3}\left(4 x^{2}-7 x+2\right)$

What do you do when substitution gives you a value in the form $\frac{k}{0}$, where $k$ is any non-zero real number?

Example 7: Evaluate: $\lim _{x \rightarrow 4} \frac{x+8}{x-4}$.

## Indeterminate Forms

What do you do when substitution gives you the value $\frac{0}{0}$ ?
This is called an indeterminate form. It means that you are not done with the problem! You must try another method for evaluating the limit!!

See if you can factor the function. If you can, you may be able to reduce the fraction and then substitute:

Example 8: Evaluate: $\lim _{x \rightarrow 1} \frac{x^{2}+3 x-10}{x^{2}-4}$

Example 9: Evaluate: $\lim _{x \rightarrow 3} \frac{x^{2}-5 x+6}{x-3}$

So far we have looked at problems where the target number is a specific real number. Sometimes we are interested in finding out what happens to our function as $x$ increases (or decreases) without bound.

## Limits at Infinity

Example 11: Consider the function $f(x)=\frac{3 x^{2}}{x^{2}+5}$. What happens to $f(x)$ as we let the value of $x$ get larger and larger?

| $x$ | 10 | 50 | 100 | 1000 | 10,000 | 100,000 | $1,000,000$ | $10,000,000$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f(x)$ |  |  |  |  |  |  |  |  |

We say that a function $f(x)$ has the limit $L$ as $x$ increases without bound (or as $x$ approaches infinity), written $\lim _{x \rightarrow \infty} f(x)=L$, if $f(x)$ can be made arbitrarily close to $L$ by taking $x$ large enough.

We say that a function $f(x)$ has the limit $L$ as $x$ decreases without bound (or as $x$ approaches negative infinity), written $\lim _{x \rightarrow-\infty} f(x)=L$, if $f(x)$ can be made arbitrarily close to $L$ by taking $x$ to be negative and sufficiently large in absolute value.

We can also find a limit at infinity by looking at the graph of a function:
Example 12: Evaluate: $\lim _{x \rightarrow \infty} \frac{6 x-7}{2 x}$


We can also find limits at infinity algebraically:

Example 13: Evaluate: $\lim _{x \rightarrow \infty}\left(5 x^{3}-8 x-3\right)$

Limits at infinity problems often involve rational expressions (fractions). The technique we can use to evaluate limits at infinity is to divide every term in the numerator and the denominator of the rational expression by $x^{n}$, where $n$ is the highest power of $x$ present in the denominator of the expression.

Then we can apply this theorem:
Theorem: Suppose $n>0$. Then $\lim _{x \rightarrow \infty} \frac{1}{x^{n}}=0$ and $\lim _{x \rightarrow-\infty} \frac{1}{x^{n}}=0$, provided $\frac{1}{x^{n}}$ is defined.
After applying this limit, we can determine what the answer should be.
Example 14: Evaluate: $\lim _{x \rightarrow \infty} \frac{5 x^{2}+2 x+6}{2 x^{2}-7 x+1}$

Often students prefer to just learn some rules for finding limits at infinity.
The highest power of the variable in a polynomial is called the degree of the polynomial. We can compare the degree of the numerator with the degree of the denominator and come up with some generalizations.

If the degree of the numerator is smaller than the degree of the denominator, then $\lim _{x \rightarrow \infty} \frac{f(x)}{g(x)}=0$.

If the degree of the numerator is the same as the degree of the denominator, then you can find $\lim _{x \rightarrow \infty} \frac{f(x)}{g(x)}$ by making a fraction from the leading coefficients of the numerator and denominator and then reducing to lowest terms.

If the degree of the numerator is larger than the degree of the denominator, then it's best to work out the problem by dividing each term by the highest power of $x$ in the denominator and simplifying. You can then decide if the function approaches $\infty$ or $-\infty$, depending on the relative powers and the coefficients.

The notation, $\lim _{x \rightarrow \infty} f(x)=\infty$ indicates that, as the value of $x$ increases, the value of the function increases without bound. This limit does not exist, but the $\infty$ notation is more descriptive, so we will use it.

Example 15: Evaluate: $\lim _{x \rightarrow-\infty} \frac{3 x^{3}+2 x-4}{x^{2}+2 x+5}$

Example 16: Evaluate: $\lim _{x \rightarrow \infty} \frac{9 x^{2}-x+1}{5 x^{2}+2 x+5}$

Example 17: Evaluate : $\lim _{x \rightarrow \infty} \frac{x-7}{x^{2}+9 x-3}$

Example 18: Evaluate: $\lim _{x \rightarrow-\infty} \frac{3 x^{4}-4 x+5}{2 x-7 x^{2}}$

From this lesson, you should be able to:
State the two basic problems of the calculus
Define limit, indeterminate form
Find a limit as $x$ approaches a target number from a graph of $f$
State when a limit fails to exist
Evaluate limits where substitution gives $\frac{k}{0}, k \neq 0$
Evaluate limits by substitution or by factoring
Evaluate limits at infinity

