Bad Derivatives, Picture Calculus and Crazy Modeling

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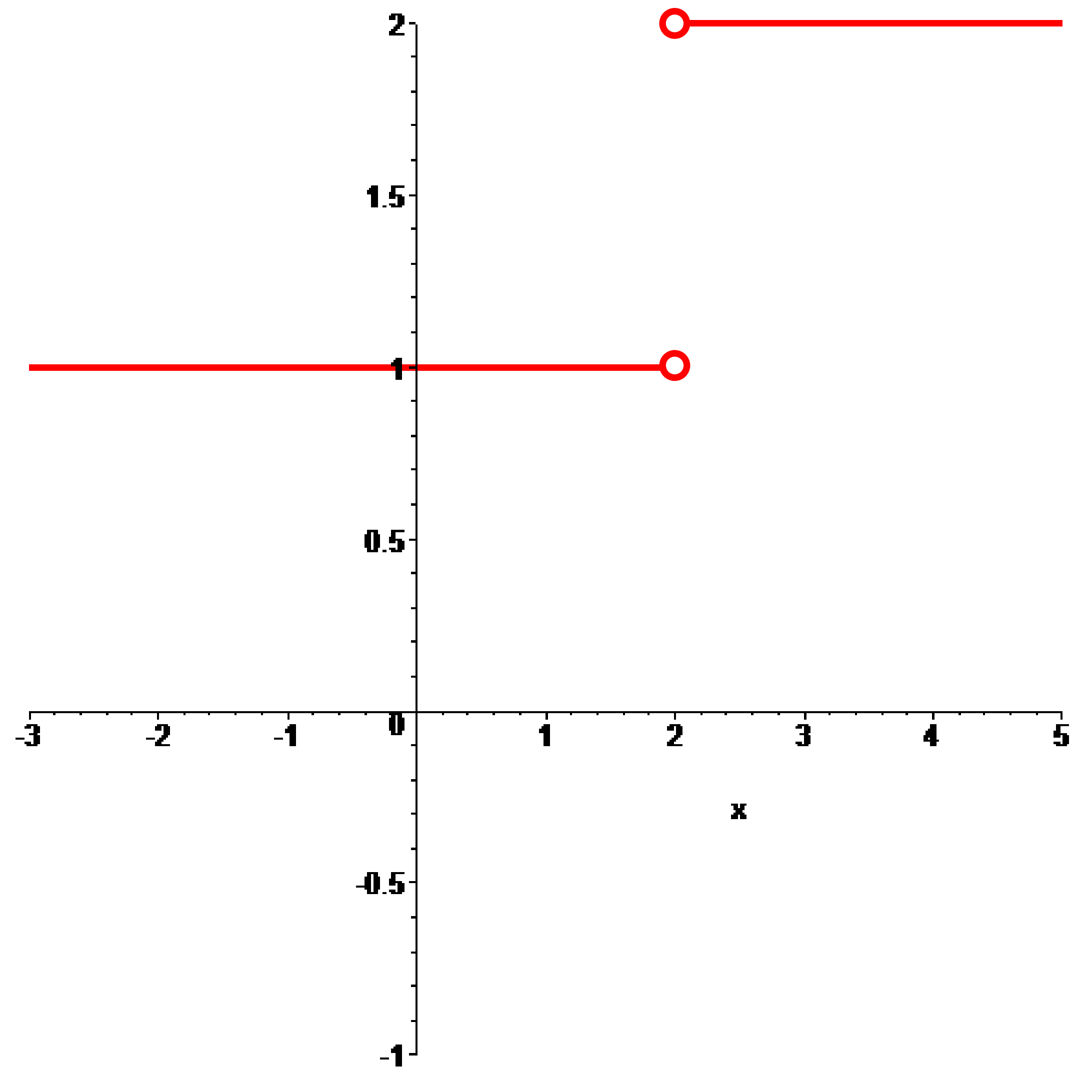
Shhhhhhhhhhhhhhhhhh.......

Make a mental note of your answers to the following four questions. Do not share your answers with others.

Can the graph on the right be the graph of a derivative of a function?



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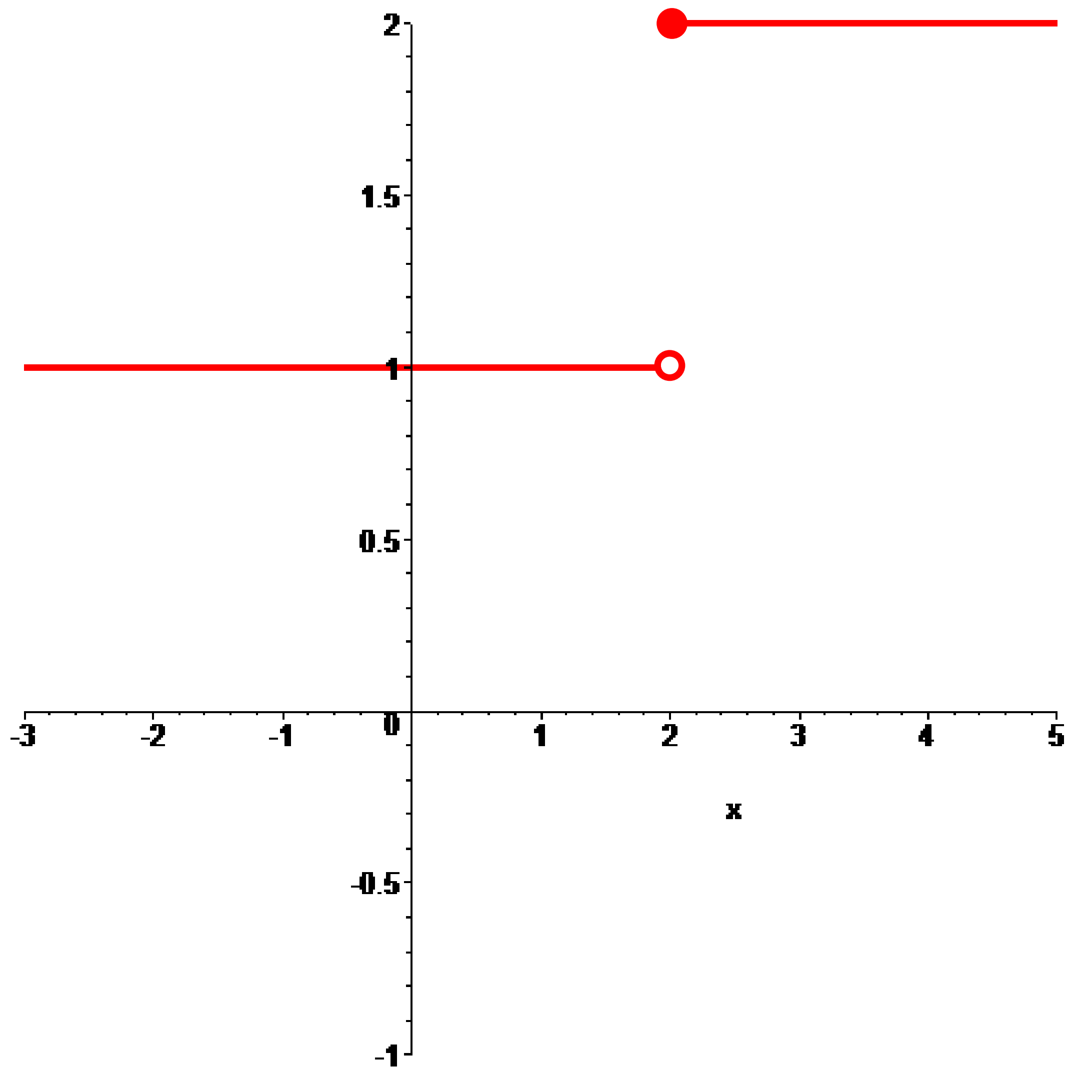

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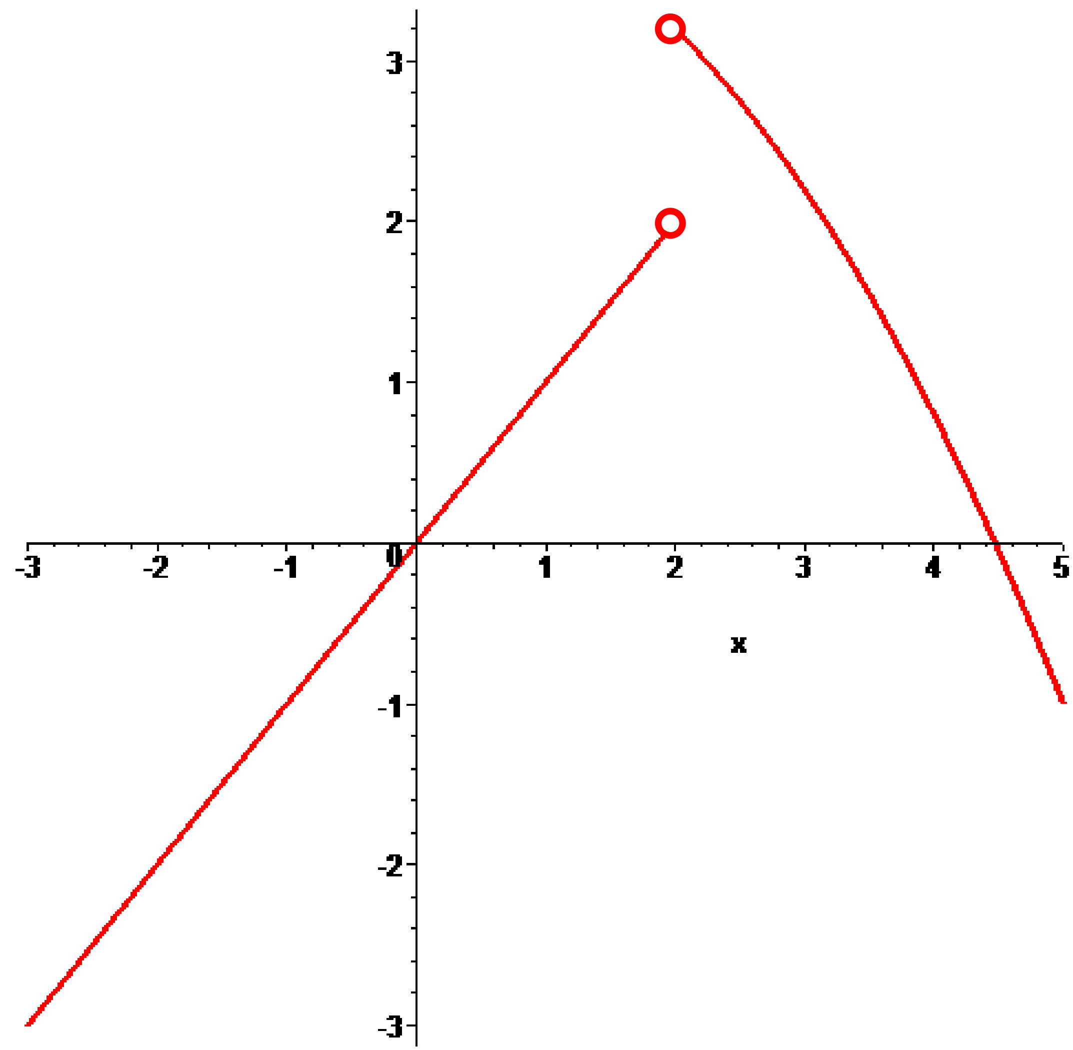

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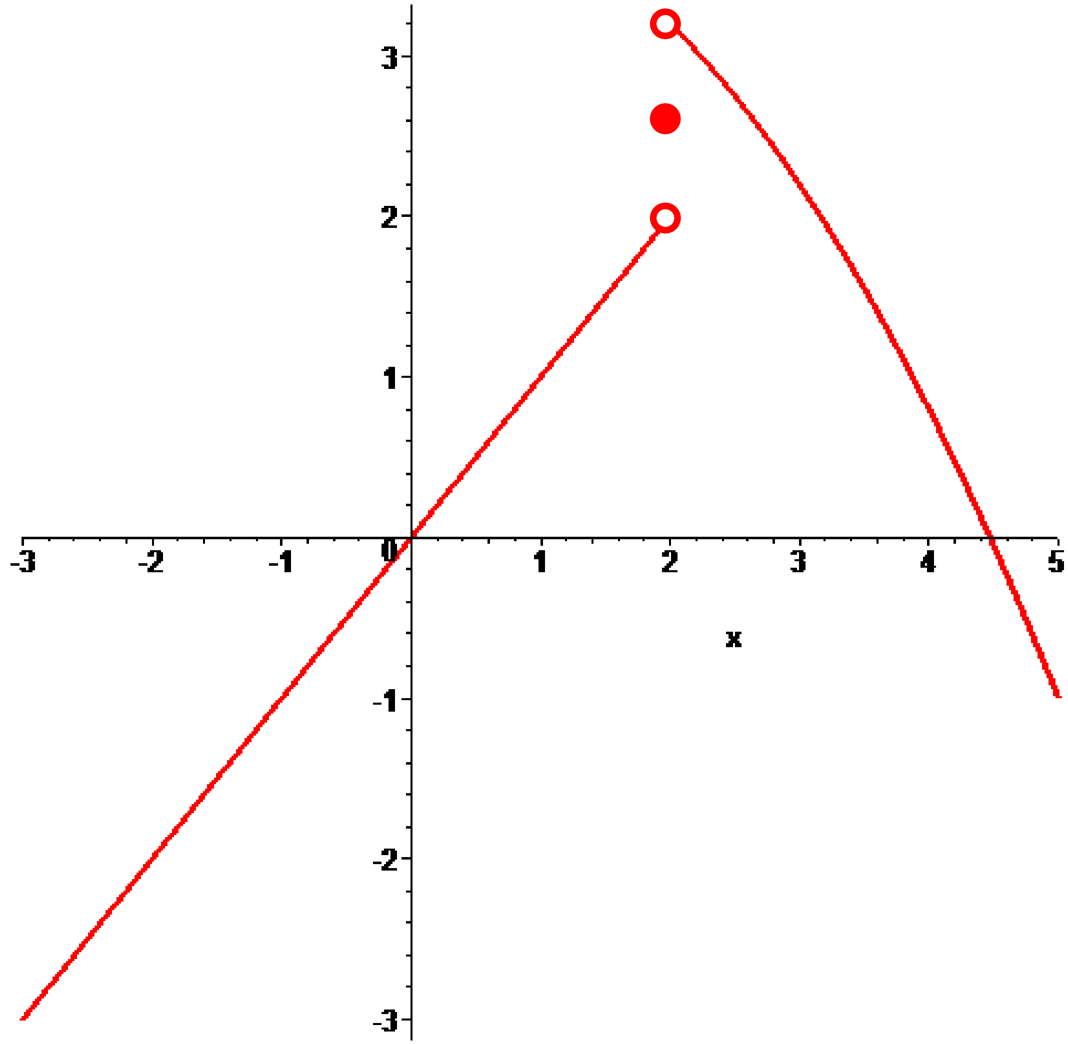


Can a differentiable function defined on *R* have a discontinuous derivative?

Show the function on the right is differentiable for all x. Then give the derivative of the function and discuss the discontinuities of the derivative if any.

$$g(x) = \begin{cases} x^2 \cos(1/x), & x > 0 \\ 0, & x \leq 0 \end{cases}$$

If the derivative is defined everywhere, then how bad can it be?

If $f'(x)$ is defined for all x then
$f'(x)$ *satisfies* the intermediate value property,
(huh??)
and consequently,
$f'(x)$ cannot have jump discontinuities
or infinite discontinuities
(go back and review your previous answers)

Darboux' Theorem:

Let f be differentiable on $[a,b]$. Then f' satisfies the intermediate value property on $[a,b]$. i.e. if M is any value between $f'(a)$ and $f'(b)$ then there is a value c between a and b so that $f'(c) = M$.

pf. Spse $f'(a) < f'(b)$. Pick $f'(a) < M < f'(b)$.

Set $g(x) = f(x) - Mx$.

① show that g has its absolute minimum in (a,b).

② How does this give the result?

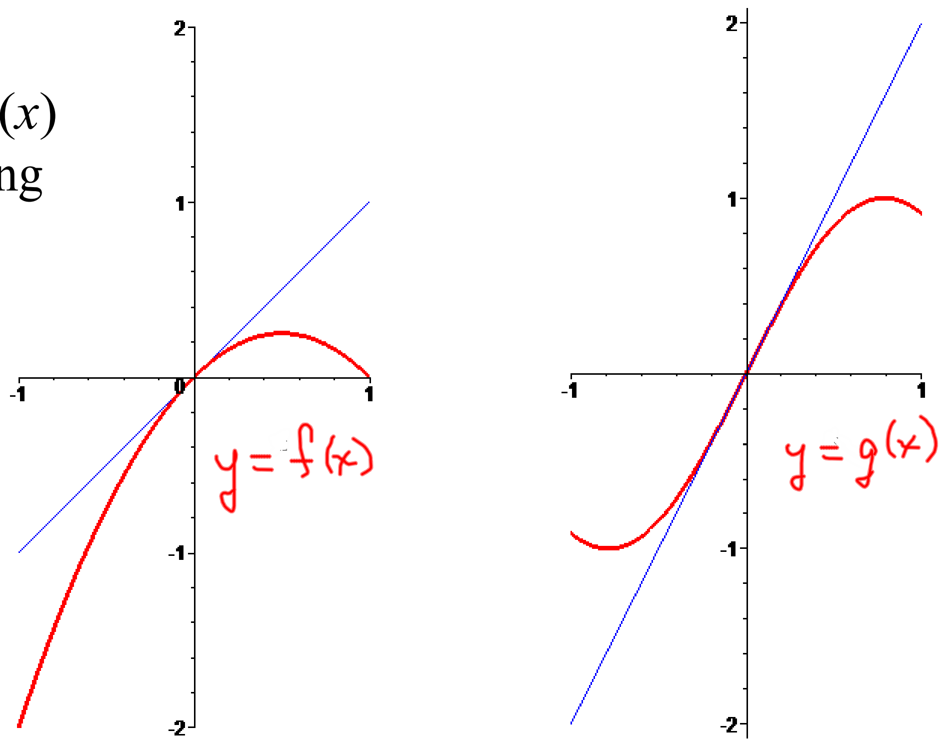
Question: Does every function that satisfies the intermediate value property have an anti-derivative?

Answer: No

$$g(x)=\begin{cases}\frac{1}{x}\cos(\ln(x)), & x>0\\ 0, & x\le 0\end{cases}$$

Picture Calculus

The graphs of $y = f(x)$ and $y = g(x)$ are shown on the right in red along with their tangent lines at $x = 0$.



Determine

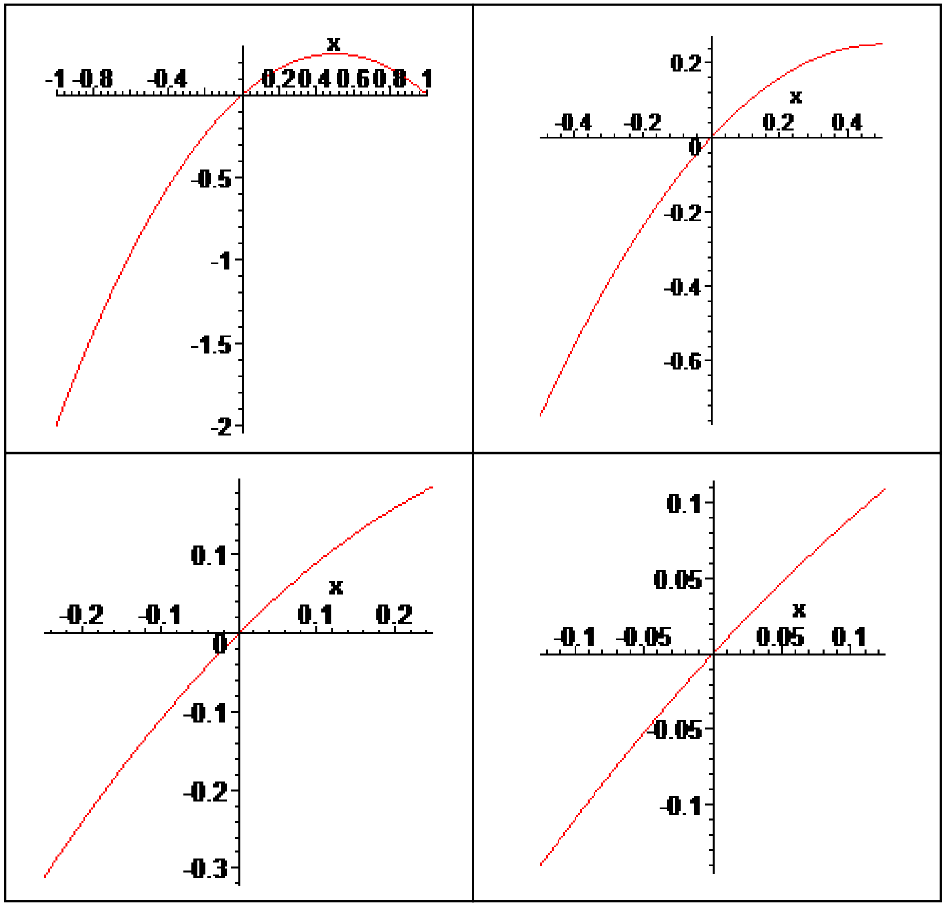
a. $$\lim_{x\to 0}\frac{g(2x)}{f(3x)}$$

b. $$\lim_{x\to 0}\frac{g(x)-x}{f(3x)}$$

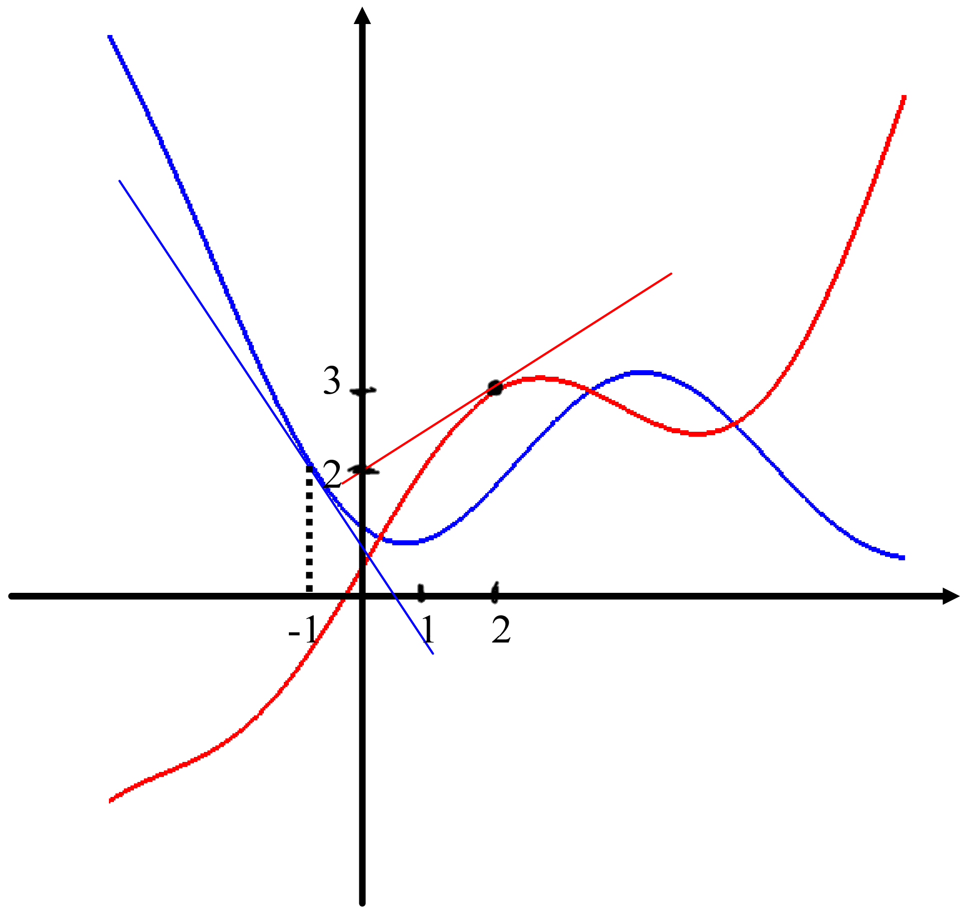
c. $$\lim_{x\to 0}\frac{g(3x)-x}{f(x)-3x}$$

Four graphs of $y = f(x)$ are shown on the right in red.

What fundamental calculus concept is being illustrated?



The graphs of $y = f(x)$ and $y = g(x)$ are shown on the right in red and blue respectively, along with the tangent line to g at $x = -1$ and the tangent line to f at $x = 2$.



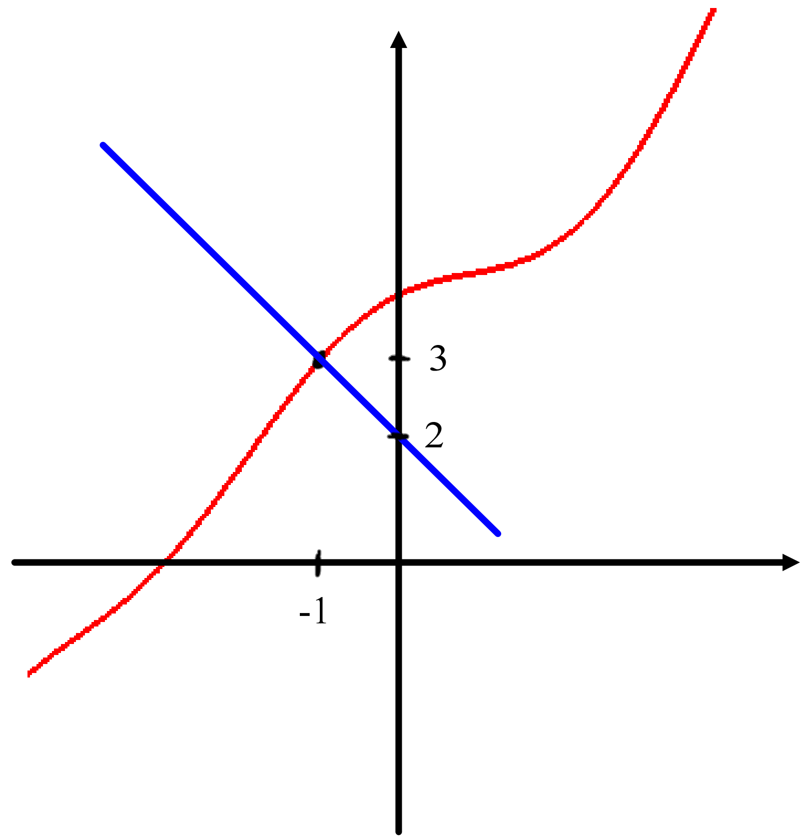
Let $h(x) = f(g(x))$.

Give $h'(-1)$.

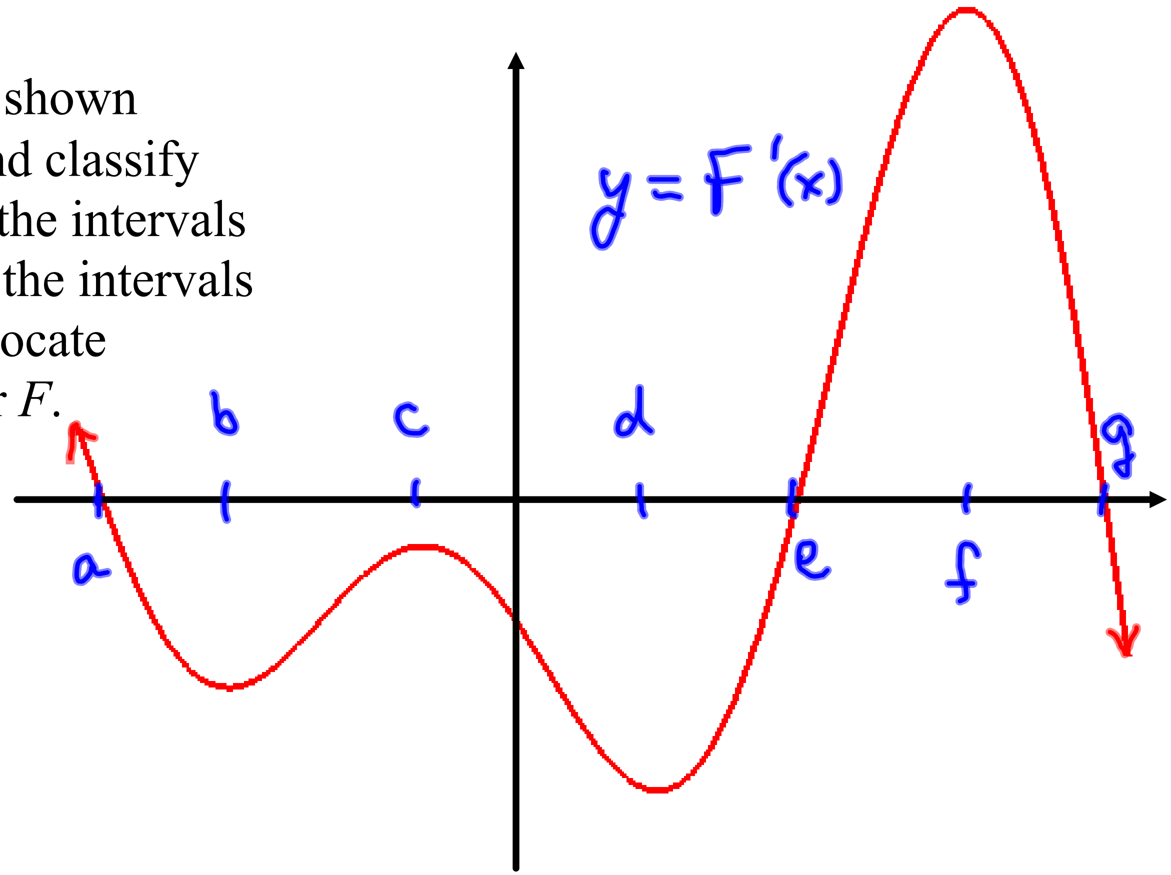
The graph of $y = f(x)$ is shown on the right in red along with its normal line at $x = -1$.

Let $h(x) = f^{-1}(x)$.

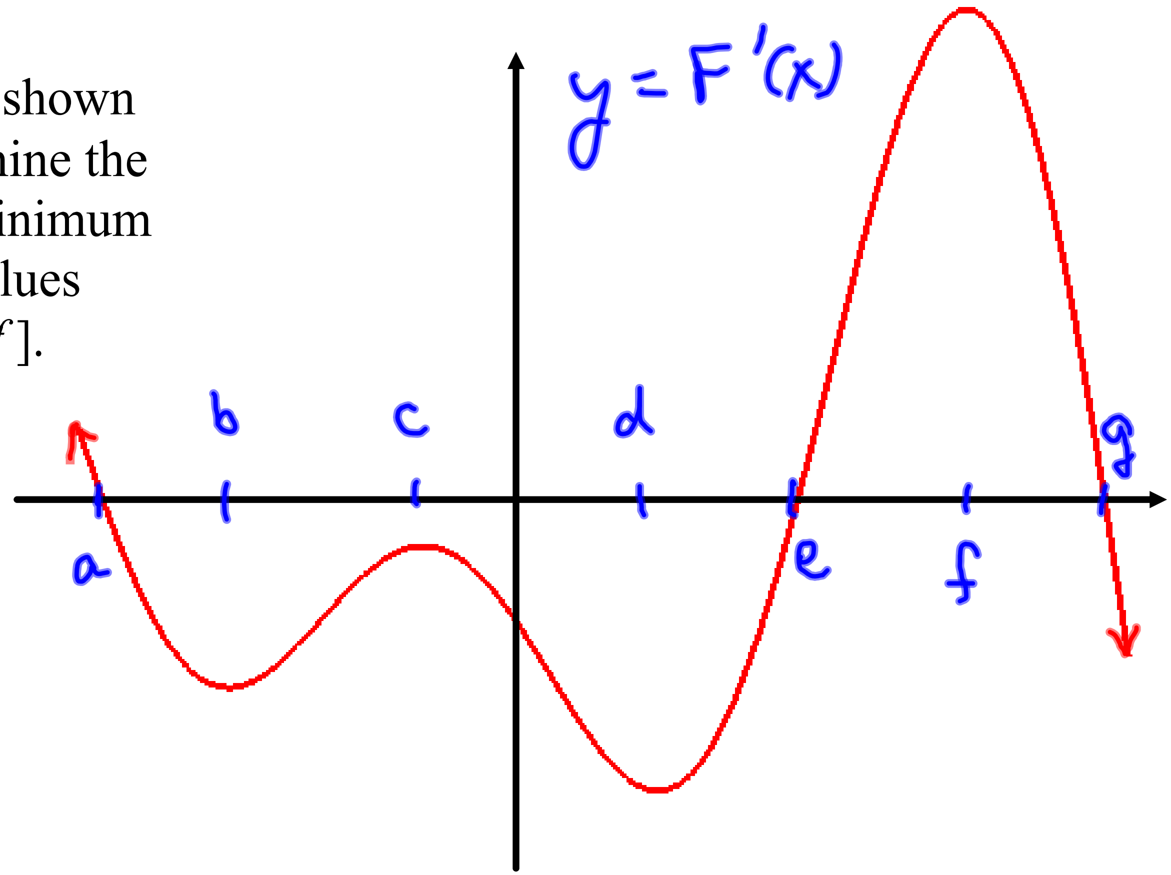
Give $h'(3)$.



The graph of $y = F'(x)$ is shown on the right in red. Find and classify the critical numbers, give the intervals of increase/decrease, give the intervals of concave up/down, and locate any points of inflection for F.



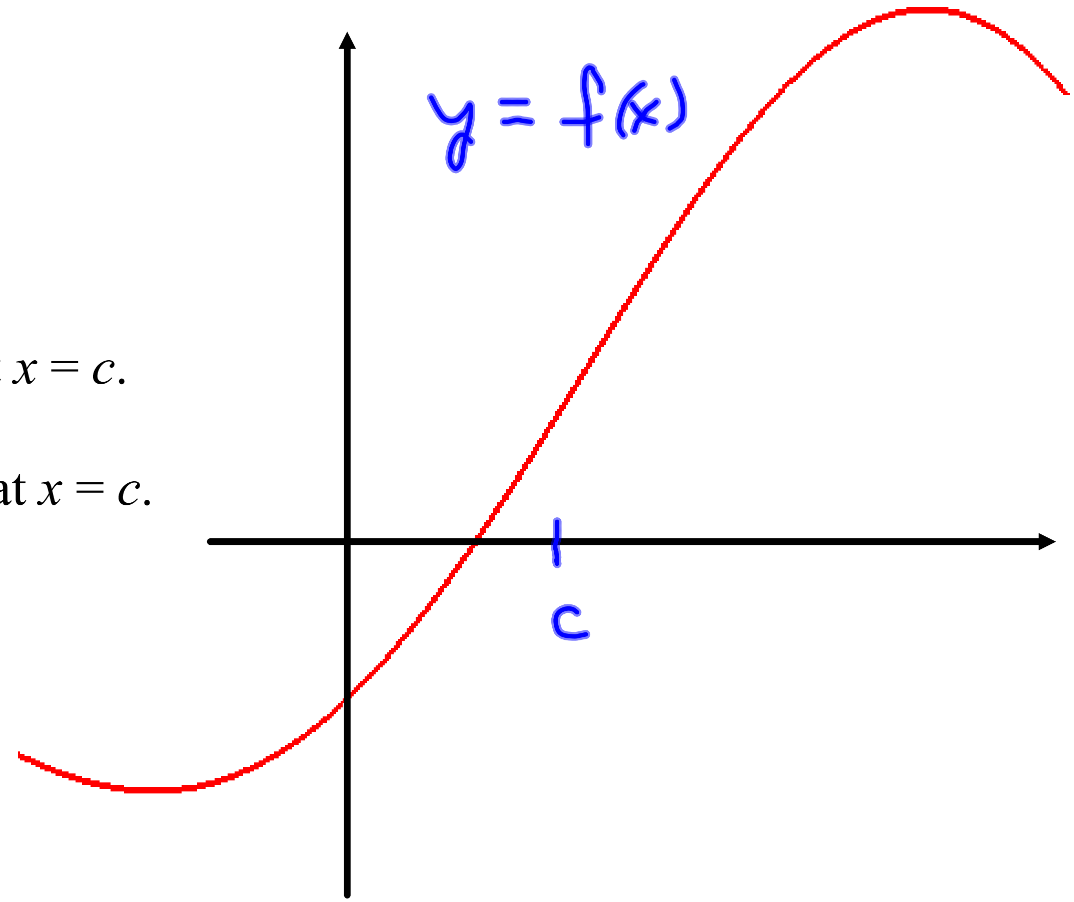
The graph of $y = F'(x)$ is shown on the right in red. Determine the location of the absolute minimum and absolute maximum values of F on the interval $[a, f]$.



The graph of $y = f(x)$ is shown on the right in red.

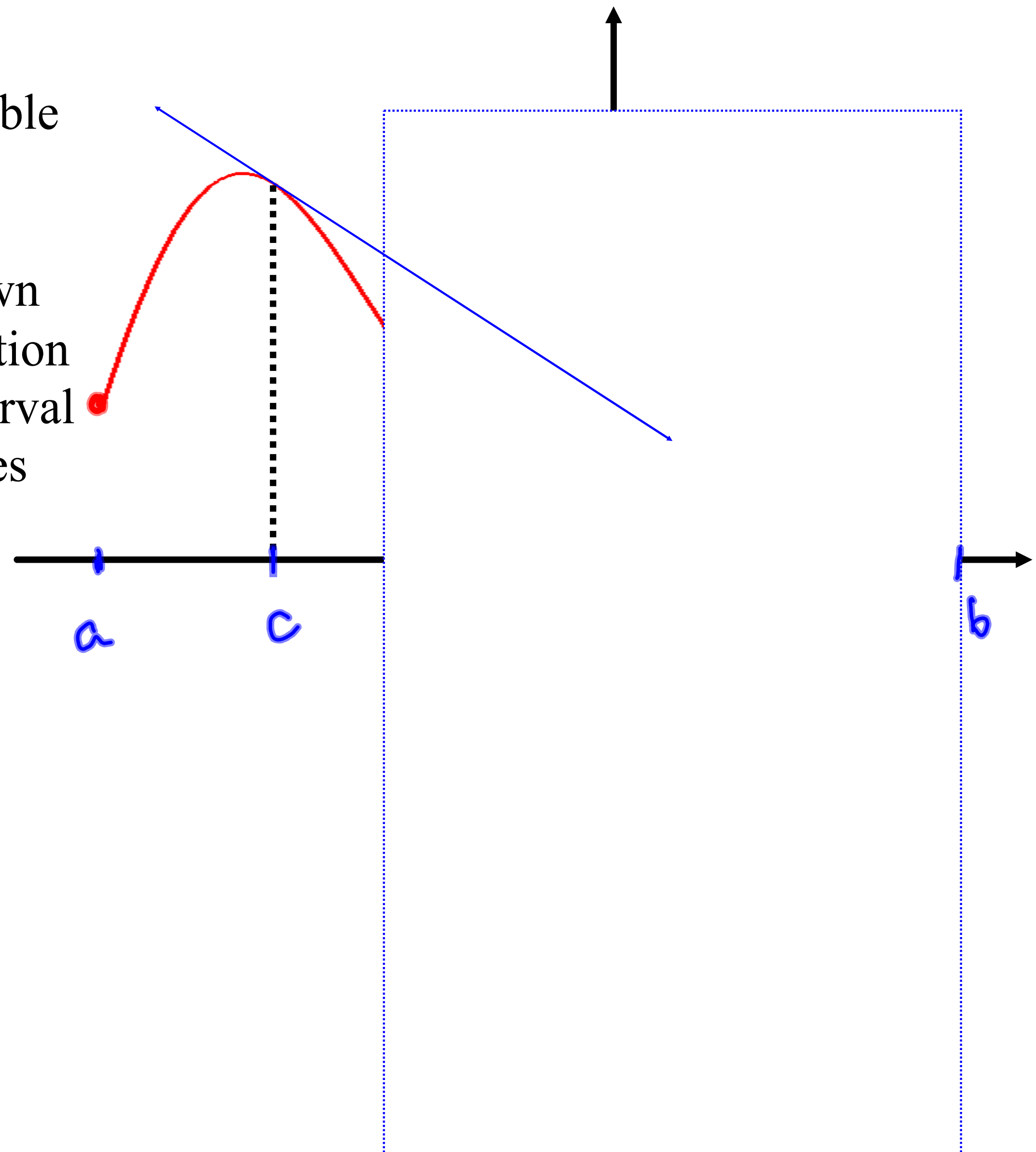
f has a ________________ at $x = c$.

f' has a ________________ at $x = c$.



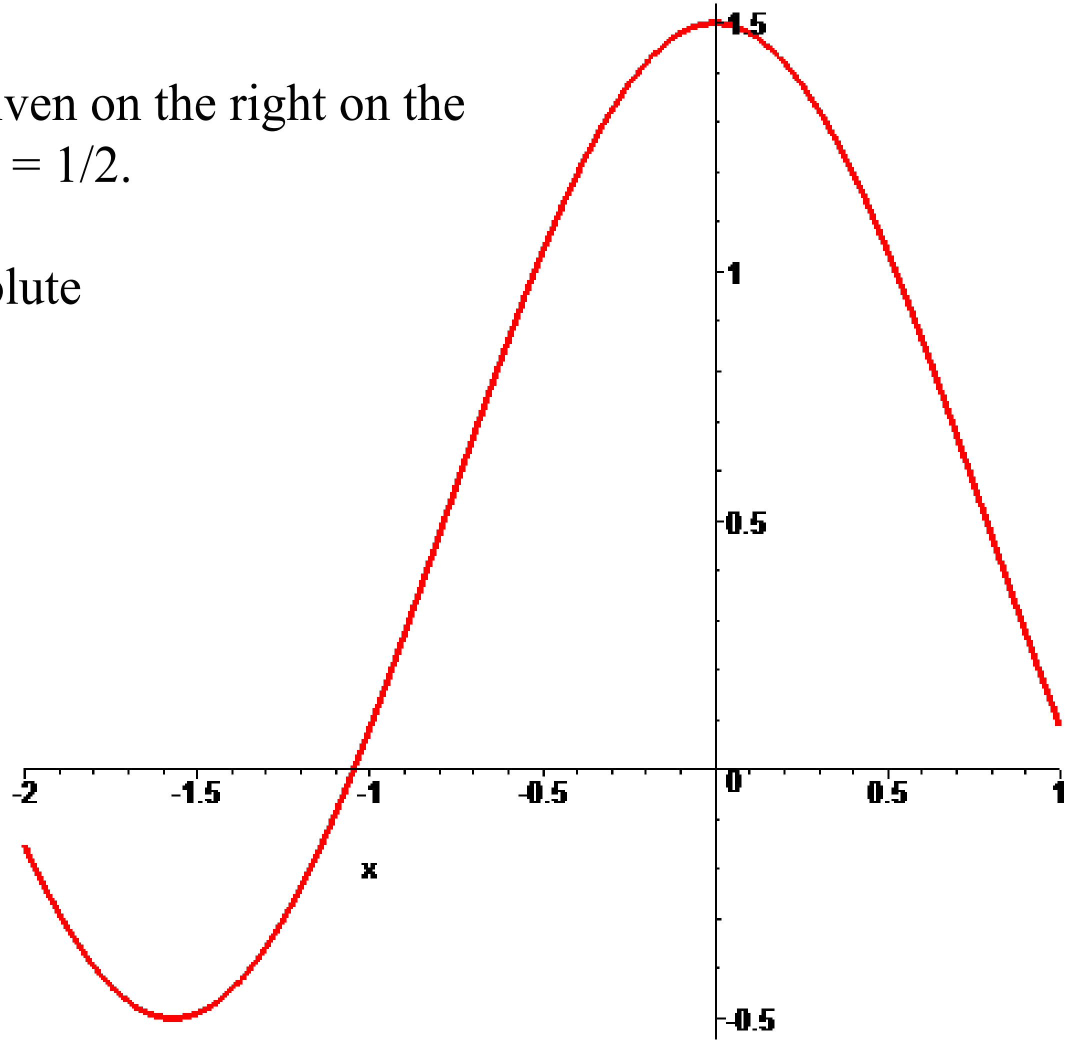
The function f is differentiable on the interval $[a,b]$.

The graph of $y = f(x)$ is shown on the right in red with a portion of the graph hidden over interval. The value of c shown satisfies the mean value theorem for derivatives on $[a,b]$. Prove that $f(x) = 0$ has a solution on the interval $[a,b]$.

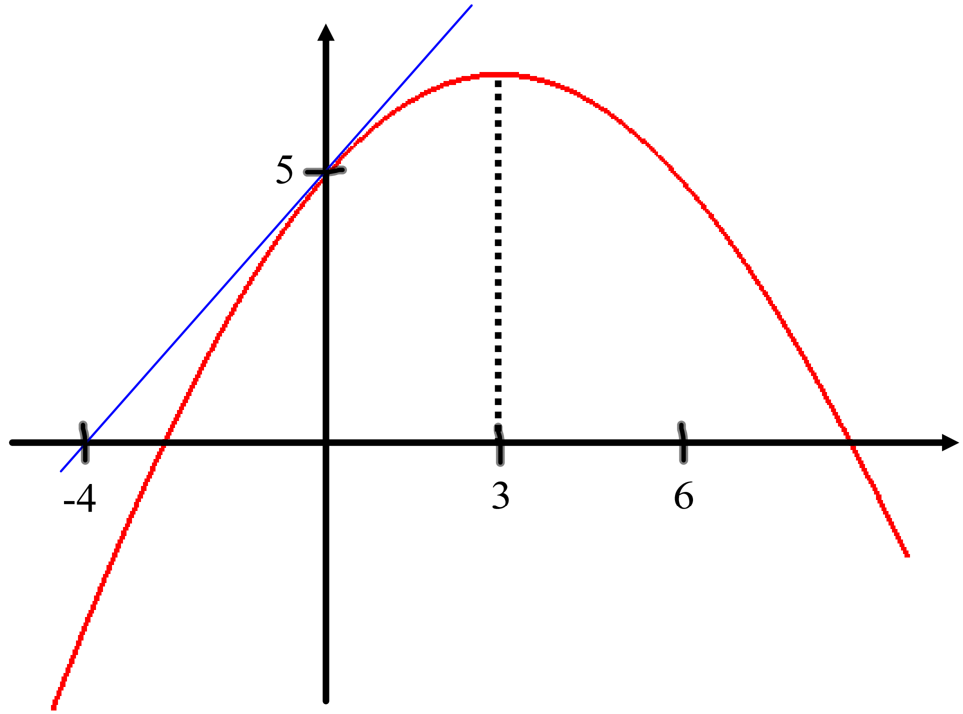


The graph of $y = f''(x)$ is given on the right on the interval [-2,1]. Also, $f'(-2) = 1/2$.

Give the location of the absolute extrema for f on [-2,1].



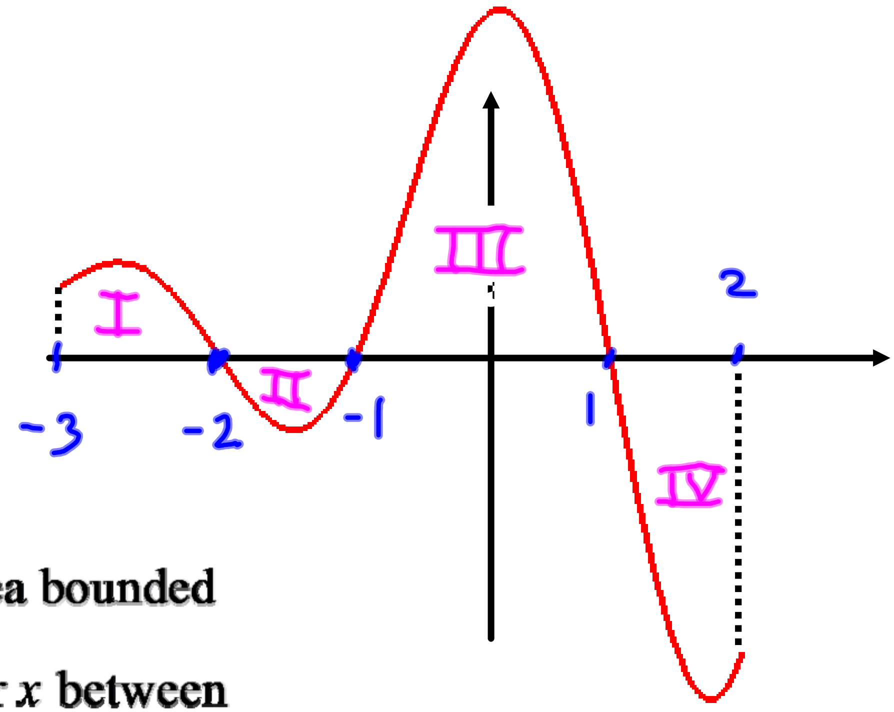
The graph of $y = f(x)$ is shown on the right on $[a,b]$ in red along with the tangent line at $x = 0$.



Suppose Newton's method is used to approximate the positive root of f from a guess of $x = 6$. $= x_0$

What is the result of one iteration?

The graph of $y = f(x)$ is shown on the right, along with the regions labeled I, II, III and IV. Suppose area(I) = 2, area(II) = 1, area(III) = 6, and area(IV) = 5.



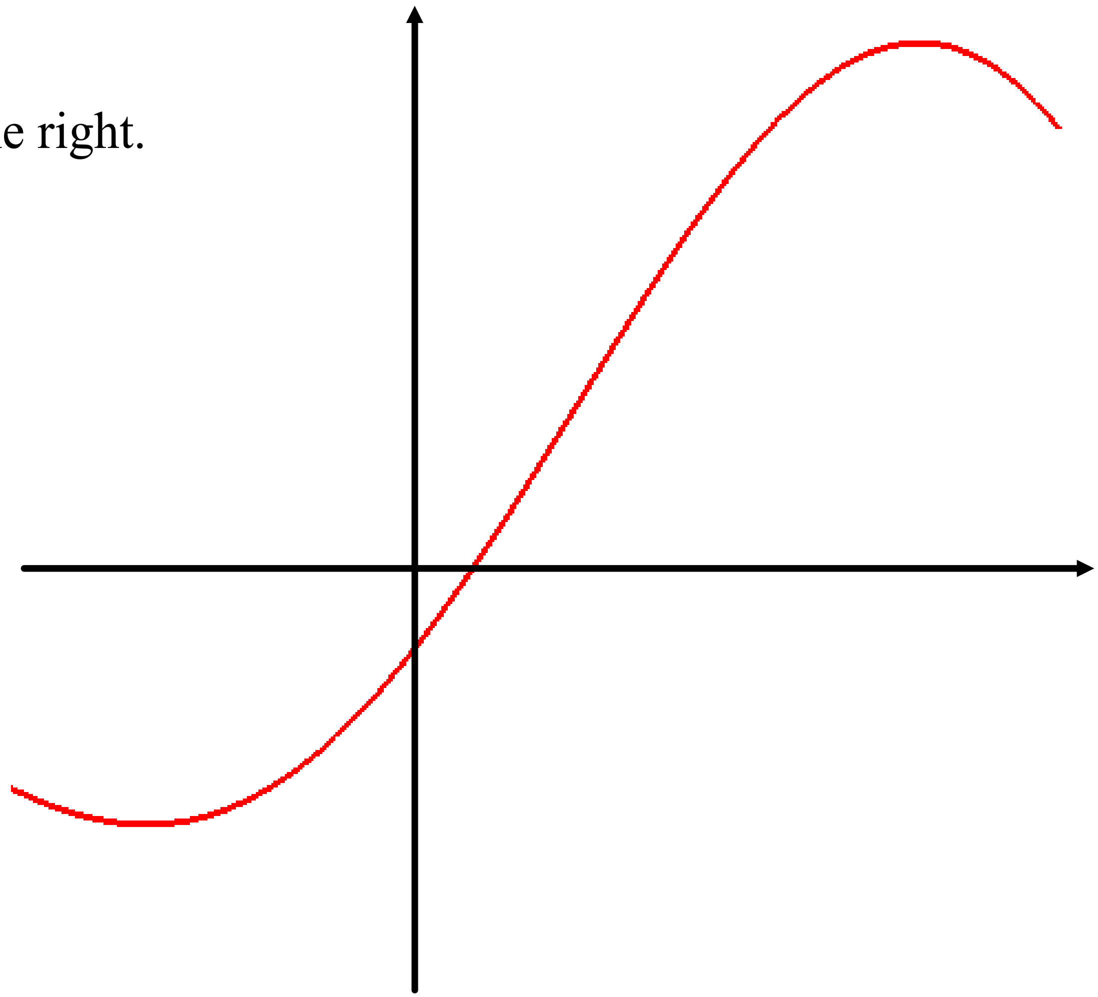
Give

$\int_{-3}^{1} f(x)\,dx$, $\int_{-2}^{2} f(x)\,dx$, and the area bounded between the graph and the x-axis for x between -3 and 2.

The graph of f is given on the right.

Give the graph of

$$\frac{d}{dx}\int_{-x}^{3} f(t)\,dt$$

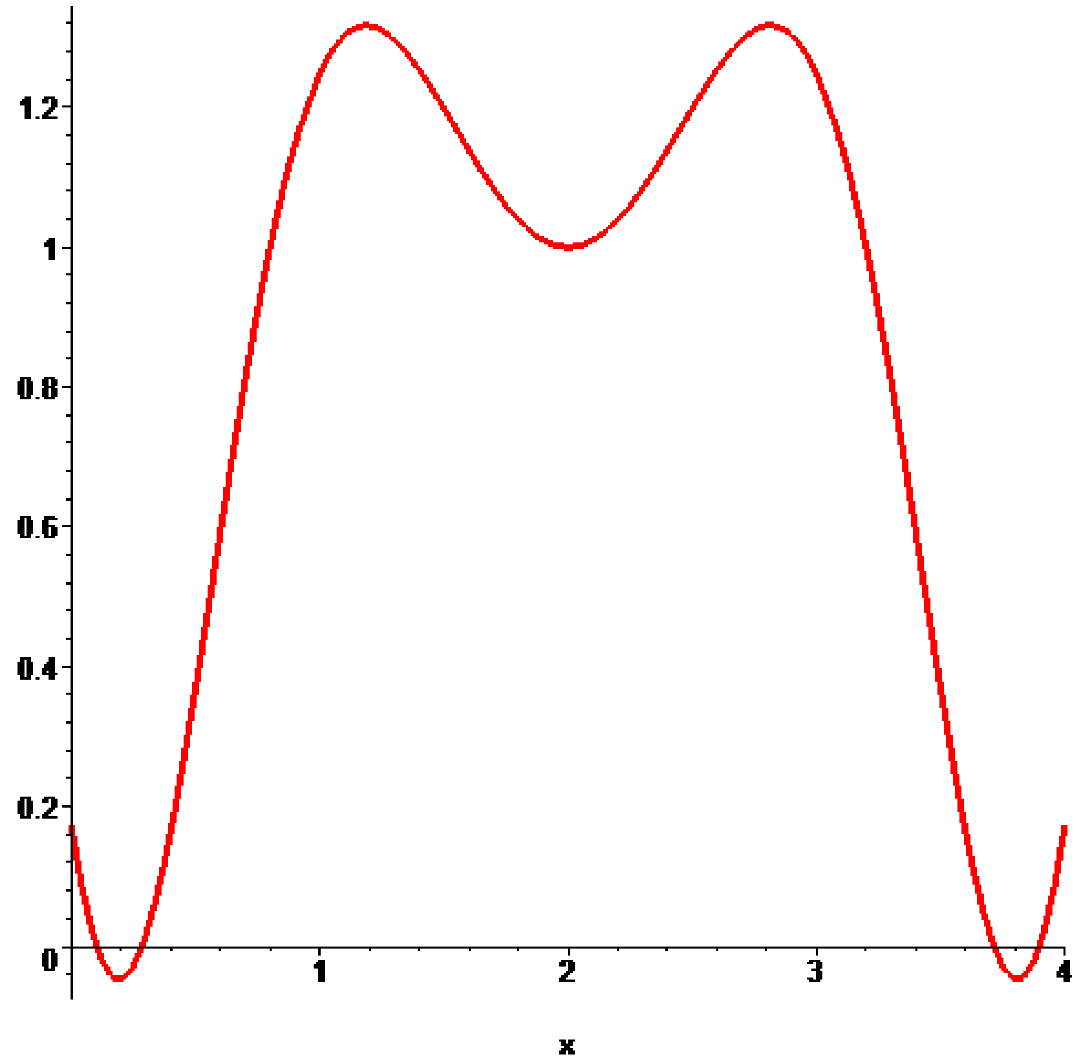


The graph of f is given on the right over the interval [0,4].

$$\int_0^4 f(x)dx = \frac{10}{3}$$

Give

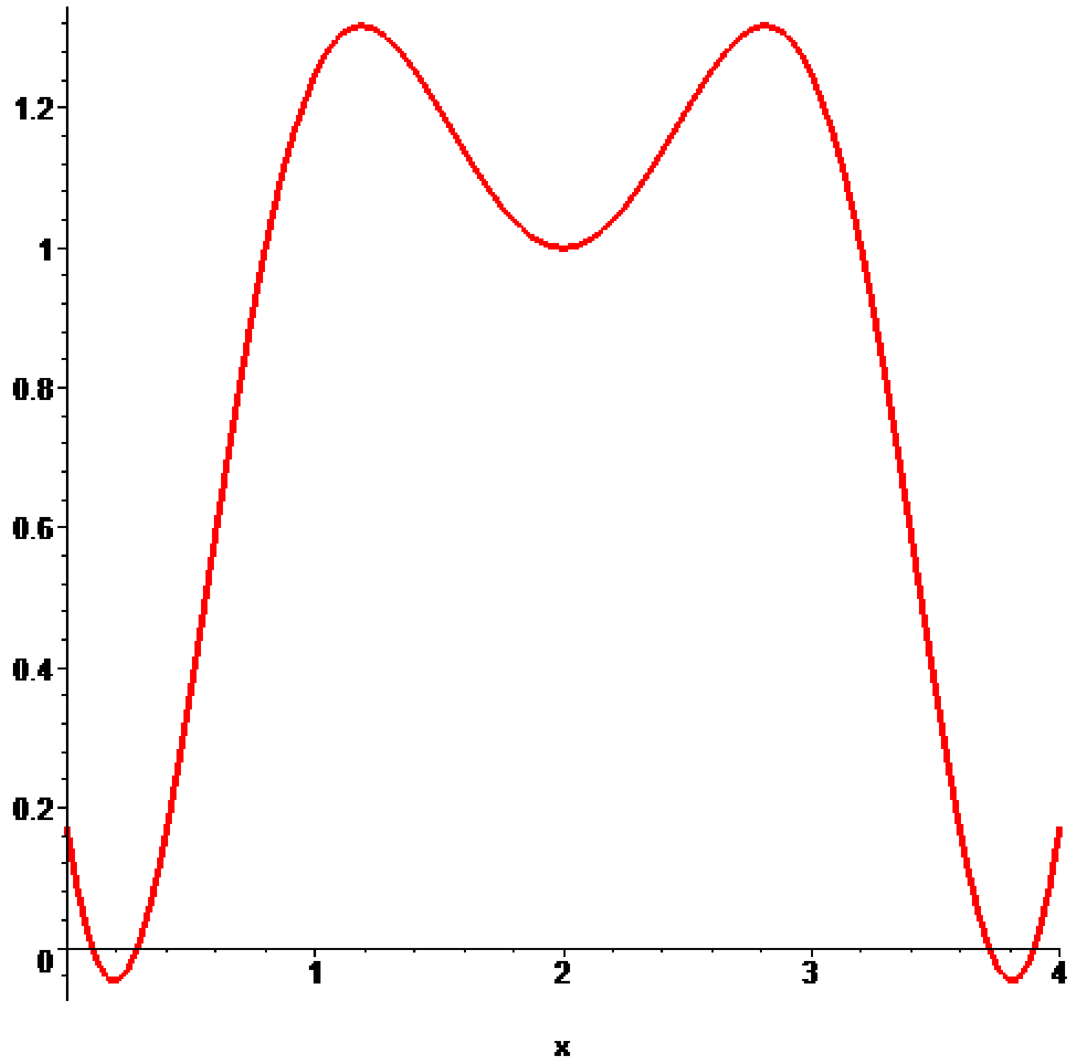
$$\int_0^4 x\, f(x)dx$$



The graph of f' is given on the right over the interval [0,4], and $f(2) = 1$.

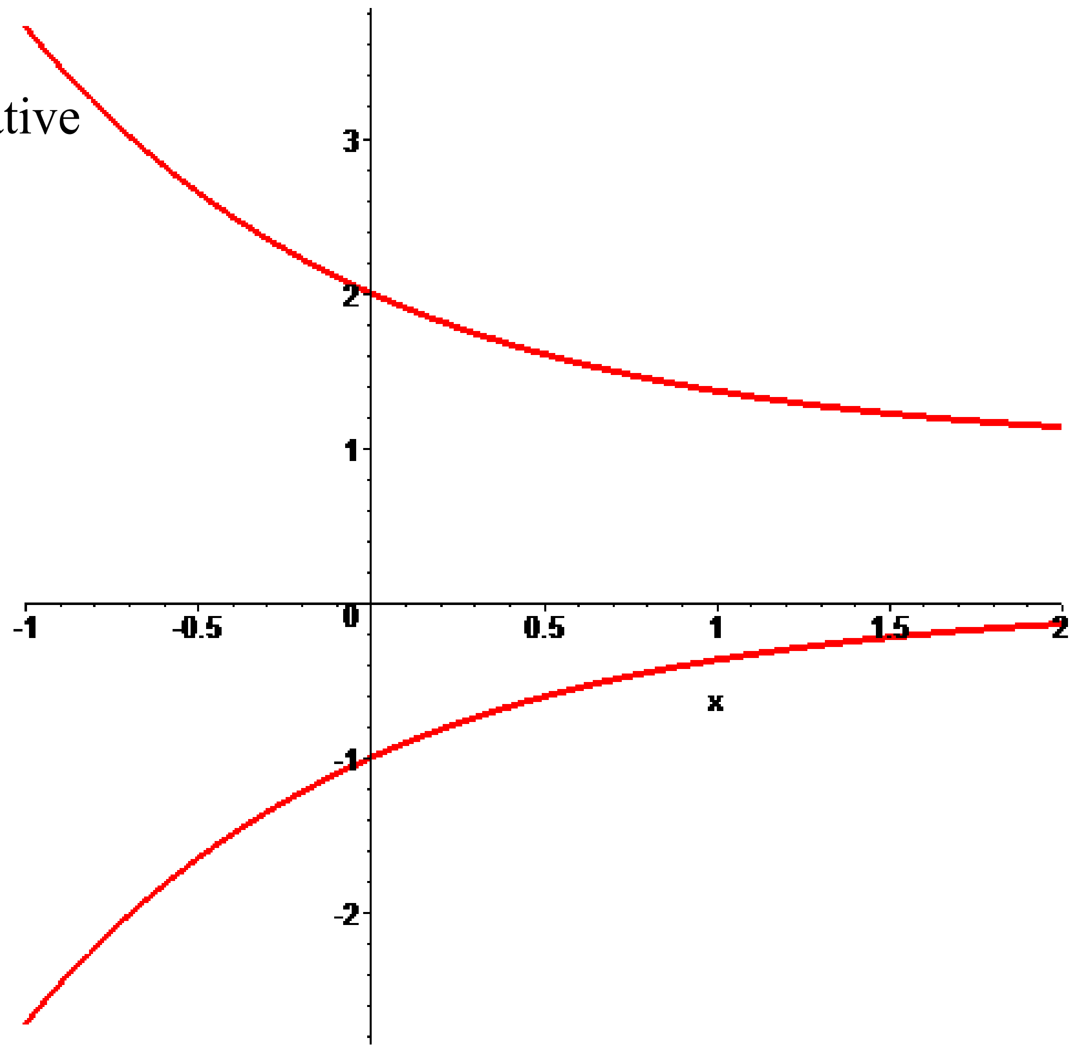
Give

$$\int_0^4 f(x)dx$$



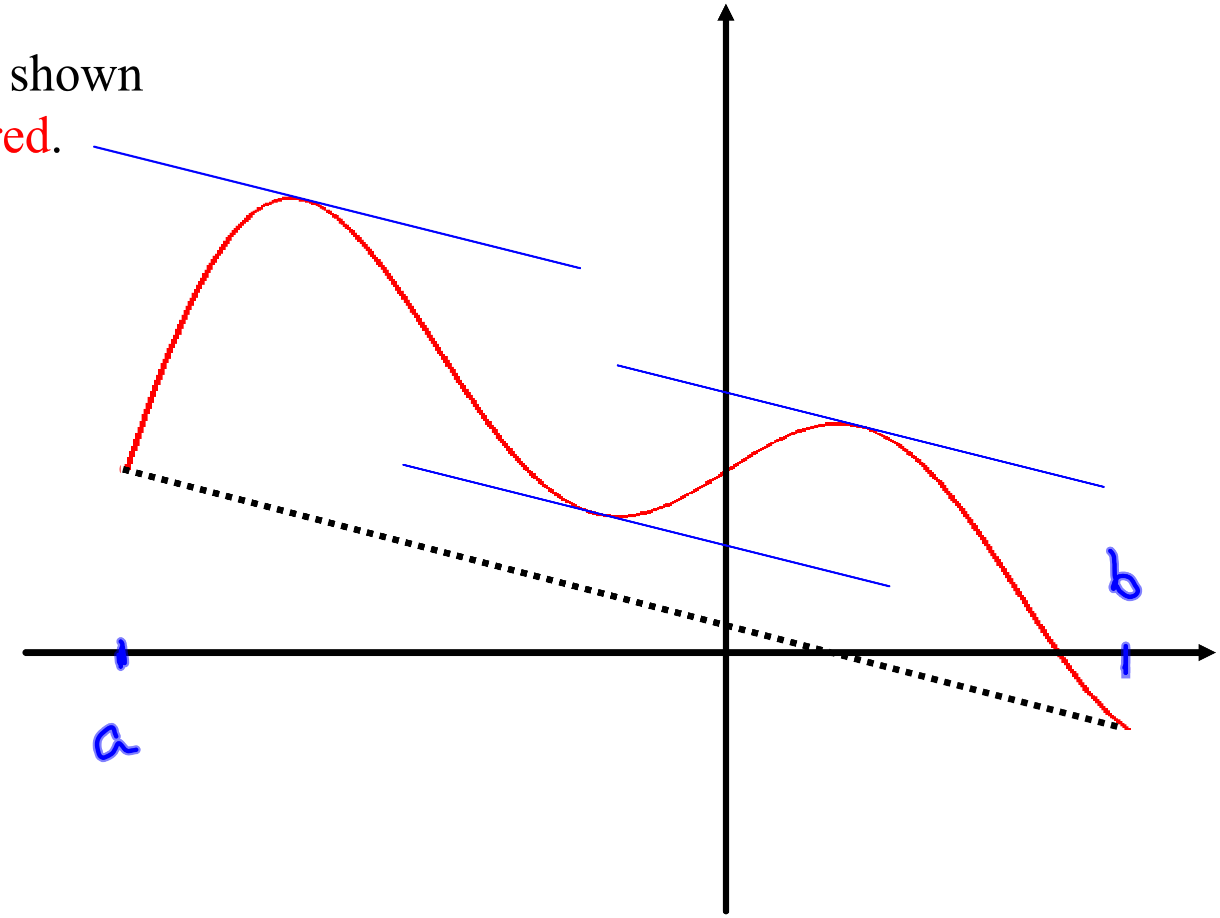
The graph of f and its derivative are given on the right.

Give a formula for $f(x)$.



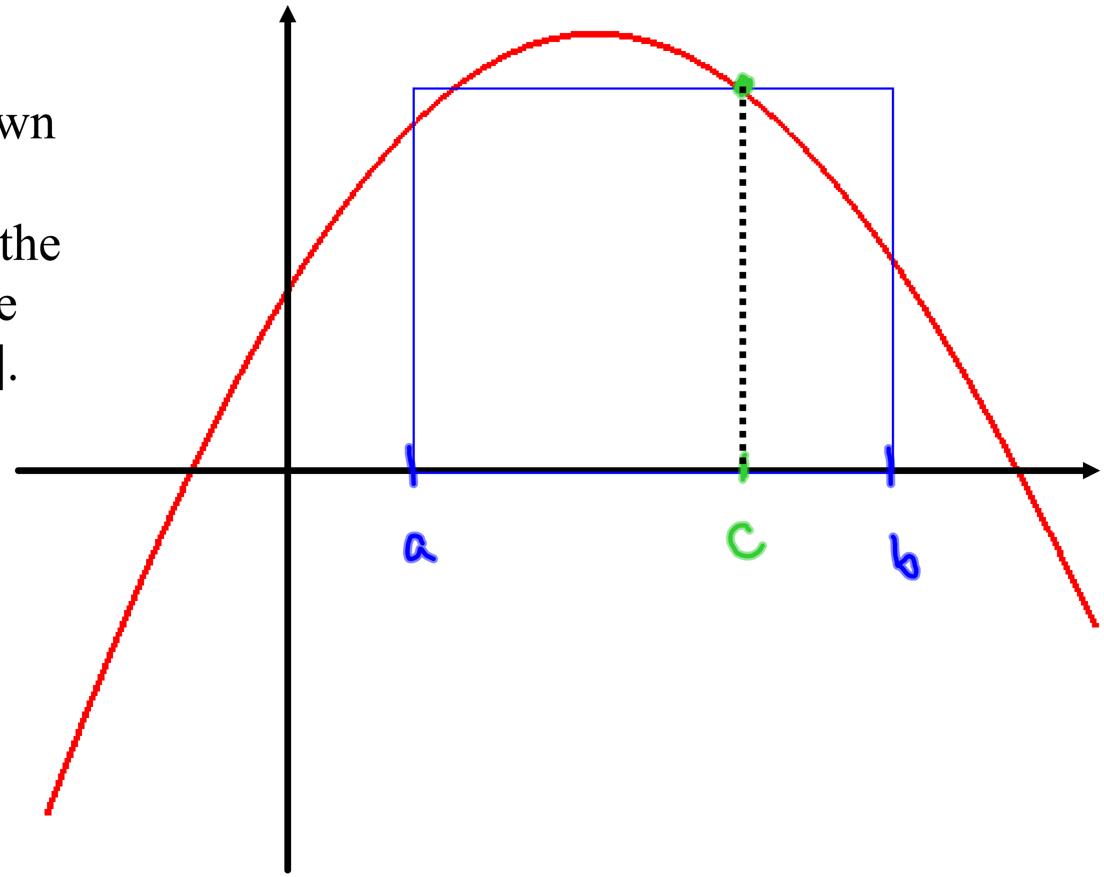
The graph of $y = f(x)$ is shown on the right on $[a,b]$ in red.

What theorem is being illustrated?



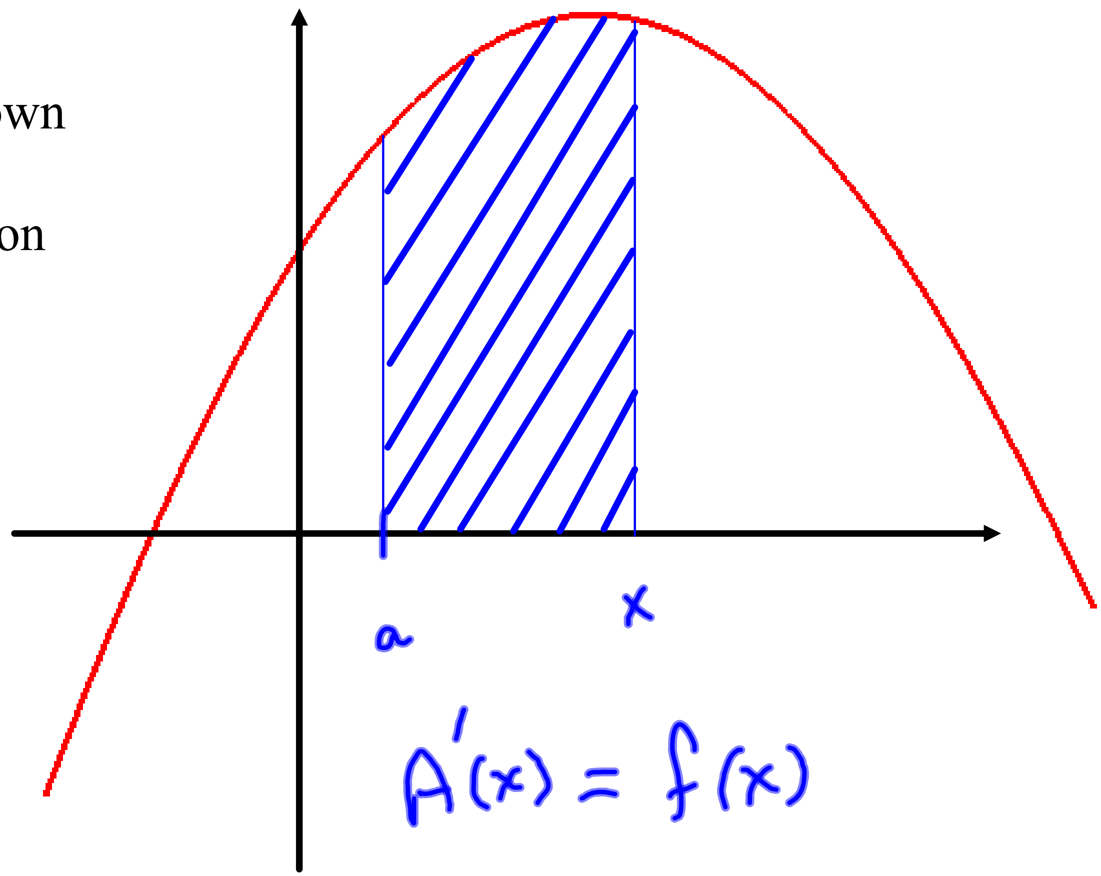
The graph of $y = f(x)$ is shown on the right in red. The area of the blue rectangle is the same as the area between the graph and the x-axis on $[a,b]$.

What theorem is being illustrated?

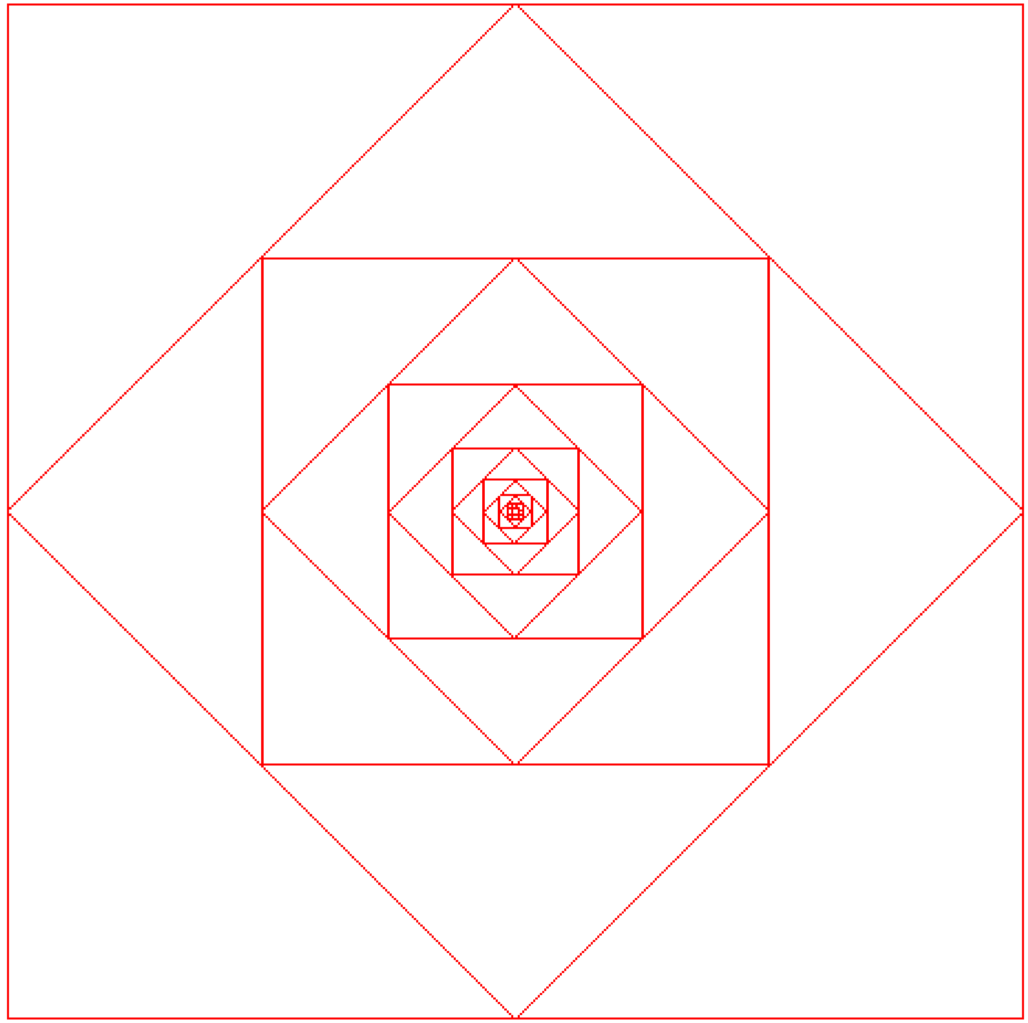


The graph of $y = f(x)$ is shown on the right in red. The area of the blue shaded region is $A(x)$.

What theorem is being illustrated?



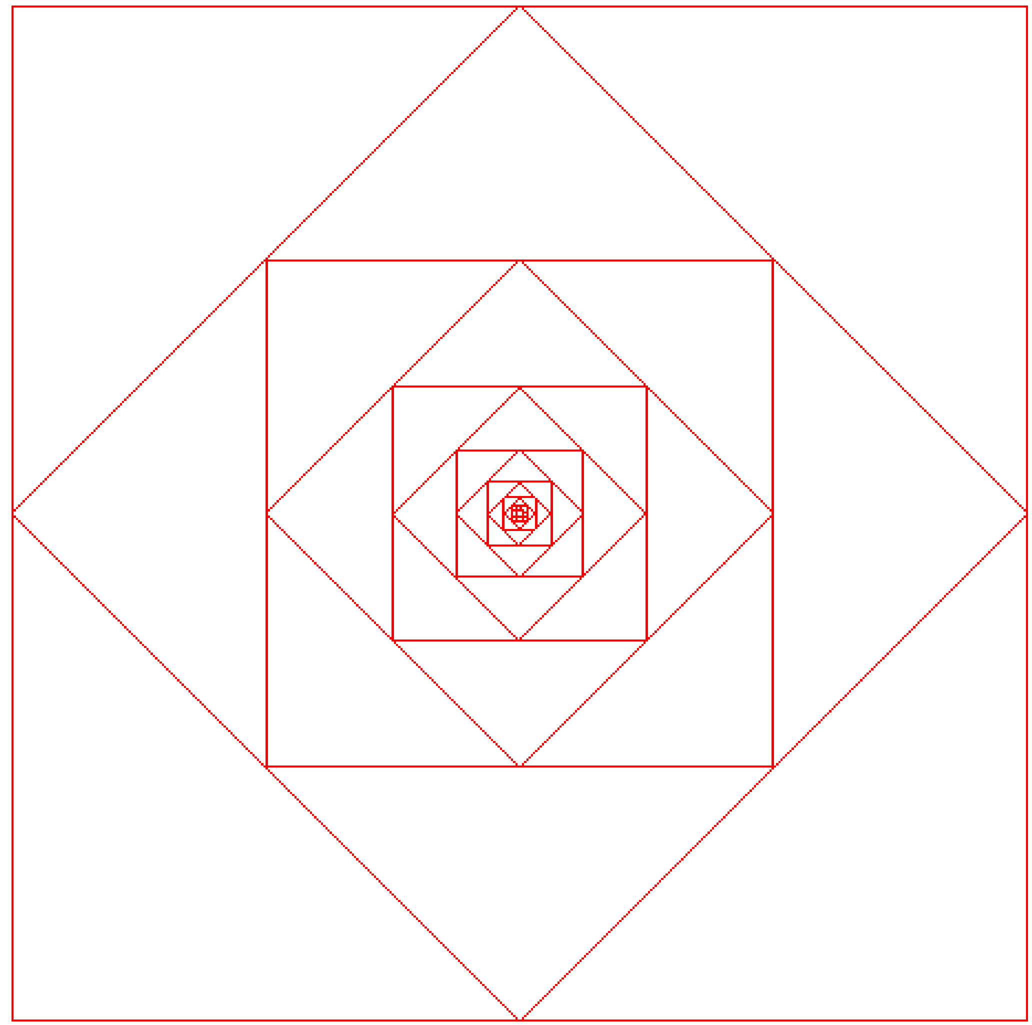
The picture on the right shows an infinite sequence of squares, with each successive square imbedded in the previous square. The outer square has side length x.



Notice that the figure implies the sum of the areas of the right triangles in the figure is x^2.

What geometric series does this sum yield?

The picture on the right shows an infinite sequence of squares, with each successive square imbedded in the previous square. The outer square has side length x.



a. Give the sum of the circumferences of the squares.

b. Give the sum of the circumferences of the right triangles.

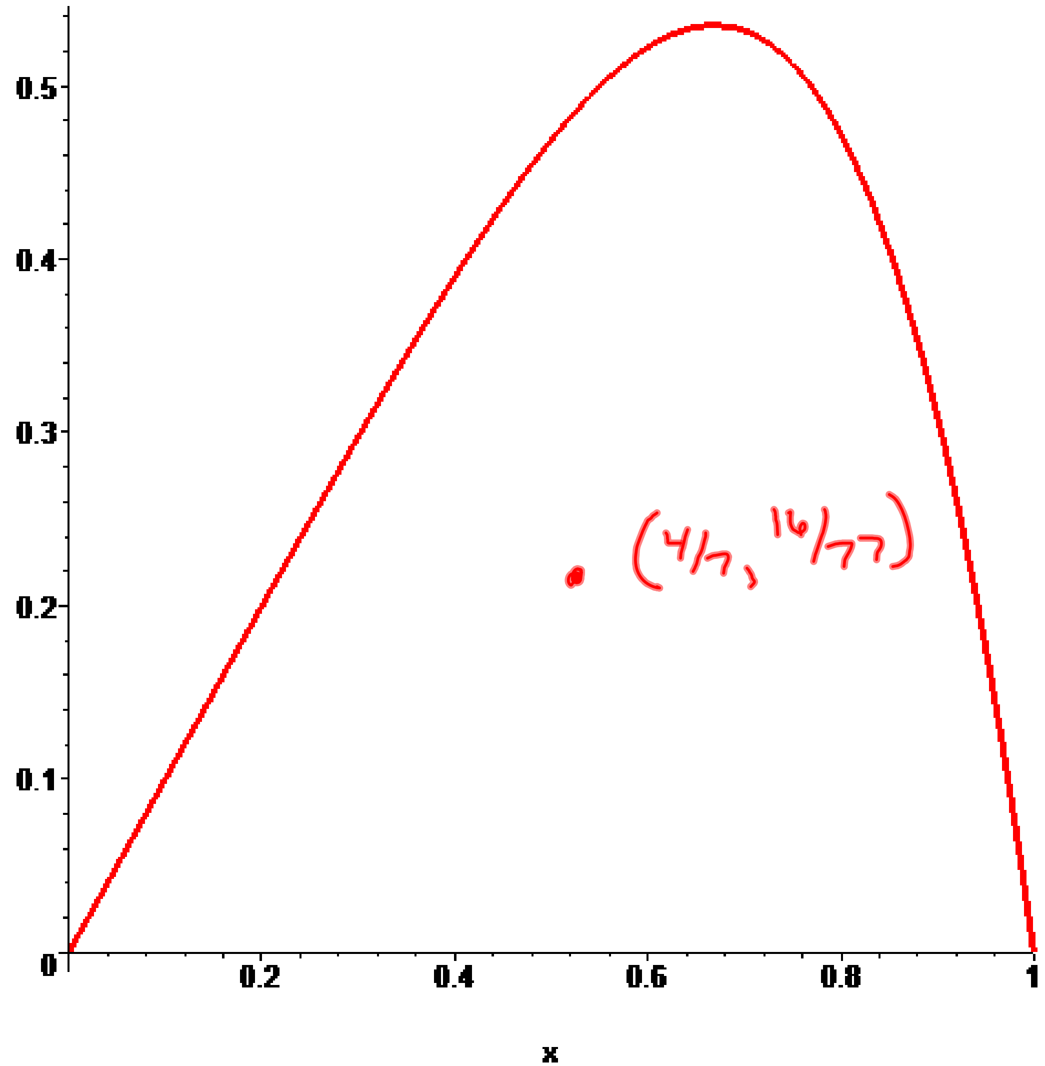
c. Give the sum of the areas of the squares.

The graph of f is given on the right. Let R be the region bounded between the graph and the x-axis. The area of this region is 1/3.

The centroid of R is (4/7 , 16/77).

What is the volume generated when R is rotated around the x-axis?

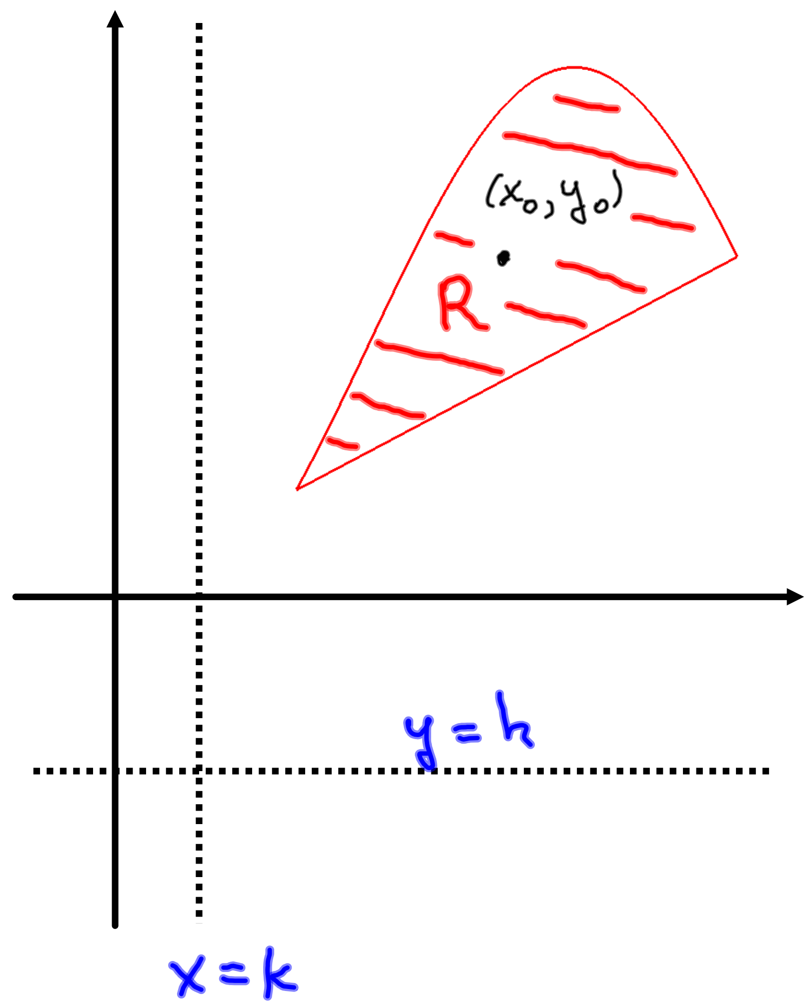
What is the volume generated when R is rotated around the y-axis?



Pappus's Theorem:

If a connected region R does not intersect the line $y = h$, then the volume of the solid generated by rotating R around $y = h$ is $2\pi\,|\,y_0 - h\,|\,\text{Area}(R)$, where y_0 is the y coordinate of the centroid of R.

If a connected region R does not intersect the line $x = k$, then the volume of the solid generated by rotating R around $x = k$ is $2\pi\,|\,x_0 - k\,|\,\text{Area}(R)$, where x_0 is the x coordinate of the centroid of R.

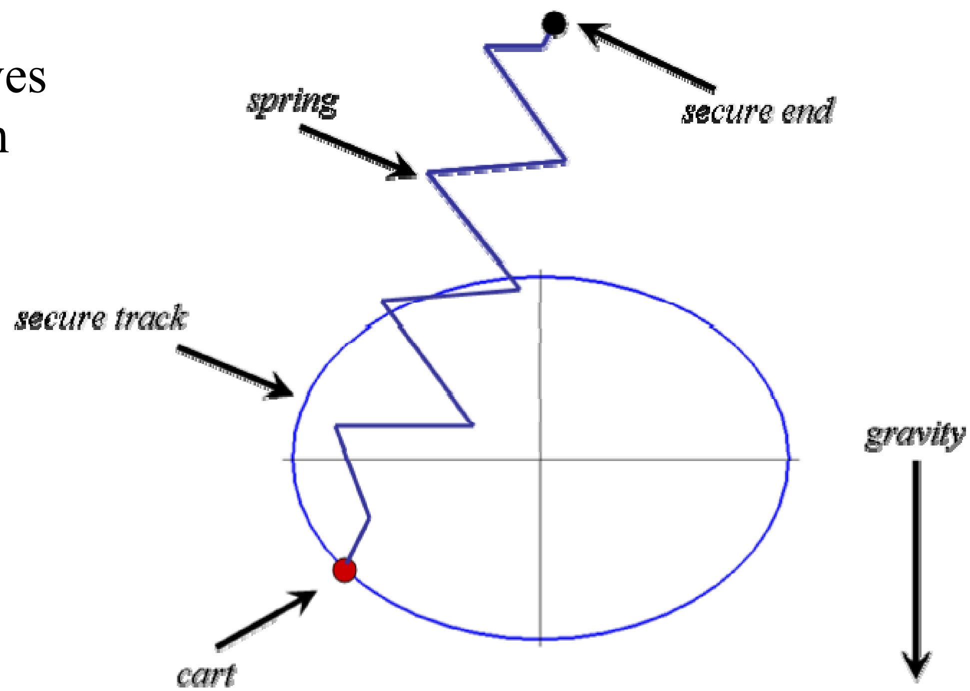


Crazy Modeling

Exploration: Consider the figure on the right. Assume the cart moves friction free along with track, with the only forces acting on the cart being gravity and the spring.

Determine whether there is a shape for the track so that the speed of the cart along the track is always constant.

Assume the force associated with the spring obeys Hooke's Law.



If we assume the spring has its fixed end at the origin, then our calculations show that tracks given implicitly by the equation

$$C = -g\,y + \frac{k}{m}\left(L\sqrt{x^2 + y^2} - \frac{1}{2}\left(x^2 + y^2\right)\right)$$

can lead to this behavior.

Here, C is a constant, g is the gravitational constant, m is mass, L is the natural length of the spring, and k is the spring constant from Hooke's law.